

# Creep and Shrinkage of Concrete

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## 29 CONTINUOUS RETARDATION SPECTRUM FOR SOLIDIFICATION THEORY OF CONCRETE CREEP

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### Abstract

*The recently proposed use solidification theory for the aging aspect of concrete creep makes it possible to use continuous retardation spectrum associated with the Kelvin chain model for nonaging creep. Application to the log-power creep law and a rate-type formulation yields an efficient and complete model for basic creep.*

**Keywords:** Concrete, Basic Creep, Viscoelasticity, Retardation Spectrum, Numerical Integration, Aging.

### 1 Introduction

The solidification theory (Bažant and Prasanna, 1989) considers the material parameters to be constant but the volume fraction of the age-independent constituent in the material depends on age. Thus, there are two separate problems in the formulation of solidification theory. The first is how to describe the variation of the volume fraction of the solidifying nonaging material constituent. The second is how to characterize nonaging creep for the purposes of large-scale numerical analysis and correlate this characterization to some physical theories.

In a preceding study (Bažant and Prasanna, 1989), both problems are resolved using the Kelvin chain with a finite number of Kelvin units characterized by a discrete spectrum of retardation times. However, as is well known from classical (nonaging) viscoelasticity, identification of a broad discrete spectrum from test data is an ill-posed problem because different retardation times can give almost equally good fits of the measured creep curves. Thus, the discrete retardation times must be chosen (suitably, with certain restrictions), and this arbitrariness of choice is disturbing.

The purpose of this study is to show briefly how to formulate the solidification theory with a continuous retardation spectrum, how this spectrum can be easily and unambiguously identified from arbitrary measured creep curves, and how it then can be easily converted to a discrete spectrum for numerical purposes. The identification of the continuous spectrum will be based on Tschoegl's (1989) work on viscoelasticity of polymers. Attention will be limited to concrete creep in absence of moisture exchange and at constant temperature. A detailed presentation of the present theory will be made elsewhere (Bažant and Xi, 1993).

### 2 Generalized Kelvin Chain Model for Nonaging Basic Creep

For the nonaging Kelvin chain model with  $N$  Kelvin units, the compliance function

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is given by the Dirichlet series:

$$J(\xi) = \sum_{\mu=1}^N \frac{1}{E_{\mu}} (1 - e^{-\xi/\tau_{\mu}}) = \sum_{\mu=1}^N A_{\mu} (1 - e^{-\xi/\tau_{\mu}}) \quad (1)$$

where  $\xi = t - t'$ ,  $t$  = time (age of concrete),  $t'$  = time (age) at the moment of loading,  $\tau_{\mu} = \eta_{\mu}/E_{\mu}$  = retardation times ( $\mu = 1, \dots, N$ );  $E_{\mu}$  and  $\eta_{\mu}$  = the elastic modulus of spring and the viscosity of dashpot for the  $\mu$ th Kelvin unit, and  $A_{\mu} = 1/E_{\mu}$ . In Eq. 1,  $\tau_{\mu}$  can be chosen but the choice must satisfy certain well-known restrictions (e.g. Bažant, ed., 1988). The values of  $A_{\mu}$ , which characterize the deformation change during the time leg corresponding to  $\tau_{\mu}$ , have to be determined by optimum fitting of the measured creep curves.

In the previous studies, certain semiempirical formulae have been derived to evaluate  $A_{\mu}$  from the available creep curves (Bažant and Prasannan, 1989). However, when a slightly different creep law is required, those formulae are not valid. Another problem is that although a set of the optimized parameters would suffice to correctly describe the given creep behavior, this set is not unique, depending on the given value of  $\tau_2$ . To deal with general creep laws and to avoid the weak points mentioned (including the ill-posedness or non-uniqueness), an effective method is to introduce a continuous Kelvin chain model in which  $A_{\mu}$  becomes a continuous retardation spectrum.

### 3 Continuous Retardation Spectra and Inverse Transformation Method

Eq. 1 may be approximated in a continuous form:

$$J(\xi) = \int_{-\infty}^{\infty} L(\tau) (1 - e^{-\xi/\tau}) d(\ln \tau) \quad (2)$$

where  $L(\tau)$  is called the continuous retardation spectrum,  $L(\tau) = \tau/E_{\mu}$ , having the same meaning in the logarithmic time scale as  $A_{\mu}$  in the actual time scale (Eq. 1). Many studies have been undertaken to deduce  $L(\tau)$  from the known compliance function of the material. We will adopt a very efficient general method developed by Tschögl (1989).

