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NEW CONCEPT OF NONLOCAL CONTINUUM DAMAGE: CRACK INFLUENCE FUNCTION

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ABSTRACT: The spatial averaging integral and weight function that have so far been employed in the macroscopic nonlocal continuum models for strain-softening damage have been obtained by semi-intuitive arguments and justified by the need to limit damage localization in numerical computations. Some micromechanics arguments have recently also been presented, but microcrack interactions and growth have been neglected. The present conference paper gives a preliminary, abbreviated exposition of a new micromechanics analysis of a system of interacting and growing microcracks. The result is a new form of the spatial integral for nonlocal damage, in which the nonlocal stress field is a solution of a Fredholm integral equation over the neighborhood of a point. While in previous formulations the weight function has been assumed as a scalar, it is now found to be a tensor. Furthermore, the weight function is not axisymmetric (isotropic) but varies with the spherical angles (i.e., is anisotropic) and exhibits sectors of shielding and amplification. For long distances, the weight function decays as r^{-2} in two dimensions and as r^{-3} in three dimensions. Application of the Gauss-Seidel iteration method, which can be conveniently combined with iterations in each loading step of a nonlinear finite element code, simplifies the handling of the nonlocality since the nonlocal inelastic stress increments can be evaluted explicitly using directly the crack influence function. A detailed article is in preparation.

1. INTRODUCTION

As is now generally accepted, finite element analysis of strain-softening damage, including its final stage—fracture, requires the use of some type of nonlocal continuum (Bažant, 1984; Bažant, Belytschko and Chang, 1984). One effective type is the nonlocal damage concept, in which the local damage or fracturing strain in the incremental stress-strain relation is replaced by its spatial average. The arguments for this formulation (Pijaudier-Cabot and Bažant, 1987, Bažant and Pijaudier Cabot, 1988), by now well proven by extensive computational experience (Bažant and Ožbolt, 1990, 1992) have been mainly computational—the need to limit localization of strain-softening damage to zones of nonzero volume, while the physical explanation has been mainly phenomenologic and empirical. Intuitively, it has been expected that the main source of nonlocality must be the interaction between adjacent microcracks. Certain micromechanics arguments based on a system of microcracks have been shown to lead to the nonlocal damage concept (Bažant, 1991). However, interpretation of these arguments for the purpose of finite element analysis has not been clear and the interactions among the microcracks with simultaneous crack growth during the loading steps have not been taken into account.

The objective of the present conference paper is to give a preliminary, abbreviated sketch of an improved micromechanics analysis that takes into account microcrack interactions and growth and predicts the proper form of the nonlocal weight function for the spatial averaging. A detailed article on this new development is in preparation.

2. NONLOCAL DAMAGE

Finite element analysis of inelastic solids is generally carried out in small loading steps, for each of which the given local constitutive law is reduced to the incremental form

$$\Delta\boldsymbol{\sigma} = \boldsymbol{E} : (\Delta\boldsymbol{\epsilon} - \Delta\boldsymbol{\epsilon}'') = \boldsymbol{E} : \Delta\boldsymbol{\epsilon} - \Delta\boldsymbol{S} \tag{1}$$

in which $\Delta\boldsymbol{\sigma}, \Delta\boldsymbol{\epsilon}$ = increments of the stress and strain tensors, $\boldsymbol{E}$ = fourth-rank tensor of elastic moduli of uncracked material, $\Delta\boldsymbol{\epsilon}''$ = inelastic strain increment tensor, and $\Delta\boldsymbol{S}$ = inelastic stress increment tensor. In a nonlocal continuum formulation, Eq. (1) is replaced by

$$\Delta\boldsymbol{\sigma} = \boldsymbol{E} : \Delta\boldsymbol{\epsilon} - \Delta\overline{\boldsymbol{S}} \tag{2}$$

where $\Delta\overline{\boldsymbol{S}}$ is the nonlocal inelastic stress increment tensor,

$$\Delta\overline{\boldsymbol{S}}(\boldsymbol{x}) = \int_V \alpha(\boldsymbol{x}, \boldsymbol{\xi})\Delta\boldsymbol{S}(\boldsymbol{\xi})\mathrm{d}V(\boldsymbol{\xi}) \tag{3}$$

in which V = volume of the body; $\boldsymbol{x}, \boldsymbol{\xi}$ = coordinate vectors; and $\alpha(\boldsymbol{x}, \boldsymbol{\xi})$ = given weight function, which must satisfy the normalizing condition

$$\int_V \alpha(\boldsymbol{x}, \boldsymbol{\xi})\mathrm{d}V(\boldsymbol{\xi}) = 1 \tag{4}$$

