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INVITED PAPER

36 SIZE EFFECTS ON FRACTURE AND LOCALIZATION: APERÇU OF RECENT ADVANCES AND THEIR EXTENSION TO SIMULTANEOUS FATIGUE AND RATE-SENSITIVITY

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1. Summary and Review of Recent Results

The size effect, defined as the dependence of nominal strength (i.e., the maximum load divided by structure dimension and thickness) of geometrically similar structures on their size, is quite pronounced in all brittle heterogenous materials that exhibit distributed cracking, yet it is almost uniformly ignored in the current practice. The usual finite element codes without some localization limiter are unable to describe it because of spurious localization of strain-softening damage. Until a few years ago it has been generally believed that the size effect is explained by Weibull-type statistical theory of random strength, and can thus be relegated to the safety factors. This is true, however, only for purely brittle structures that fail at initiation of macroscopic crack growth, as is the case for steel structures.

Due to their heterogeneity and large size of the fracture process zone, quasibrittle structures, including reinforced (and some plain) concrete structures as well as rock masses, modern tough ceramics and ice plates, typically grow large stable fractures prior to reaching their maximum load. This desirable behavior causes a strong deterministic size effect which overshadows the statistical size effect. Explanation: in a larger structure, the energy release due to fracture comes from a larger volume, and since a unit fracture extension dissipates a fixed amount of energy, the energy density in a larger structure (and thus also the nominal stress) must be lower.

The present lecture reviews recent advances in which the size effect law for quasibrittle structures (Fig. 1) has been theoretically formulated and experimentally validated (see Bibliography). It also explains how the knowledge of this law can be exploited to characterize and measure the material fracture energy with other nonlinear fracture characteristics, and to define a brittleness number of a structure. Further it shows how the finite element method can capture the size effect if some form of a nonlocal continuum damage concept is introduced

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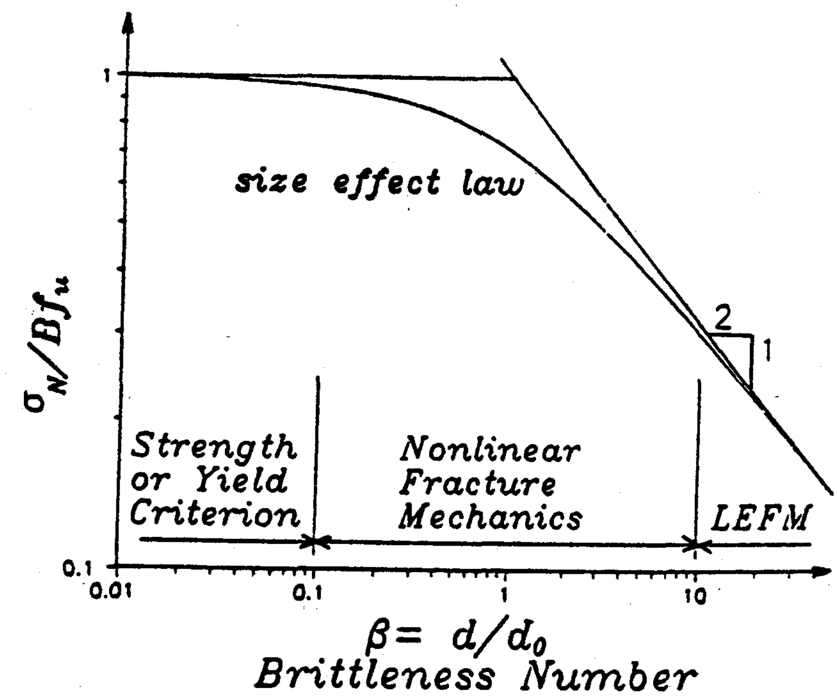

Fig.1. Size effect law for quasi-brittle structures.

