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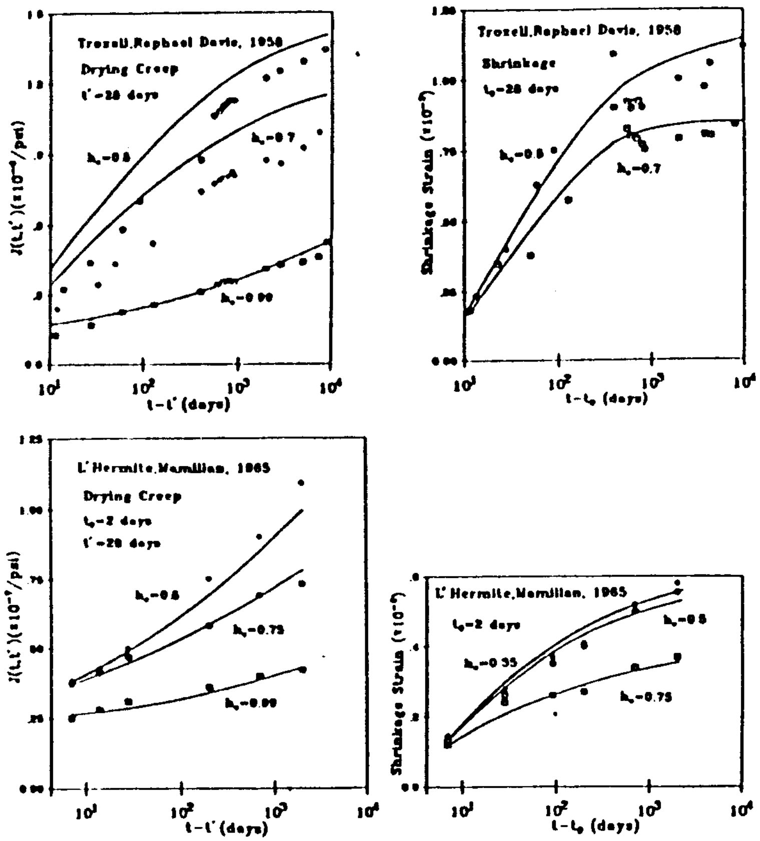


Fig. 4

Compressive drying creep and shrinkage test data by Troxell et al., and L'Hermite, Mamillan. (solid lines - fits by present theory)

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## IS STRAIN-SOFTENING MATHEMATICALLY ADMISSIBLE?

Z. P. Bažant[1] and T. P. Chang[2]

The phenomenon of strain-softening, i.e., the decrease of stress at increasing strain, is now well documented experimentally for concrete (e.g., Reinhardt and Cornelissen, 1984) and is known to exist in heterogeneous brittle materials in general. In particular, the strain-softening zone within these materials is known to be of finite size and to dissipate a finite amount of energy. However, when strain-softening is applied in conjunction with the classical local continuum concept, the strain-softening zone is found to localize, in those simple cases for which exact solutions have been found, into a zone of zero thickness (a surface), see Ref. 2. Consequently, numerical solutions by finite elements which utilize the strain-softening concept, do not yield realistic results. They do converge with mesh refinement, however, to physically unrealistic solutions for which the strain-softening zone is of zero thickness and a zero energy is dissipated by failure [2].

It can be shown that the problem is due to the use of the classical, local continuum concept. Heterogeneous materials should properly be modeled by some type of a nonlocal continuum, in which the stress at a certain point is not a function of the strain at the same point but a function of the strain distribution over a certain representative volume of the material centered at that point, i.e., the average over that volume. Such ideas have been promulgated by numerous workers in continuum mechanics, beginning with Kröner, Levin, Kunin, Krumhansl, and Eringen. It appeared, however, that the classical nonlocal continuum model which these workers advanced cannot be used to model strain-softening since it leads to a numerically unstable behavior. It has been shown [3] that the proper form of the equation of motion of the nonlocal continuum should have the form

$$(1 - c)\, D_j\, \sigma_{ij} + c\, \tau_{ij,j} = \rho\, \ddot{u}_i \qquad (1)$$

in which $\tau_{ij}$ is the local stress tensor, a smoothed microscopic quantity; $\sigma_{ij}$ is called the broad-range stress tensor and represents the stress resultants over a certain characteristic volume $V$ of the heterogeneous material; $u_i$ is the Cartesian displacement (macroscopic, smoothed), $\rho$ is the average mass density, superior dots