3. SPATIAL INTERACTIONS DUE TO GROWING MICROCRACKS

Consider an elastic solid containing many microcracks, numbered as $\mu = 1, ...N$, subjected to a given strain increment $\Delta\boldsymbol{\epsilon}$. On the macroscale the microcracks are considered to be smeared, as the solid is treated as a continuum. Exploiting the principle of superposition, we may decompose the loading step into two substeps: (I) In the first substep, the cracks are imagined temporarily "frozen", that is, they can neither grow and open wider nor shorten and close. The stress increments are given by $\boldsymbol{E} : \Delta\boldsymbol{\epsilon}$, which is represented by the line segment $\overline{13}$ (Fig. 1) having the slope of the initial elastic modulus E. These stress increments are transmitted across the temporarily frozen cracks. (II) In the second substep, the cracks are unfrozen and the stresses transmitted across the cracks are released. If the cracks did not grow (nor new cracks were nucleated), the unfreezing at constant macrostrain $\boldsymbol{\epsilon}$ would cause the stress drop $\overline{34}$ down to point 4 on the secant line $\overline{01}$, and the change of state of the solid would be calculated by applying the opposite of this stress drop onto the crack surfaces. However, when the cracks grow (and new cracks nucleate), a larger stress drop defined by the local strain-softening constitutive law and represented by the segment $\Delta\boldsymbol{S} = \overline{32}$ in Fig. 1 takes place. Thus, the normal surface tractions

$$\Delta p_\mu = \boldsymbol{n}_\mu \Delta\boldsymbol{S}_\mu \boldsymbol{n}_\mu \tag{5}$$

representing the normal component of $\Delta\boldsymbol{S}$, must be considered in the second substep as loads Δp_μ that are applied onto the crack surfaces, the unit normal of which is denoted as $\boldsymbol{n}_\mu$; a product with no product sign denotes here a product of tensors contracted on one index (often written as the dot product, whereas here we omit the dot).

We introduce two important simplifications: (1) Although the stress transmitted across the temporarily frozen crack varies along the crack, we consider only its average, i.e., Δp_μ is constant along each crack and is approximately based on the value of the macroscale stress drop $\Delta\boldsymbol{S}$ at the center

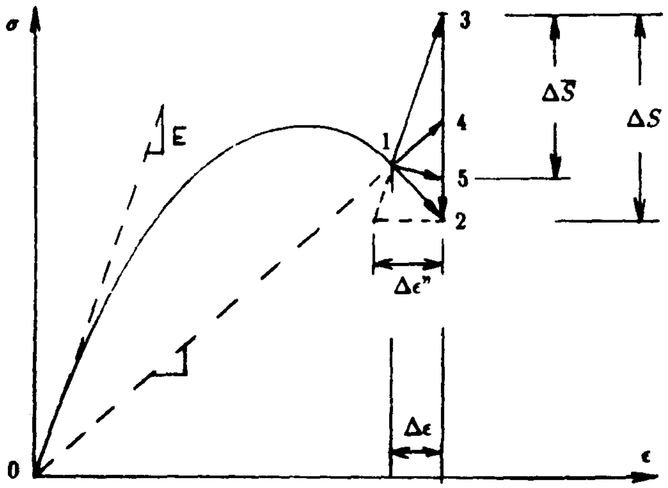

Figure 1: Local and Nonlocal Inelastic Stress Increments During the Loading Step.

of the crack. This approximation was introduced by Kachanov (1987), who numerically demonstrated that the error is negligible except for the unimportant case when the minimum distance between two cracks is at least an order of magnitude less than their size. (2) We consider only Mode I crack openings, i.e. neglect the shear modes (Mode II and III). This is probably justified by the high relative surface roughness of the microcracks (especially in a material such as concrete), which must be expected to prevent any significant relative slip of the crack faces (the Mode II or III macro-displacements that can occur on macroscopic cracks are mainly the result of Mode I openings of microcracks in the fracture process zone that are inclined with respect to the macrocrack).