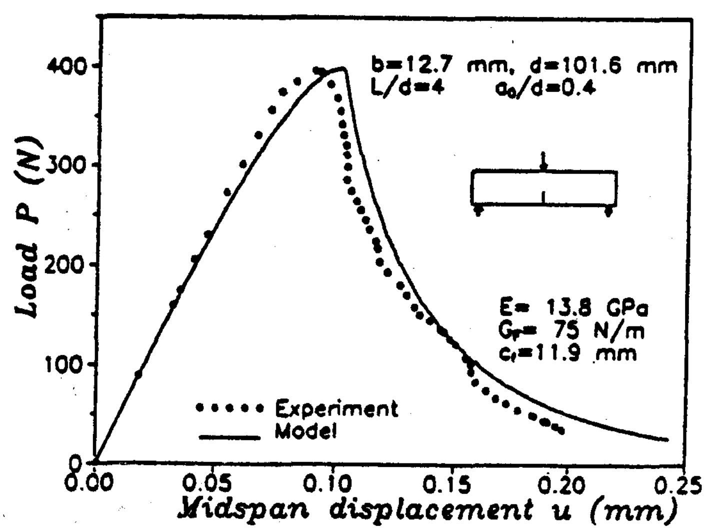

Fig.2. Prediction of the load-deflection curve from measured maximum loads of rock fracture specimens of different sizes, in comparison with the load- deflection curve measured by Bažant, Gettu, and Kazemi (1990).

(Fig. 3). The size effect in fatigue fracture, rate sensitivity and the influence of creep are discussed in detail, and a new formulation is presented. Finally, it is argued that all the articles of concrete design codes dealing with brittle failures have to be revised.

Although the size effect due to energy release appears to dominate in most structures, the statistical nature of material strength cannot be denied. However, the existing theory, based on Weibull (1939), is theoretically inadequate because it ignores the stress redistributions due to fracture growth prior to maximum load, which cause a strong deterministic size effect.

It is shown that a proper remedy is achieved by postulating that the failure probability of small representative volumes of the material depends on the nonlocal strain (i.e., the spatial average of strain over the representative volume) rather than the local stress. Asymptotic integration of the Weibull-type probability integral over the structure volume leads to a refined size effect law amalgamating the deterministic and statistical size effects (Fig. 4). The previously derived size effect law is obtained as the limiting case.

New measurements of size effect in diagonal shear failure of reinforced concrete beams without stirrups (Fig. 1) verify the theory and prove that Weibull-type theory is inapplicable.

2. Some New Results on the Influence of Loading Rate and Fatigue on Fracture Growth and Size Effect

After a broad review of the results recently achieved at Northwestern University, which have been or are being published in detail elsewhere and have been summarized above, we now present in detail some new results on the influence of loading rate and fatigue on fracture properties, including the size effect.

2.1 Effect of Material Creep and Fracture Growth Velocity

First we summarize the formulation presented at a recent conference (Bažant, 1990). Fracture of concrete is known to exhibit a significant sensitivity to the rate of loading. For concrete, the influence of the rate of loading on fracture propagation is even more pronounced and is further compounded by viscoelasticity of the material in the entire structure. To calculate the response of a structure, as well as to be able to evaluate laboratory measurements, the most important is the determination of the load or reaction P as a function of the load-point displacement u and time t for a prescribed loading regime. For the special case of rate-independent elastic behavior, the load-deflection curve of a fractured specimen is given by the following well-known relations (Bažant and Cedolin, Sec. 12.4)

$$u = \frac{1}{E}P\bar{C}(a), \quad \bar{C}(a) = \bar{C}_0 + \frac{2}{b}\phi(\alpha), \quad \phi(\alpha) = \int_0^{\alpha} [k(\alpha')]^2 d\alpha' \tag{1}$$