Using Eq. 2, and setting  $\tau = 1/\zeta$  with  $d(\ln \tau) = -d(\ln \zeta)$ , we get

$$f(\xi) = - \int_0^{\infty} \zeta^{-1} L(\zeta^{-1}) e^{-\xi \zeta} d\zeta = J(\xi) - \int_0^{\infty} L(\zeta^{-1}) \zeta^{-1} d\zeta \quad (3)$$

$f(\xi)$  is the Laplace transform of  $\zeta^{-1} L(\zeta^{-1})$ , and  $\xi$  is the transform variable. Now the important point is that the transform can be inverted by Widder's (1971) inversion formula, based on an asymptotic method. The inversion operator is

$$L(\tau) = \lim_{k \rightarrow \infty} \frac{(-k\tau)^k}{(k-1)!} f^{(k)}(k\tau) \quad (4)$$

The approximate spectrum of order  $k$  is obtained by using a finite value of  $k$ . The compliance data are entered through Eq. 3, and then numerical differentiation of  $f^{(k)}(k\tau)$  yields  $L(\tau)$ . But experimental data generally exhibit random scatter and thus are not precise enough to allow taking higher derivatives except perhaps the second. Instead of numerical differentiation of the test data, one must differentiate a smooth continuous compliance functions that matches the experimental data well enough.

At this point, the only problem that remains is to decide what kind of compliance function should be chosen. For concrete, the basic features that are exhibited by most sets of data (Bažant and Kim, 1991) are that the short-term creep follows the power curve while the long-term creep follows the logarithmic curve. This means that the

log-power law is a simple yet reasonable representation of the compliance function for concrete.

### 4 Application to Nonaging Log-Power Creep Law

Let us now apply this formulation to the log-power, which law reads

$$J(\xi) = q_2 \ln \left[ 1 + \left( \frac{\xi}{\lambda_0} \right)^n \right] \quad (5)$$

Here, empirically, one can use for most concretes  $\lambda_0 = 1$ , while the value of  $q_2$  depends on the type of concrete. Then for  $k = 3$ , Eq. 4 yields

$$L(\tau) = \left[ \frac{-2n^2(3\tau)^{2n-3}[n-1-(3\tau)^n]}{[1+(3\tau)^n]^3} + \frac{n(n-2)(3\tau)^{n-3}[n-1-(3\tau)^n] - n^2(3\tau)^{2n-3}}{[1+(3\tau)^n]^2} \right] \frac{(3\tau)^3}{2} q_2 \quad (6)$$

This is the approximate retardation spectrum of order 3, which seems sufficient for practical purposes. According to the data fitting in Bažant and Prasannan (1989b),  $n$  is approximately a constant. In the case of small  $n$ , the terms with  $n$  in Eq. 6 may be neglected without much loss of accuracy. Combining this with some other simplifications of Eq. 6, a simple approximation to the retardation spectrum can be obtained:

$$L(\tau) \approx q_2 n(1-n) \frac{(3\tau)^n}{1+(3\tau)^n} \quad (7)$$

For very large  $\tau$ , the spectrum  $L(\tau)$  approaches a constant.

For a crude compliance function that would be suited for a design code, Eq. 7 is preferable because of its simple form. For computational analysis, such as finite element analysis, or when  $n$  is large ( $n > 0.45$ ), Eq. 6 ought to be used.

For the purpose of numerical computation, one can subdivide  $\ln \tau$  into time intervals  $\Delta(\ln \tau_{\mu}) = \ln(10)\Delta(\log \tau_{\mu})$  and thus approximate the integral in Eq. 2 by a finite sum:

$$J(\xi) = \sum_{\mu=1}^N A_{\mu} (1 - e^{-\xi/\tau_{\mu}}) \quad (8)$$

where  $A_{\mu} = L(\tau_{\mu}) \ln(10) (\log \tau_{\mu})$ ,  $L(\tau_{\mu})$  is given by Eq. 6 or Eq. 7, and  $\Delta(\log \tau_{\mu})$  is the time interval between two adjacent Kelvin units in the logarithmic scale. Computational experience shows that intervals  $\Delta(\log \tau_{\mu}) = \log(10) = 1$  give sufficiently smooth creep curves (or compliance function), while greater separations of  $\tau_{\mu}$  give creep curves of bumpy appearance.