Using the superposition method formulated by Kachanov (1967) and also employed by Datsyshin and Savruk (1973), Gross (1982), Chudnovsky and Kachanov (1983), Chudnovsky et al. (1987), Chen (1984), and Horii and Nemat-Nasser(1985) (and in a displacement version also by Collins, 1963), the opening and stress intensity factor of crack μ may be characterized by a uniform crack traction Δp_μ that is acting on single crack in an infinite solid (with elastic moduli $\boldsymbol{E}$) and is solved from the superposition relation:

$$\Delta\bar{p}_\mu = \Delta p_\mu + \sum_{\nu\neq\mu} \Lambda_{\mu\nu}\Delta\bar{p}_\nu \tag{6}$$

in which $\mu = 1,...N, \nu = 1,...N$, and $\Lambda_{\mu\nu}$ are the crack influence coefficients representing the average stress at the frozen crack μ caused by a unit uniform pressure applied on unfrozen crack ν. The summation in Eq. (6) skips $\nu = \mu$ because $\Lambda_{\mu\mu}$ has no meaning; however, for the sake of convenience we will define $\Lambda_{\mu\mu} = 0$, which permits taking the sum in Eq. (6) over all $\nu = 1,...N$. Then, using Eq. (5) (and assuming that the crack orientations $\boldsymbol{n}_\mu, \boldsymbol{n}_\nu$ do not rotate during the loading step), we may write

$$\boldsymbol{n}_\mu \Delta\bar{\boldsymbol{S}}_\mu \boldsymbol{n}_\mu = \boldsymbol{n}_\mu \Delta \boldsymbol{S}_\mu \boldsymbol{n}_\mu + \sum_{\nu=1}^{N} \Lambda_{\mu\nu} \boldsymbol{n}_\nu \Delta\bar{\boldsymbol{S}}_\nu \boldsymbol{n}_\nu \tag{7}$$

(When the crack orientations rotate, it is necessary to move the increment operators Δ in front of the products $\boldsymbol{n}_\mu \Delta\bar{\boldsymbol{S}}\boldsymbol{n}_\nu$.) The values of $\Delta\bar{\boldsymbol{S}}_\mu$ are graphically represented in Fig. 1 by the segment $\Delta\bar{S} = \overline{35}$, which can be smaller or larger than segment $\overline{32}$.

Now we will introduce two simplifying assumptions: (1) The influence of the microcracks at point $\boldsymbol{\xi}$ of the macro-continuum upon the microcracks at point $\boldsymbol{x}$ of the macro-continuum is determined by the dominant microcrack orientation at each point, and (2) the dominant microcracks for each loading step are those that are normal to the direction of the maximum principal macro-stress increment $\Delta\bar{S}^I$

at each point. Then, $\boldsymbol{n}_\mu$ corresponds to the principal stress direction and Eq. (7) becomes:

$$\Delta\overline{S}^I_\mu = \Delta S^I_\mu + \sum_{\nu=1}^{N} \Lambda_{\mu\nu}\Delta\overline{S}^I_\nu \tag{8}$$

On the macroscale, on which the microcracks are smeared, the continuum counterpart of this equation may be written as

$$\Delta\overline{S}^I(\boldsymbol{x}) = \Delta S^I(\boldsymbol{x}) + \int_V \Lambda(\boldsymbol{x},\boldsymbol{\xi})\Delta\overline{S}^I(\boldsymbol{\xi})\,\mathrm{d}V(\boldsymbol{\xi}) \tag{9}$$

where $\Lambda(\boldsymbol{x},\boldsymbol{\xi}) = \mathcal{E}(\Lambda_{\mu\nu})/V_c$, V_c is a constant that may be interpreted as the volume per crack, and $\mathcal{E}$ is a statistical averaging operator which yields the average (moving average) over a certain appropriate neighborhood of point $\boldsymbol{x}$ or $\boldsymbol{\xi}$. Such statistical averaging is a necessary part of the homogenization procedure that determines the smoothing macro-continuum because in a random crack array the characteristics of the individual cracks must be expected to exhibit enormous random scatter. Eq. (9) represents a Fredholm integral equation (integral equation of the second kind with a square-integrable kernel) for the unknown function $\Delta\overline{S}^I(\boldsymbol{x})$, assuming that the function $\Delta S^I(\boldsymbol{x})$ is given. Its solution can be written as

$$\Delta S^I(\boldsymbol{x}) = \Delta\overline{S}^I(\boldsymbol{x}) - \int_V K(\boldsymbol{x},\boldsymbol{\xi})\Delta\overline{S}^I(\boldsymbol{\xi})\mathrm{d}V(\boldsymbol{\xi}) \tag{10}$$

in which function $K(\boldsymbol{x},\boldsymbol{\xi})$ is the resolvent of the kernel $\Lambda(\boldsymbol{x},\boldsymbol{\xi})$. This resolvent could be calculated numerically in advance of the nonlocal finite element analysis, however the use of the last equation seems incovenient since the resolvent surely would not allow a simple physical interpretation and a closed-form expression.