$$P = b\sqrt{d}\,\frac{K_I^R(c)}{k(\alpha)} \tag{2}$$

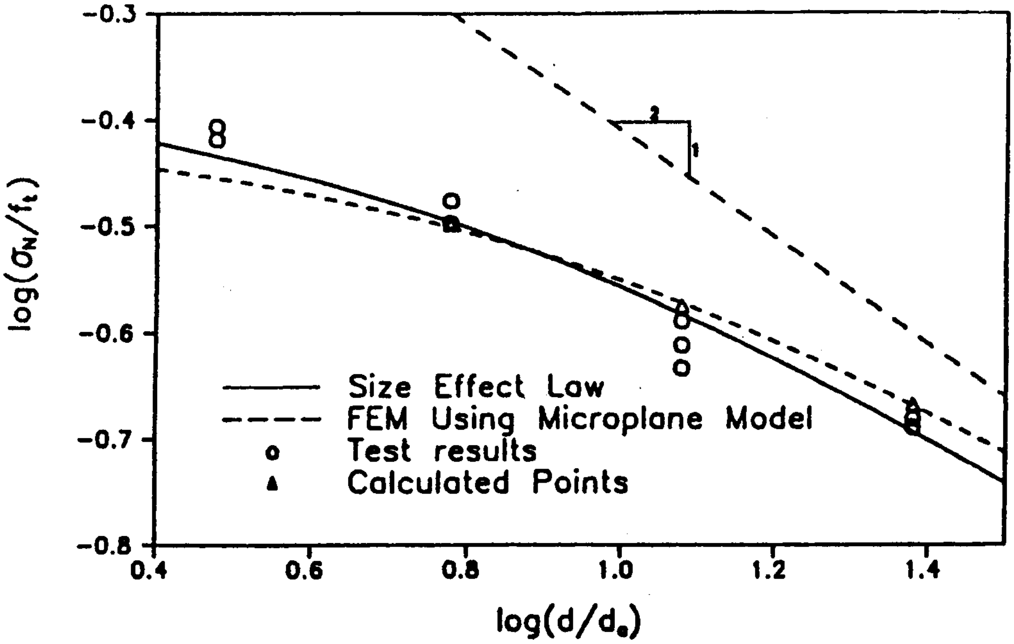

Fig.3. Nominal strengths of fracture specimens of different sizes calculated by a finite element program based on nonlocal microplane material model, compared with the size effect law and with experimental measurements of Bažant and Pfeiffer (1987).

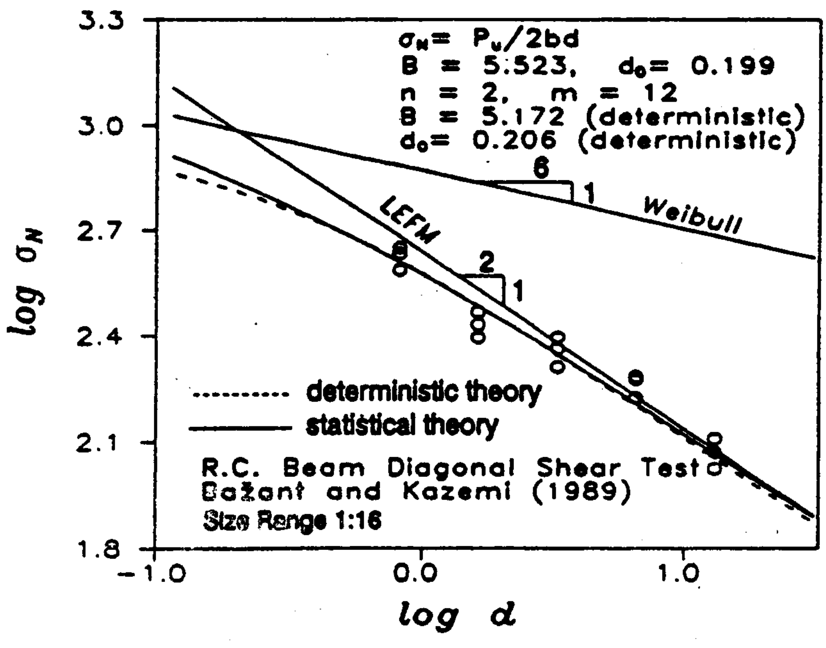

Fig.4. Statistical generalization of the size effect law (Bažant and Xi, 1990).

(see e.g., Bažant and Cedolin, 1991, Sec. 12.4), in which E = Young's elastic modulus, $\bar{C}$ = secant compliance of the cracked structure for a unit value of E (the actual compliance is $C = \bar{C}E$), $\bar{C}_0$ = initial elastic compliance for a unit E-value before any crack has formed, $\alpha = a/d$ = relative crack length, d = characteristic dimension (size) of the structure, b = structure thickness, $K_I^R(c)$ = R-curve (resistance curve) of K_I determined for the given specimen geometry and material type (which may be determined from size effect tests), and $G(\alpha)$ = nondimensionalized energy release rate defined by writing $K_I = P\,k(\alpha)/b\sqrt{d}$ (the energy release rate then is $G = K_I^2/E = P^2 g(\alpha)/E\,b^2 d$, $g(\alpha) = [k(\alpha)]^2$). (Note that $R(c) = [K_I^R(c)]^2/E$ = energy R-curve.) For plane strain conditions, E needs to be replaced by $(1-\nu^2)$.