Fig. 1 compares  $\ln(1 + \xi^n)$  with Eq. 8, in which  $L(\tau)$  is replaced by the continuous spectrum, Eq. 6. It can be seen that the compliance functions obtained from the continuous spectrum agree with the log-power curve very well, which proves that the approximation of order 3 is accurate enough. Fig. 2 shows the compliance functions obtained from the simplified continuous spectrum, Eq. 7. It is clear that, for small  $n$ , Eq. 7 is also accurate enough.

Another advantageous feature of the continuous retardation spectrum is that some physical characteristics of creep can be obtained merely by comparison of the intensity of the spectrum within a certain retardation time range. For instance, Fig. 3 shows the retardation spectra for various values of  $n$ . The curve for  $n = 0.04$  shows that creep intensity (value of retardation spectrum) can be considered almost uniform within the

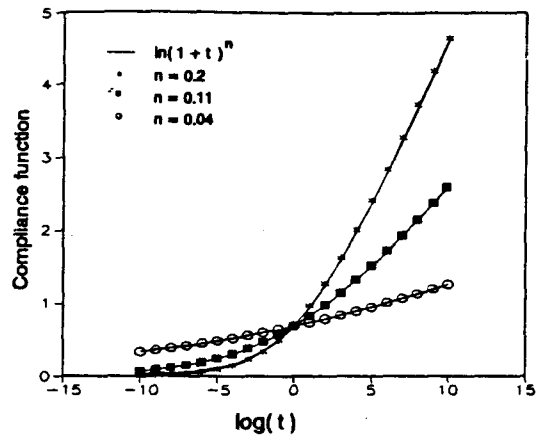


Fig. 1 Comparison of compliance function

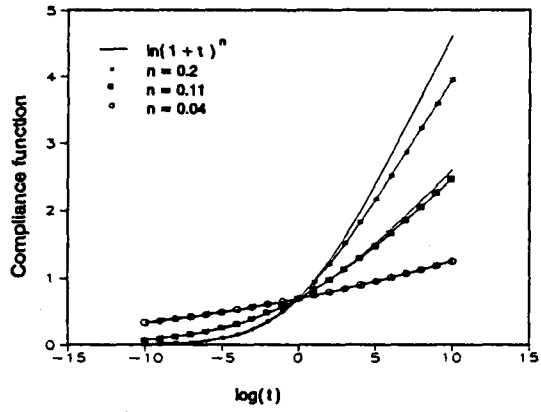


Fig. 2 Comparison of log-power law with simplified compliance function

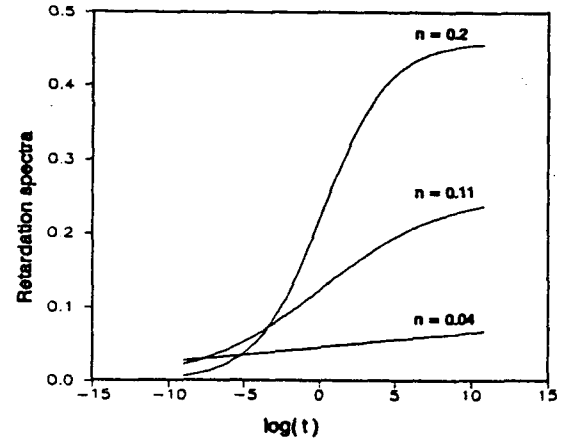


Fig. 3 Retardation spectra with various n

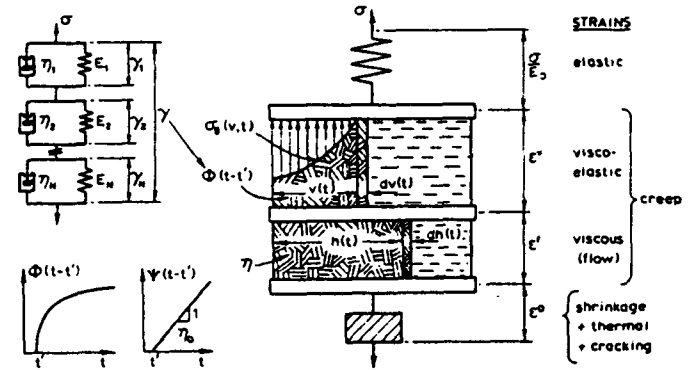


Fig. 4 Model for role of solidification in creep

time range  $10^{-5} - 10^5$ . By checking the compliance function in Fig. 1 (curve with  $n = 0.04$ ), it is seen that the creep indeed proceeds smoothly. However, the curve with  $n = 0.2$  in Fig. 3 shows a relatively strong intensity in the time range  $10^1 - 10^{10}$ , which means that a significant part of the total creep will be delayed to the long-time range, and that the creep in the short-time range will be relatively small. By checking the compliance curve for  $n = 0.2$  in Fig. 1, there is indeed a sharp increase of creep in the long-time range. So, in addition to computational advantages, the continuous retardation spectrum also reflects the creep intensity in various time ranges.