With the notation

$$\Psi_{\mu\nu} = \delta_{\mu\nu} - \Lambda_{\mu\nu} \tag{11}$$

where $\delta_{\mu\nu}$ = Kronecker delta, Eq. (8) can be transformed to

$$\sum_\nu \Psi_{\mu\nu}\Delta\overline{S}^I_\nu = \Delta S^I_\mu \tag{12}$$

The macro-continuum countepart of this discrete matrix relation is

$$\int_V \Psi(\boldsymbol{x},\boldsymbol{\xi})\Delta\overline{S}^I(\boldsymbol{\xi})\,\mathrm{d}V(\boldsymbol{\xi}) = \Delta S^I(\boldsymbol{x}) \tag{13}$$

which represents an integral equation of the first kind for unknown function $\Delta\overline{S}^I(\boldsymbol{\xi})$. Obviously, $\Psi(\boldsymbol{x},\boldsymbol{\xi}) = \delta(\boldsymbol{x}-\boldsymbol{\xi}) - \Lambda(\boldsymbol{x},\boldsymbol{\xi})$ where $\delta(\boldsymbol{x}-\boldsymbol{\xi})$ = Dirac delta function in two or three dimensions; indeed, substitution of this expression into Eq. (13) yields Eq. (9).

Defining the inverse square matrix:

$$[\alpha_{\mu\nu}] = [\Psi_{\mu\nu}]^{-1} \tag{14}$$

we may write the solution of the equation system (12) as

$$\Delta\overline{S}^I_\mu = \sum_\nu \alpha_{\mu\nu}\Delta S^I_\nu \tag{15}$$

On the macroscale, the continuum counterpart of the last equation is

$$\Delta\overline{S}^I(\boldsymbol{x}) = \int_V \alpha(\boldsymbol{x},\boldsymbol{\xi})\Delta S^I(\boldsymbol{\xi})\mathrm{d}V(\boldsymbol{\xi}) \tag{16}$$

where $\alpha(\boldsymbol{x},\boldsymbol{\xi}) = \mathcal{E}(\alpha_{\mu\nu})/V_c$. The kernel α of this equation represents the resolvent of the kernel $\Psi(\boldsymbol{x},\boldsymbol{\xi})$ of Eq. (13). Furthermore, $\alpha(\boldsymbol{x},\boldsymbol{\xi}) = \delta(\boldsymbol{x}-\boldsymbol{\xi}) - K(\boldsymbol{x},\boldsymbol{\xi})$, because substitution of this equation into Eq. (16) furnishes Eq. (10).

In the previous formulation, the requirement that nonlocal averaging must not alter a uniform stress field led to the normalizing condition (4). For Eq. (20) and (19), the same requirement leads to the conditions:

$$\int_V \Lambda(x,\xi)dV(\xi) = 0; \qquad \sum_{\nu=1}^{N} \Lambda_{\mu\nu} = 0 \tag{17}$$

The state of uniform stress and strain is, of course, associated with a perfectly random array of infinitely many cracks that is statistically uniform on the macroscale, and thus the last condition would hold for such an array only. Furthermore, as a consequence of the condition of preservation of a uniform stress field, and in agreement with the foregoing relations, we must also have

$$\int^V K(x,\xi)\mathrm{d}V(\xi) = 0; \qquad \int^V \Psi(x,\xi)\mathrm{d}V(\xi) = 1; \qquad \sum_{\nu=1}^{\infty} \Psi_{\mu\nu} = 1 \tag{18}$$

Eq. (16) is the typical form of averaging [Eq. (3)] introduced in the nonlocal continuum damage model (Pijaudier-Cabot and Bažant, 1987; Bažant and Pijaudier-Cabot, 1989; Bažant and Ožbolt, 1990, 1991, 1992). This form of averaging was introduced semi-intuitively and was justified by numerical experience. The foregoing derivation lends this type of averaging micromechanics basis. However, because of the matrix inversion according to Eq. (14), it seems difficult to ascribe to the weight function $\alpha(x,\xi)$ a simple physical meaning in micromechanics terms and deduce a simple closed-form expression for this function. We will now try to achieve this goal in a different way.