To incorporate into these equations the effect of (aging) linearly viscoelastic behavior outside the fracture process zone, we proceed as follows. For uniaxial stress, the aging viscoelastic stress-strain relation is

$$\epsilon(t) = \int_{t_0}^{t} J(t,t')d\sigma(t') \tag{3}$$

(Stieltjes integral). Here shrinkage and thermal expansion are neglected; σ, ϵ = uniaxial stress and strain, $J(t,t')$ = given compliance function of the material that characterizes creep, representing strain at age t caused by a unit uniaxial stress applied at age t'. Eq. 3 may be written in an operator form as $\epsilon(t) = \underline{E}^{-1}\sigma(t)$ where $\underline{E}^{-1}$ is the creep operator defined by this equation. The load-displacement relation may be obtained from the corresponding elastic relation by replacing $1/E$ with the corresponding creep operator $\underline{E}^{-1}$. Thus, from Eq. 1,

$$u(t) = \int_{t_0}^{t} J(t,t')d\{P(t')\bar{C}[a(t')]\} \tag{4}$$

For the purpose of numerical solution, time t is subdivided by discrete times $t_r (r = 0,1,2,3\ldots)$ into time steps $\Delta t_r = t_r - t_{r-1}$. Time t_0 represents the age at the first loading. Using the trapezoidal rule (the error of which is proportional to Δt^2), we may approximate Eq. 4 as

$$u_r = \sum_{s=1}^{r} J_{r,s-\frac{1}{2}}(P_s\bar{C}_s - P_{s-1}\bar{C}_{s-1}) \tag{5}$$

where subscript r refers to time t_r and $s-\frac{1}{2}$ refers to time $t_s - \Delta t_s/2$; $P_s = P(t_s)$; $\bar{C}_s = \bar{C}[a(t_s)]$; $J_{r,s-\frac{1}{2}} = J(t_r, t_{s-\frac{1}{2}})$; and the initial load value is $P(t_0) = P_0 = 0$. Writing Eq. 5 for t_{r-1} instead of t_r, i.e., $u_{r-1} = \sum_{s=1}^{r-1} J_{r-1,s-\frac{1}{2}}(P_s\bar{C}_s - P_{s-1}\bar{C}_{s-1})$ and subtracting this from Eq. 5 one gets

$$\Delta u_r = \frac{1}{E_r^w}(P_r\bar{C}_r - P_{r-1}\bar{C}_{r-1}) + \Delta u_r^{creep} \tag{6}$$

in which $\Delta u_r = u_r - u_{r-1}$, $1/E_r^w = J_{r,r-\frac{1}{2}}$ and

$$\Delta u_r^{creep} = \sum_{s=1}^{r-1}(J_{r,s-\frac{1}{2}} - J_{r-1,s-\frac{1}{2}})(P_s\bar{C}_s - P_{s-1}\bar{C}_{s-1}) \quad \text{for } r > 2 \tag{7}$$

(this is analogous to the general relations in Bažant, ed., 1988, p. 116). It may be noted that the accuracy can be somewhat improved if, instead of $J_{r,r-\frac{1}{2}}$, the effective modulus approximation, $1/E''_r = J_{r,r-1} = J(t_r, t_{r-1})$, is used for the last step. Eq. 7 is applicable only for $r > 2$. For the first two steps,

$$\begin{aligned} &\text{for } r = 1: \ \Delta u_1 = u_1 = J_{1,0} P_1 \bar{C}_1 \\ &\text{for } r = 2: \ \Delta u_2 = J_{2,1}(P_2\bar{C}_2 - P_1\bar{C}_1) + (J_{2,\frac{1}{2}} - J_{1,\frac{1}{2}})P_1\bar{C}_1 \end{aligned} \tag{8}$$