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refer to time derivatives, and latin lower case subscripts refer to Cartesian coordinates $x_i$ $(i = 1,2,3)$. Furthermore, $c$ is an empirical coefficient between 0 and 1 (typically $c = 0.05$) which characterizes the subdivision of the total stress between the local stress and the broad-range stress. The broad-range stress is related to the mean strain $\bar{\epsilon}_{km}$ as follows

$$\sigma_{ij} = \bar{C}_{ijkm}\,(\bar{\epsilon})\,\bar{\epsilon}_{km} \qquad (2)$$

in which

$$\bar{\epsilon}_{km}(x) = D_m\, u_i(x) = \frac{1}{V}\int_{V(x)} \frac{\partial u_i(x')}{\partial x_i}\, dV' = \frac{1}{V}\int_{S(x)} u_j(x')\, n_i(x')\, dS' \qquad (3)$$

Here $\bar{C}_{ijkm}$ represents the 4th-order tensor of secant moduli, which depend on the mean strain tensor $\bar{\epsilon}$, $V(x)$ is the characteristic volume of the material centered at the point of location vector $x$, $S(x)$ is the surface of this volume, $n_i(x')$ are the direction cosines of a unit outward normal of this surface, and $D_m$ is a gradient averaging operator defined by Eq. 3.

The foregoing continuum equations differ from those for the classical local continuum in two significant respects. First, one must use not only the broad-range stresses but also the local stresses, or else a numerical discretization would exhibit instabilities. Second, the gradient averaging operator is used not only in the definition of mean strain (Eq. 3), as is done in the classical local continuum theory, but also in the equation of motion. Substutiting Eqs. 2-3 into Eq. 1, one then gets a symmetric operator of the form $D_j\, \bar{C}_{ijkm}\, D_m\, u_k$, while the classical local continuum theory yields the operator $\partial_j\, \bar{C}_{ijkm}\, D_m\, u_k/\partial x_m$. The latter operator is nonsymmetric, and it leads to nonsymmetric finite element matrices even if the material properties are assumed to be elastic, which is obviously an unacceptable feature of the classical local continuum theory.

The continuum equations of motion (Eqs. 1-3) can be shown to follow unequivocally from the following hypothesis: the microscopic (smoothed) stress at point x depends on the change of distance between symmetrically located points lying a distance $\ell$ apart and does not depend on the change of distance between any other two points lying a finite distance apart.

The discretization of the foregoing continuum equations of motion by finite elements leads to a system of elements which all have a characteristic size, $\ell$, which is fixed even if the mesh is refined; see Fig. 1. The elements are overlapping, i.e., are imbricated, and therefore, the present new version of nonlocal continuum is called the imbricate continuum. The imbricate arrangement of finite elements is quite easy to program. The only difference from the usual finite element programs is in the integer matrix specifying the node numbers belonging to the individual finite elements. The imbricate element arrangement is obtained only when the mesh size is smaller than the characteristic length ℓ. For element sizes equal or larger than ℓ the imbricate continuum reduces to the usual finite element model. If the elements are larger than ℓ, a consistent modeling of strain-softening requires adjusting the strain-softening relations so as to assure correct energy dissipation, as is done in the blunt crack band theory.

Figures 2-3 show some of the results obtained in Ref. 4. Analyzed is the wave propagation in a strain-softening one-dimensional bar whose ends are subjected to a constant velocity motion away from the bar beginning with time $t = 0$. The boundary velocity is such that the initial waves propagating into the bar are just below the elastic limit. When these waves meet at mid-span, strain-softening is produced. For the case of classical local continuum an exact solution of this problem is possible [2]. The finite element model of local continuum converges to this exact solution, which is characterized by a strain-softening zone of zero thickness in the middle of the bar. The solution does not depend continuously on the boundary velocity, nor does it depend continuously on the strain-softening slope. Figs. 2-3 shows the results for the present imbricate continuum, assuming the characteristic length to be 1/5 of the bar length, and the stress-strain relation to be bilinear, with a stress reduction to zero. Fig. 2 shows the profiles of mean strains at various times, for N = 5, 15, 45, and 195 elements of equal size among the bars. We see that the strain-softening region is now finite, and the results converge. Fig. 3 shows the energy consumed by failure, and it is noteworthy that this energy converges to a finite value rather than zero.