4. GAUSS-SEIDEL ITERATIVE METHOD APPLIED TO NONLOCAL AVERAGING

In finite element programs, nonlinearity is typically handled by iterations of the loading steps. Let us, therefore, examine the iterative solution of Eq. (12), which represents a system of N linear algebraic equations for N unknowns $\Delta\overline{S}^I_\mu$, assuming ΔS^I_μ to be given. The matrix of $\Psi_{\mu\nu}$ must be symmetric and positive definite on physical grounds. It that case the iterative solution by Gauss-Seidel iterative method converges, and in the r-th iteration, the new, improved values of the unknowns, labeled by subscript $(r+1)$, are calculated from the previous values, labeled by subscript (r), according to the following relations: (e.g., Rektorys, 1969, Collatz, 1960)

$$\Delta\overline{S}^I_{\mu(r+1)} = \Delta S^I_\mu + \sum_\mu \Lambda_{\mu\nu}\Delta\overline{S}^I_{\nu(r)} \tag{19}$$

The values of ΔS^I_μ may be used as the initial values of $\Delta\overline{S}^I_{\mu(r+1)}$ in the first iteration.

The macro-continuum counterpart of the Gauss-Seidel iterative method, which converges to the solution of the Fredholm integral equation (9), is analogous to Eq. (19) and is given by the relations:

$$\Delta\overline{S}^I_{(r+1)}(x) = \Delta S^I(x) + \int_V \Lambda(x,\xi)\Delta\overline{S}^I_{(r)}(\xi)\mathrm{d}V(\xi) \tag{20}$$

This is the relation that ought to be used in finite element programs with iterations in each step. We see that the form of averaging is different from that currently used, given by Eq. (16). The integral term is additive to the initial values, while in Eq. (15) the integral gives the entire values of the inelastic stress increments. This difference must be associated with a difference in the basic properties of the weight functions $\alpha(x,\xi)$ and $\Lambda(x,\xi)$, which has already been indicated in Eq. (17).

5. NONLOCAL WEIGHT FUNCTION AS CRACK INFLUENCE FUNCTION

By virtue of applying the Gauss-Seidel iterative method, the coefficients $\Lambda_{\mu\nu}$, obtained from the stress field of one crack in an infinite elastic solid, can be used in an explicit evaluation of the nonlocal inelastic stress increments. For the purpose of macro-continuum representation, however, certain properties of this field must be simplified in two ways. **First, it is not possible** to deal with cracks of finite size, having (in two dimensions) two distinct crack tips. **Second**, it is necessary to eliminate the singularity of the stress field near the crack tips, even if the tips have fracture process zones of negligible size (in the homogenizing operation that establishes the macro-continuum, the singularities with a finite spacing yield a nonsingular bounded field). The first condition can be met by taking the long-distance (rather than near-tip) asymptotic field of a crack in infinite elastic solid. For the case of two dimensions, the following rather simple expressions have been derived for this field:

$$\sigma_{xx} = \frac{\sigma a^2}{r^2}\left(\frac{\cos 2\theta}{2} - \sin\theta\sin 3\theta\right), \qquad \sigma_{yy} = \frac{\sigma a^2}{r^2}\left(\frac{\cos 2\theta}{2} + \sin\theta\sin 3\theta\right) \tag{21}$$

$$\tau_{xy} = \frac{\sigma a^2}{r^2}\sin\theta\cos 3\theta$$

where σ = uniform pressure applied at the crack faces; $2a$ = crack length; subscripts x, y refer to cartesian coordinates with origin at the crack center and axis y normal to the crack; σ_{xx} and σ_{yy} are the normal stresses, τ_{xy} is the shear stress; and r, θ are polar coordinates with origin at the crack center and the polar angle θ measured from axis x.