The foregoing analysis must now be generalized by incorporating a law for the growth of crack length $a(t)$. Materials that, under a constant very fast loading rate, follow linear elastic fracture mechanics exhibit, at slower rates, crack growth that approximately obeys the law

$$\dot{a} = \kappa_c \left(\frac{K_I(P)}{K_{If}}\right)^n \exp\left[-\frac{U}{R_0}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right] \tag{9}$$

in which K_I = stress intensity factor produced by load P, U = activation energy of crack growth, R_0 = gas constant, T = absolute temperature, T_0 = reference temperature, and κ_c, n = empirical material constants. The applicability of this well-known relation to concrete has been verified in Bažant and Prat (1988), but without considering different specimen sizes and the nonlinearity of fracture. To take into account nonlinear fracture properties and obtain the correct transitional size effect (agreeing with the size effect law), Eq. 9 must be generalized as:

$$\dot{a} = \kappa_c \left(\frac{K_I(P)}{K_I^R(c)}\right)^n \exp\left[-\frac{U}{R_0}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right] = \Phi(P,a), \quad \text{with } K_I(P) = \frac{Pk(\alpha)}{b\sqrt{d}} \tag{10}$$

where Φ is a function of P and a, as defined by this equation, and $K_I^R(c)$ is the given R-curve (determined in advance for the given structure geometry). For the time step (t_{r-1}, t_r), Eq. 10 yields

$$\Delta a_r = \Phi(P_{r-\frac{1}{2}}, a_{r-\frac{1}{2}})\Delta t_r \tag{11}$$

The following algorithm may be used in every time step Δt_r (for $r > 2$), in which the previous values $a_0, a_1, \ldots a_{r-1}; P_0, \ldots P_{r-1}; u_0, \ldots u_{r-1}$ are already known.

1. Setting $P_{r-\frac{1}{2}} = P_{r-1}$ and $a_{r-\frac{1}{2}} = a_{r-1}$, calculate $\Delta a_r = \Phi(P_{r-*}, a_{r-1})\Delta t_r$ (Eq. 11) as the first estimate, and set $a_{r-\frac{1}{2}} = a_{r-1} + \frac{1}{2}\Delta a_r, a_r = a_{r-1} + \Delta a_r$. Evaluate $\bar{C}_r = \bar{C}(a_r)$ from Eq. 1.
2. Loop on iterations.
3. Calculate Δu_r^{creep} from Eq. 7. Then, using prescribed Δu_r (or prescribed P_r), calculate P_r (or Δu_r) from Eq. 6. Then calculate Δa_r from Eq. 11 and obtain updated values $a_{r-\frac{1}{2}} = a_{r-1} + \frac{1}{2}\Delta a_r$ and $a_r = a_{r-1} + \Delta a_r$. Evaluate updated $\bar{C}_r = \bar{C}(a_r)$ from Eq. 1.

4. Check the given tolerance criterion, requiring that the absolute value of the change of P_r in the last iteration be less than $|eP_r|$ where e is a given small number (e.g., $e = 10^{-6}$). If violated, go to 2 and start the next iteration. If satisfied, go to 1 and start the first iteration of the next time step Δt_{r+1}.

The special case in which the crack propagation law is time-dependent but the material behavior is time-independent (elastic) is obtained from the preceding algorithm if one sets $J(t,t') = 1/E$ for $t \geq t'$.

Another special case arises when the crack propagates according to the classical time-independent law of fracture mechanics (with R-curve) while the material creeps. This situation is approached when $\dot{a} \to 0$ for $K_I < K_I^R(c)$, and $\dot{a} \to \infty$ for $K_I > K_I^R(c)$, i.e., when for $K_I < K_I^R(c)$ the rate $\dot{a}$ is almost 0 and for $K_I > K_I^R(c)$ the rate $\dot{a}$ is extremely large. Such behavior is obtained from Eq. 10 when $n \to \infty$. The foregoing algorithm, however, cannot be expected to converge well for extremely large n values, and it is preferable to obtain the limiting case of time-independent crack propagation law directly. In that case, for a propagating crack we simply have the condition $K_I = K_I^R(c)$, which replaces Eqs. 9-11 while Eqs. 4-8 remain applicable. The values of $\bar{C}_r$ and $\bar{C}_s$ in Eqs. 6-8 must now be such that the relation $P_r = b\sqrt{d}\, K_I^R(c_r)/k(\alpha_r)$ be always satisfied when $c_r = a_r - a_0$, $\alpha_r = a_r/d$. From this relation, the load increment is

$$\Delta P_r = b\sqrt{d}\left(\frac{K_I^R(c_{r-1} + \Delta a_r)}{k\left(\alpha_{r-1} + \frac{\Delta a_r}{d}\right)} - \frac{K_I^R(c_{r-1})}{k(\alpha_{r-1})}\right) \tag{12}$$