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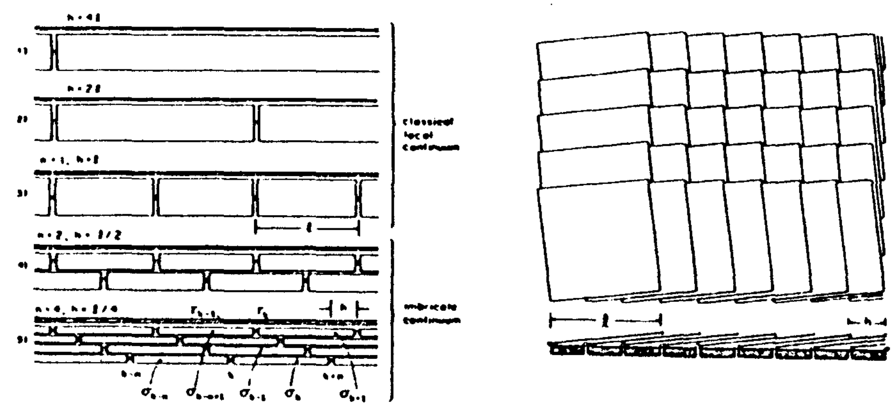

Fir. 1

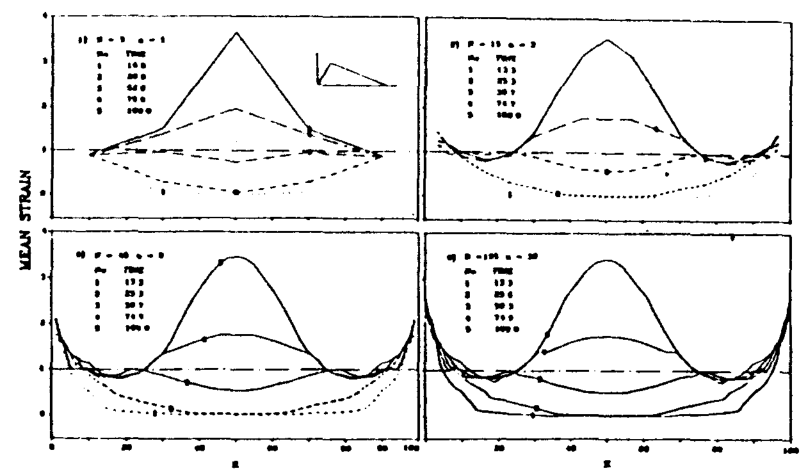


Fig. 2

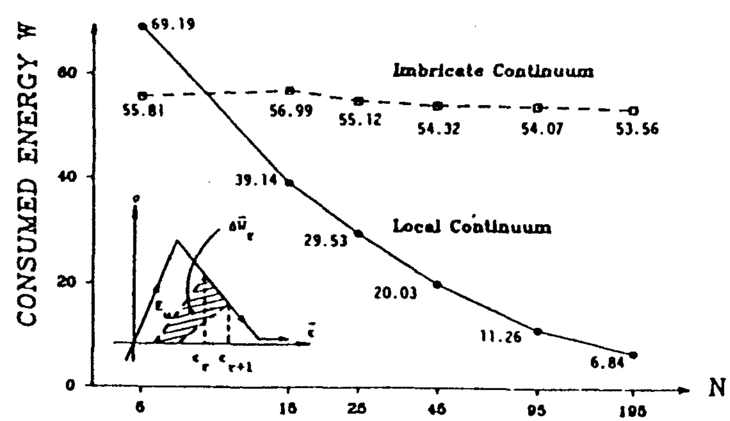


Fig. 3