The long-range asymptotic stress field has a r^{-2} singularity at the crack center, but it is anyway invalid very near the crack. But the near-tip asymptotic fields cannot be used for the macro-continuum, owing to the $r^{-1/2}$ singularities at the crack tips. These near-tip singularities get obviously wiped out by the statistical averaging operation $\mathcal{E}$. As a result of this averaging, and also because of material heterogeneity (such as the aggregates in concrete), the stress field that is embodied in the weight function Λ ought to be bounded and nearly uniform near the center point $r = 0$. On the other hand, the long-range asymptotic properties of this stress field ought to be preserved because the averaging operator $\mathcal{E}$ covers only a limited neighborhood of the given point x.

Aside from the foregoing modifications, it is convenient to reformulate Eq. 9, in which the differences in the principal stress orientations at points x and ξ are reflected in the values of scalar Λ. For computations, it is more convenient to center the coordinates $x \equiv x_1, y \equiv x_2$ at point ξ such that axis y coincides with the direction of the maximum principal local inelastic stress increment $\Delta S^I(\xi)$, and express the corresponding effects (nonlocal inelastic stress increments) $\Delta S(x)$ in terms of their components in common structural coordinates $X_I, I = 1,2$ ($X_1 \equiv X, X_2 \equiv Y$). In this manner, Eq. (20) for Gauss-Seidel-type iterations is reformulated (in tensorial component form) as follows:

$$\Delta\overline{S}_{IJ(r+1)}(x) = \Delta S_{IJ}(x) + \int_V R_{IJkl}(\xi)L_{kl}(x,\xi)\Delta\overline{S}^I_{(r)}\mathrm{d}V(\xi) \tag{22}$$

in which $R_{IJkl} = c_{Ik}c_{Jl}$ = coordinate rotation tensor (square matrix when the stress tensors are programmed as column matrices); c_{Ik}, c_{Jl} = coefficients of rotation transformation (components of new axis direction vectors) from local coordinates x_i at point ξ (having in general a different orientation at each ξ) to common structural coordinates X_I; subscripts I, J or k, l refer to cartesian components in the common structural coordinates or in the local coordinates at ξ; and L_{kl} = components of a tensorial nonlocal weight function analogous to scalar function Λ. Function L_{kl} will in this case be more aptly named the *crack influence function.*

A strict implementation of Gauss-Seidel iterations suggests programming an iteration loop for Eq. (22) within the loop for the iterations of the loading step. However, one common iteration loop, which is computationally muchmore efficient, can probably serve both purposes. Then, of course, the iteration solution is not exactly the Gauss-Seidel method because the strains are also being updated during the iterations. It remains to verify computationally whether convergence would still be achieved.

The simplest possible expressions for the components of the crack influence function for Eq. (22) satisfying the foregoing considerations regarding the adaptation of the long-range stress field in Eq. (21) appear to be as follows:

$$L_{11}(x,\xi) = \frac{(\kappa\ell)^2}{r^2+\ell^2}\left(\frac{\cos 2\theta}{2} - \sin\theta\sin 3\theta\right), \quad L_{22}(x,\xi) = \frac{(\kappa\ell)^2}{r^2+\ell^2}\left(\frac{\cos 2\theta}{2} + \sin\theta\sin 3\theta\right)$$

$$L_{12}(x,\xi) = \frac{(\kappa\ell)^2}{r^2+\ell^2}(\sin\theta\cos 3\theta) \tag{23}$$

in which κ is an empirical constant such that $\kappa\ell$ represents the average or effective crack length a for the macro-continuum (in theory, it seems this value should be increased during the loading process); and ℓ is an empirical constant, having the role of nonlocal characteristic length, that eliminates the r^{-2} singularity of the long-range asymptotic stress field and at the same time reflects the effect of statistical averaging of the inhomogeneities (such as aggregates in concrete, or grain or crystal size in ice, ceramics and rocks); it may perhaps be taken equal to the larger of the crack size and the maximum inhomogeneity (aggregate) size.

The scalar crack influence function $\Lambda(\boldsymbol{x},\boldsymbol{\xi})$ in (20) or (16) can be obtained from Mohr circle: $2\Lambda(\boldsymbol{x},\boldsymbol{\xi}) = (\sigma_{xx}+\sigma_{yy}) + (\sigma_{xx}-\sigma_{yy})\cos 2(\psi-\theta) - 2\tau_{xy}\sin 2(\psi-\theta)$ in which θ,ψ are the angles of the principal stress directions at points $\boldsymbol{\xi},\boldsymbol{x}$ with the line connecting these two points. Substituting here for σ_{xx}, etc., the expressions in (21) or (23), one get an expression which, as Planas (1992) pointed out, can be reduced by trigonometric transformations to the following simple and symmetric form, in which $f(r) = a^2/r^2$ or $(\kappa\ell)^2/2(r^2+\ell^2)$:

$$\Lambda(\boldsymbol{x},\boldsymbol{\xi}) = -f(r)\left[\cos 2\theta + \cos 2\psi + \cos 2(\theta+\psi)\right] \tag{24}$$