At the same time, Eq. 6 may be rewritten in the form

$$\Delta u_r = \frac{1}{E_r''}\left[(P_{r-1} + \Delta P_r)\bar{C}(a_{r-1} + \Delta a_r) - P_{r-1}\bar{C}_{r-1}\right] + \Delta u_r'' \tag{13}$$

Before starting the solution of time step Δt_r, the values of E_r'', $\Delta u_r''$, P_{r-1}, a_{r-1}, c_{r-1}, α_{r-1} are known, and thus Eqs. 12-13 contain two unknowns: Δa_r and either ΔP_r or Δu_r. If ΔP_r is prescribed, then Δa_r may be solved first from Eq. 12 (e.g., by Newton iterations) and then Δu_r from Eq. 13. If Δu_r is prescribed, then Eqs. 12 and 13 represent two simultaneous nonlinear equations for Δa_r and ΔP_r, which may be solved iteratively.

The rate-of-loading effect on fracture has been studied experimentally by the size effect method. The most interesting result (Bažant and Gettu, 1989) is that the effective length of the fracture process zone decreases as the loading rate increases, and thus the response is getting more brittle, closer to LEFM. This is seen in Fig.5 which shows that, for specimens of three sizes (1:2:4), the max σ_N-points shift to the right (i.e., toward a higher brittleness β) as the time to reach the peak load increases (tests at constant displacements rates).

2.2 Size Effect Adjustment of Paris Law for Fatigue Fracture

Let us now consider another phenomenon causing cracks to grow — fatigue. For the moment, we ignore creep and fracture growth velocity. According to

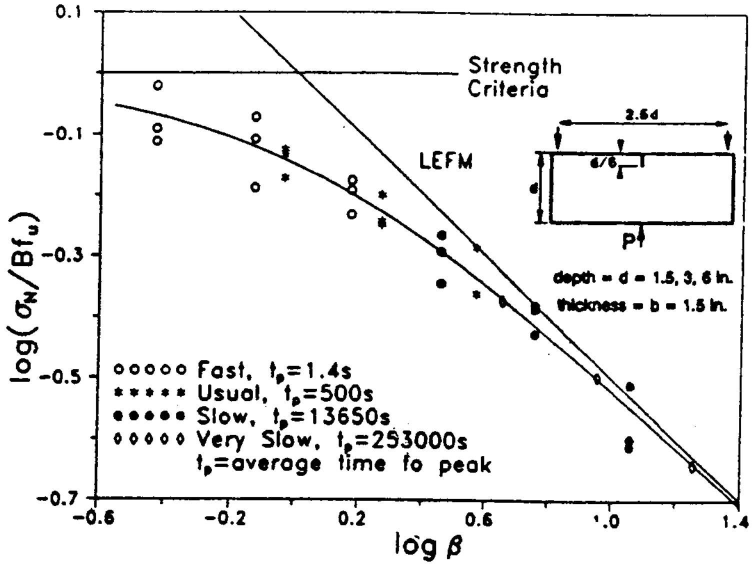

Fig.5. Size effect on nominal strengths at different loading rates, reported by Bažant and Gettu (1989).