## 5 Solidification Theory for Aging and Rate-Type Formulation

For the sake of completion of our formulation, let us briefly indicate how the foregoing formulation is combined with the solidification theory, which was presented and justified in detail in Bažant and Prasanna (1989a,b). This theory assumes that the aging property of creep is caused by the processes of hydration and polymerization of cement (the hydration is of course also manifested by the increase of strength with age). Concrete is divided into three parts: the liquid part, which cannot bear load; and two load-bearing parts exhibiting viscous flow and viscoelastic deformation. Thus, the total creep strain is composed of two terms, the viscous flow strain,  $\epsilon^f$ , and the viscoelastic strain,  $\epsilon^v$ .

The key feature of the theory is that the aging aspect of concrete creep is considered to be due to the growth of the volume fraction  $v(t)$  (Fig. 4) of the effective load-bearing portion of solidified matter (i.e. hydrated cement), representing both the increase of the volume fraction of hydrated cement and the increase of the load-bearing solid fraction caused by formation of further bonds (or polymerization of calcium silicate hydrates). The advantage is that, in this theory, the properties of the load-bearing matter are age-independent. Thus, the conventional viscoelastic (and viscoplastic) theories, as well as thermodynamic relations, can be applied. The creep strain rate corresponding to the viscoelastic solid part,  $\dot{\epsilon}^v$ , can be expressed as the product of the age-independent strain rate of solid,  $\dot{\gamma}$ , and the increase of the volume fraction  $v(t)$  of the solid (Bažant and Prasanna, 1989a,b):

$$\dot{\epsilon}^v(t) = \frac{F[\sigma(t)]}{v(t)} \dot{\gamma}(t) \quad (9)$$

where function  $F[\sigma(t)]$  is introduced to reflect nonlinear behavior at high stress (at low stress,  $F[\sigma(t)] = 1$ ).

In this formulation, all of the procedures we previously developed for the nonaging basic creep are applicable to the viscoelastic microstrain,  $\gamma^v$ . In the one-dimensional case, we may apply Eq. 8 at constant stress  $\sigma$

$$\gamma^v = \sigma \sum_{\mu=1}^N A_{\mu} (1 - e^{-t/\tau_{\mu}}) \quad (10)$$

In analogy to Eq. 9, we have for the flow term:

$$\dot{\epsilon}^f(t) = q_3 \frac{F[\sigma(t)]}{v(t)} \sigma(t) \quad (11)$$

where  $q_3$  is an empirical coefficient which depends on the composition of concrete, similar to  $q_2$  in Eq. 5.

Numerical computation, as in finite element programs, proceeds in small time steps  $\Delta t = t_{i+1} - t_i$  ( $i = 1, 2, \dots$ ) and in an incremental form, which we give for the general case of three dimensions. We can always assume that, for a sufficiently short time step  $\Delta t$ , the stresses change linearly with  $t$ . Then, solving the differential equations for each Kelvin unit, we get

$$\Delta \sigma = D_c (\Delta \epsilon_c - \Delta \epsilon_c'') \quad (12)$$

where  $E$  and matrices  $D_c = ED$ ,  $\Delta \epsilon_c''$  can be found in Bažant (1988) or Bažant and Prasanna (1989), and will not be repeated here.

Eq. 12 reduces the solution of the basic creep problem to a sequence of elastic solutions with initial strains. In the case of high stress level, nonlinearity due to  $F[\sigma]$  requires iterations of each time step to achieve good accuracy.

## 6 Conclusions

1. Despite the aging of concrete, it is possible and advantageous to use a continuous retardation spectrum for the Kelvin chain model in the solidification theory. This spectrum can be determined by the asymptotic transformation method, which is applicable for any creep law. By this method, a unique retardation spectrum can be obtained from the given compliance function.

2. Application to the log-power creep law reveals that for concrete the asymptotic retardation spectrum of order 3 is sufficiently accurate in practice. For small values of the time exponent,  $n$ , which are typical of concrete, the spectrum can be simplified and still describe the compliance function satisfactorily.

3. By discretization of the continuous retardation spectrum, a rate-type formulation for concrete creep can be obtained and combined with the solidification theory for aging. This yields a complete model for basic creep.

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