For the case of three dimensions, one may assume the cracks to be penny-shaped. The stresses around such cracks have been expressed as integrals of Bessel functions, which is cumbersome for numerical computations. Recently, however, Fabrikant (1990) ingeniously derived closed form expressions for these stresses. On the basis of his expressions, the following long-range asymptotic field for a penny-shaped crack of radius a loaded by uniform pressure σ, serving as the basis for the crack influence function, can be derived, as a generalization of Eq. (21) to three dimensions:

$$\begin{aligned}
\sigma_{\rho\rho} &= \frac{\sigma a^3}{\pi r^3}\left[(1+2\nu)\left(\sin^2\theta - \frac{2}{3}\right) + (1-2\nu-5\cos^2\theta)\sin^2\theta\right] \\
\sigma_{\phi\phi} &= \frac{\sigma a^3}{\pi r^3}\left[(1+2\nu)\left(\sin^2\theta - \frac{2}{3}\right) - (1-2\nu-5\cos^2\theta)\sin^2\theta\right] \\
\sigma_{zz} &= 2\sigma\frac{a^3}{\pi r^3}\left(\sin^2\theta - \frac{2}{3}\right), \qquad \sigma_{\rho z} = -\frac{\sigma a^3}{\pi r^3}\sin 2\theta(4-5\sin^2\theta), \qquad \sigma_{\rho\phi} = \sigma_{\phi z} = 0
\end{aligned} \tag{25}$$

in which r,θ,ϕ are the spherical coordinates attached to cartesian coordinates x,y,z at point $\boldsymbol{\xi}$, with angle θ measured from axis z which is normal to the crack at point $\boldsymbol{\xi}$; r = distance between points $\boldsymbol{x}$ and $\boldsymbol{\xi}$; and subscripts ρ,ϕ,z refer to components in cylindrical coordinates with origin at the crack center and ρ,ϕ as polar coordinates in the crack plane, angle ϕ being measured from axis x.

The expressions for the three-dimensional crack influence function $L_{\rho\rho}(\boldsymbol{x},\boldsymbol{\xi}), L_{\phi}(\boldsymbol{x},\boldsymbol{\xi}), \ldots$ may now be easily obtained by replacing in the foregoing expressions r^{-3} with $(r^3+b^3)^{-1}$ and a^3 with c^3, similarly to the adaptation leading from Eq. (21) to (23).

One can easily check that the foregoing expressions for the crack influence function satisfies the aforementioned condition that its spatial average be zero. Further note that, asymptotically for large distances r, the crack influence function decays in two dimensions as r^{-2} and in three dimensions as r^{-3}.

In contrast to the previous formulations, the weight function (crack influence function) is not axisymmetric (isotropic) but depends on the the polar or spherical angles (i.e. is ansisotropic) and exhibits a shielding sector in which the crack influence function is positive, and an amplification sector, in which it is negative. The lines separating these sectors can be shown to have, with respect to the crack plane, inclinations $\theta = 30°$ in two dimensions and $90° - \theta = 35.264°$ in three dimensions. The consequence is that interactions between adjacent cracks depend on the direction of propagation of damage with respect to the orientation of the maximum principal inelastic stress. From previous numerical experience with the nonlocal microplane model, it seems that this feature might improve representation of certain test results.

6. CONCLUSION

The present analysis shows that Kachanov's formulation of the superposition method for crack systems can be used to obtain the form of the spatial averaging operator for a nonlocal continuum. It turns out that the weight function of the nonlocal spatial integral should not be a scalar but a tensor. Furthermore, this function should not be axisymmetric (isotropic) but should depend on the polar angle or spherical angle (i.e. be anisotropic). When an iterative solution according to the Gauss-Seidel iterative method is used, the weight function represents continuum smearing of the crack influence matrix. The asymptotic decay of the weight function at long distances r should be of the type r^{-2} in two dimensions and r^{-3} in three dimensions. The spatial integral of the weight function should be zero, and sectors of shielding and amplification can be distinguished.

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