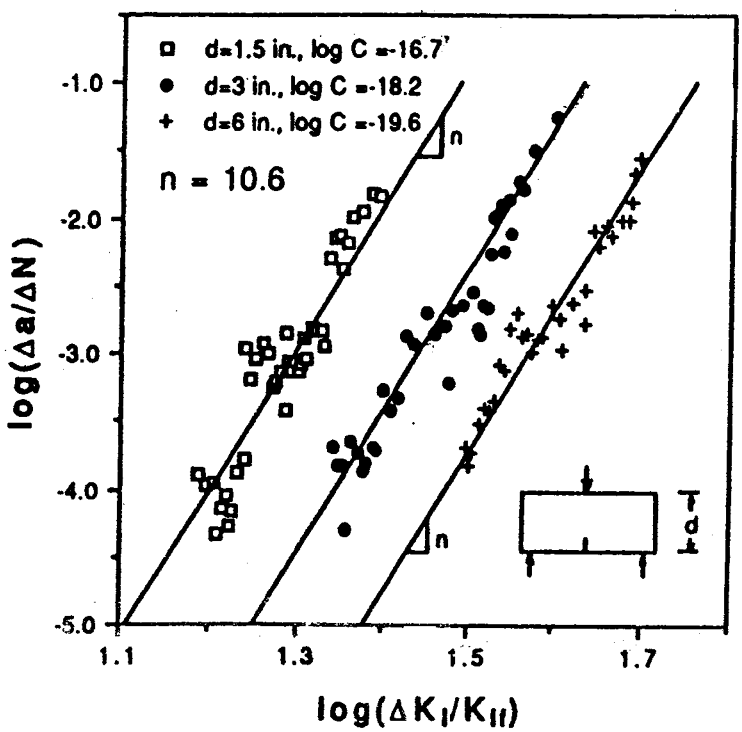

Fig.6. Test results of Bažant and Xu (1991) on fatigue fracture of concrete specimens of three different sizes and their comparison with size-adjusted Paris law.

the size effect law proposed in Bažant (1984), the nominal strength, defined as $\sigma_N = P/b\,d$ where P = maximum load, depends on structure size d approximately as

$$\sigma_N = B\, f_t'(1+\beta)^{-1/2}, \quad \beta = d/d_0 \tag{14}$$

where $B\ f_u$ and d_0 are two empirical constants and f_t' is chosen to represent the tensile strength. From this law one can show that the apparent stress intensity factor determined by methods of linear elastic fracture mechanics (LEFM) depends on the structure size as

$$K_{I_c} = K_{I_f}\sqrt{\frac{\cdot\beta}{1+\beta}} \tag{15}$$

Under repeated loading, cracks tend to grow, even if there is no creep (Eq. 3) and no time-dependent fracture growth (Eq. 9). This is described by the well-known Paris law (Paris, Gomez, Anderson, 1961; Paris, Erdogan, 1963). Applicability of this law to fatigue crack growth in concrete has been verified by Swartz and Go, 1984. Since Paris law describes the crack growth as a function of the amplitude of the stress intensity factor K_I, a question arises with respect to the size effect. In monotonic loading, the stress intensity factor does not provide sufficient characterization of fracture when different sizes are considered, as is clear from Eq. 15. The same phenomenon must be expected for cyclic fracture, especially since fracture under monotonic loading can be regarded as a limiting case of fracture under cyclic loading. Recent fatigue fracture experiments on notched concrete beams at Northwestern University have shown that the fatigue crack growth in geometrically similar specimens of different sizes can be described by the following law:

$$\frac{\Delta a}{\Delta N} = C\left(\frac{\Delta K_I}{K_{I_c}}\right)^m \tag{16}$$

in which K_{I_f} = fracture toughness for an infinitely large specimen, ΔK_I = amplitude of the stress intensity factor, K_{I_c} = apparent fracture toughness according to Eq. 15, N = number of load cycles, $\Delta a/\Delta N$ = crack length extension per cycle; and C, m = constants. For $\beta \to \infty$, this equation reduces to the well-known Paris law. For normal size concrete specimens, however, the deviations from Paris law are quite significant. This is revealed by the experimental results in Fig. 6 for three different sizes in the ratio 1:2:4. In this plot, the Paris law gives one inclined straight line of slope n, but it is seen from Fig. 6 that for each size a different straight line results. The solid straight lines represent Eq. 16.

2.3 Proposed Load-Deflection Curve Calculation for Fatigue with Size Effect

Eq. 16 is the most simple, direct generalization of Paris' law for size effect, which remains in the spirit of linear elastic fracture mechanics. It represents the fatigue counterpart of modeling the size effect in monotonic loading through an adjustment of the fracture toughness, as stated in Eq. 15. A more fundamental, and more general fracture formulation which yields the proper size effect in monotonic loading is obtained by using the R-curve, $K_I^R(c)$, which may itself be best determined from measured maximum loads of specimens of different sizes. Since monotonic fracture is a limit case of fatigue fracture when the load amplitude ΔP approaches zero, a more fundamental generalization of Paris law, which should have a greater generality than that proposed in Bažant and Xu (1991), should have the form

$$\frac{\Delta a}{\Delta N} = \kappa_f \left(\frac{\Delta K_I}{K_I^R(c)}\right)^m, \quad c = a - a_0 \tag{17}$$

where κ_f is a constant. The size effect arises from the fact that in a large specimen the same length of crack growth requires a larger ΔK_I, and thus larger ΔP and larger $\Delta \sigma_N$. Note also that with $K_I^R(c)$, Eq. 17 becomes analogous to Eq. 10.

From Eq. 1, we have, for small increments Δa at constant P, $\Delta u = P[d\bar{C}(a)/da]\Delta a/E$. Substitution for Δa according to Eq. 17 yields the deflection increment during Δt due to load cycling alone:

$$\Delta u^{cyc} = \frac{P}{E}\frac{d\bar{C}[a(t)]}{da}\Delta a^{cyc}, \quad \text{with} \quad \Delta a^{cyc} = \kappa_f \left(\frac{\Delta K_I}{K_I^R[c(t)]}\right)^m f\Delta t \tag{18}$$

where $f\Delta t = \Delta N$, f = frequency. Step-by-step integration of this equation provides the load-deflection history $u^{cyc}(t)$ under cyclic loading, with the rate effects neglected.

2.4 Proposed Generalization for Simultaneous Fatigue and Rate Effects at Various Structure Sizes

Paris' law, as well as its generalizations in Eqs. 16 and 17, does not contain time and load frequency. The crack growth depends only on the number of cycles. This means that fatigue crack growth in rate-insensitive materials is assumed to be the same as for infinitely fast cycling. Thus, for a material that is rate-sensitive, we may imagine that during a finite interval $\Delta t = t_r - t_{r-1}$ the cycles ΔN happen infinitely fast (instantly) at the middle of time interval, $t_{r-\frac{1}{2}} = (t_{r-1} + t_r)/2$, so that the creep and crack growth effect during ΔN are negligible, and all the creep and crack growth happen during the rest of the time interval while there is no load cycling. So we may superpose the deflection increments according to Eqs. 18, 11 and 6, which furnishes

$$\Delta u_r = \frac{P_r\bar{C}_r - P_{r-1}\bar{C}_{r-1}}{E''} + \Delta u_r^{creep} \frac{P_{r-\frac{1}{2}}}{E}\frac{d\bar{C}(a_{r-\frac{1}{2}})}{da}(\Delta a^{cyc} + \Delta a^{growth}) \tag{19}$$

with

$$\Delta a^{cyc} = \kappa_f f \left(\frac{\Delta K_I}{K_I^R(c_{r-\frac{1}{2}})}\right)^m \Delta t_r, \tag{20}$$

$$\Delta a^{growth} = \Phi(P_{r-\frac{1}{2}}, a_{r-\frac{1}{2}})\Delta t_r \tag{21}$$

where subscripts $r - \frac{1}{2}$ refer to the midstep.

Assuming history independence of fatigue rate and crack growth rate, we may further extend this equation to cycling of variable amplitude, and load variation during cycling. To this end, we need to replace ΔK_I with $\Delta K_{I_{r-\frac{1}{2}}}$ and f with $f_{r-\frac{1}{2}}$. Eqs. 19-21 may be integrated in a similar algorithm as described below Eq. 11. In the first iteration of the step all the unknowns with subscript $r - \frac{1}{2}$ have to be assumed equal to their values at $f_{r-\frac{1}{2}}$, and in the subsequent iterations they are based on the preceding iteration.

3. Conclusion

The size effect, which is probably the most important consequence of fracture mechanics, can be handled with great generality on the basis of the approximate size effect law proposed previously. The theory based on the size effect law, which has originally been formulated for deterministic material properties, monotonic loading, and rate-independent behavior without creep, can be extended to cover random material strength in the sense of Weibull as well as the effects of fatigue due to load cycling, material creep and rate sensitivity of fracture growth.

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