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Nonlocal model for size effect in quasibrittle failure based on extreme value statistics

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ABSTRACT: A nonlocal generalization of Weibull theory is presented to predict the probability of failure of unnotched structures that reach the maximum load before a large crack forms, as is typical of the test of modulus of rupture. The probability of failure at a material point is assumed to be a power function of the average strain in the neighborhood of that point. For normal sizes, the deterministic theory is found to dominate the mean response and govern the size effect. But the probabilistic theory can provide the entire probability distribution. For extremely large beam sizes, the statistical size effect dominates and the mean prediction approaches asymptotically the classical Weibull size effect. This fundamental feature is discussed in relation to the existing stochastic finite element models. Comparison to the existing test data demonstrates a good agreement with the theory.

1 INTRODUCTION

As became clear in the early 1980's, the size effect on the nominal strength of quasibrittle structures is in most instances predominantly deterministic. It is caused by stress redistributions and energy release associated with either the growth of a large fracture process zone (FPZ) or a long stable crack (Bažant 1984, Bažant & Chen, 1997; Bažant & Planas 1998). Since the material properties represent a random field, some aspects of the size effect should nevertheless be probabilistic. Presenting a new combined energetic-probabilistic theory and exploring where the statistical aspect is important for the mean, variance and probability distribution is objective of this paper.

A combined energetic-probabilistic theory, having the classical Weibull probabilistic theory of failure as one limit and the deterministic energetic theory as another limit, can be now developed on the basis of the available evidence. Since the Weibull theory (Weibull 1939), based on the weakest link model, deals with structures that fail before a large (macroscopic) crack can form, a nonlocal generalization of Weibull statistical theory may be developed to predict the probability of failure of unnotched structures that reach the maximum load before a large crack forms, as is typical of the test of modulus of rupture (flexural strength).

The tail of the cumulative probability distribution of material failure at one point is assumed to be a power function (characterized by Weibull modulus m and scaling parameter σ_0) of the average inelastic (or damage) strain in a neighborhood the size of which is characterized by the material characteristic length. The averaging indirectly imposes spatial statistical correlation. The deterministic size effect is automatically exhibited as the limit case of such a formulation for $m \to \infty$.

For very large sizes of unnotched structures, the statistical size effect is shown to dominate and the mean prediction approaches asymptotically the classical Weibull size effect . This is contrary to structures with notches or large stable cracks, for which the classical Weibull size effect has previously been shown to be approached asymptotically for very small, rather than very large, structure sizes. The new energetic statistical theory is shown to agree quite well with extensive test data found in the literature.

It has been argued that a sound probabilistic theory of quasibrittle failure must asymptotically approach the Weibull theory with the weakest link model (extreme value statistics) in the case that the ratio of structure size D to the characteristic length l of the material tends to ∞. The stochastic finite element method, in which the role of l is played by the autocorrelation length of the random field of material strength, does not satisfy this basic requirement, while the proposed theory does.

2 NONLOCAL WEIBULL THEORY

2.1 *Failure probability calculation: Weibull integral*

The Weibull integral for probability P_f of structural failure (Bažant & Planas 1998, ch. 12) was reformulated by Bažant & Novák (2000a,b) in a nonlocal form. In this reformulation, the local stresses are replaced by the nonlocal (spatially averaged) strains multiplied by the modulus of elasticity, as proposed by Bažant & Xi (1991). Then the multi-dimensional generalization of Weibull integral may be written as

$$P_f = 1 - \exp\left\{ - \int_V \sum_{i=1}^{n} \left\langle \frac{\overline{\sigma}_i(\boldsymbol{x})}{\sigma_0} \right\rangle^m \frac{dV(\boldsymbol{x})}{V_r} \right\} \tag{1}$$

where n = number of dimensions (1, 2 or 3), σ_0 = Weibull scaling parameter, V_r = representative volume of material (having the dimension of material length), σ_i = principal stresses ($i = 1, ...n, n$), and an overbar denotes nonlocal averaging. The failure probability now depends no longer on the local stresses $\sigma_i(\boldsymbol{x})$ but on the nonlocal stresses $\overline{\sigma}_i(\boldsymbol{x})$ which are the results of some form of spatial averaging of strains; for details see Bažant & Xi (1991), Bažant & Planas (1998, ch. 12), and Bažant & Novák (2000a,b). In the case of an unreinforced simply supported symmetric beam with a symmetric uniaxial stress field treated as two-dimensional, (1) becomes:

$$P_f = 1 - \exp\left\{ -\frac{2}{V_r} \int_0^{L/2} \int_{-s}^{D/2} \left[\frac{\overline{\sigma}(x,y)}{\sigma_0} \right]^m dx\, dy \right\} \tag{2}$$

where L = span of the beam, D = size (height) of the beam and s = shift of the neutral axis of beam caused by distributed cracking.

It might seem that the analysis of strain-softening would call for using finite elements. In the present problem of beam bending, however, this is unnecessary because only the states before a crack forms are of interest. The softening zone, restrained by the adjacent material that is in an elastic state, does not yet localize, remaining distributed over a long portion of the beam. Therefore, the classical hypothesis of cross sections remaining planar is a good approximation. of strains within the cross section. computational model adopted are given in Bažant & Novák (2000a).

2.2 *Illustration of spatial distribution of contributions to failure probability*

To clarify the basic concept, it is helpful to present at this point Figure 1, which shows the succession of breaks of material points according to the spatial distribution of the contributions to failure probability entering the integral (2). Both the three-point bending and the four-point bending cases are studied in Figure 1. For the input, consisting in the number of failed points (indicated at the top-left corner of each quarter of the beam), pure Monte Carlo simulation is performed according to the distribution of probabilities. Naturally, the first failed points appear near the midspan, in the case of three-point bending, or near the bottom face within the maximum moment region of the beam, in the case of four-point bending. As the number of failed points increases, the development of the shape of the fracture process zone, visualized by different levels of probabilities, can be observed. It should be kept in mind that the figure does not portray the sequence of failures associated with the formation of real crack. Rather, it shows merely the distribution of the contributions to failure probability intended to provide better insight into the nonlocal Weibull theory.

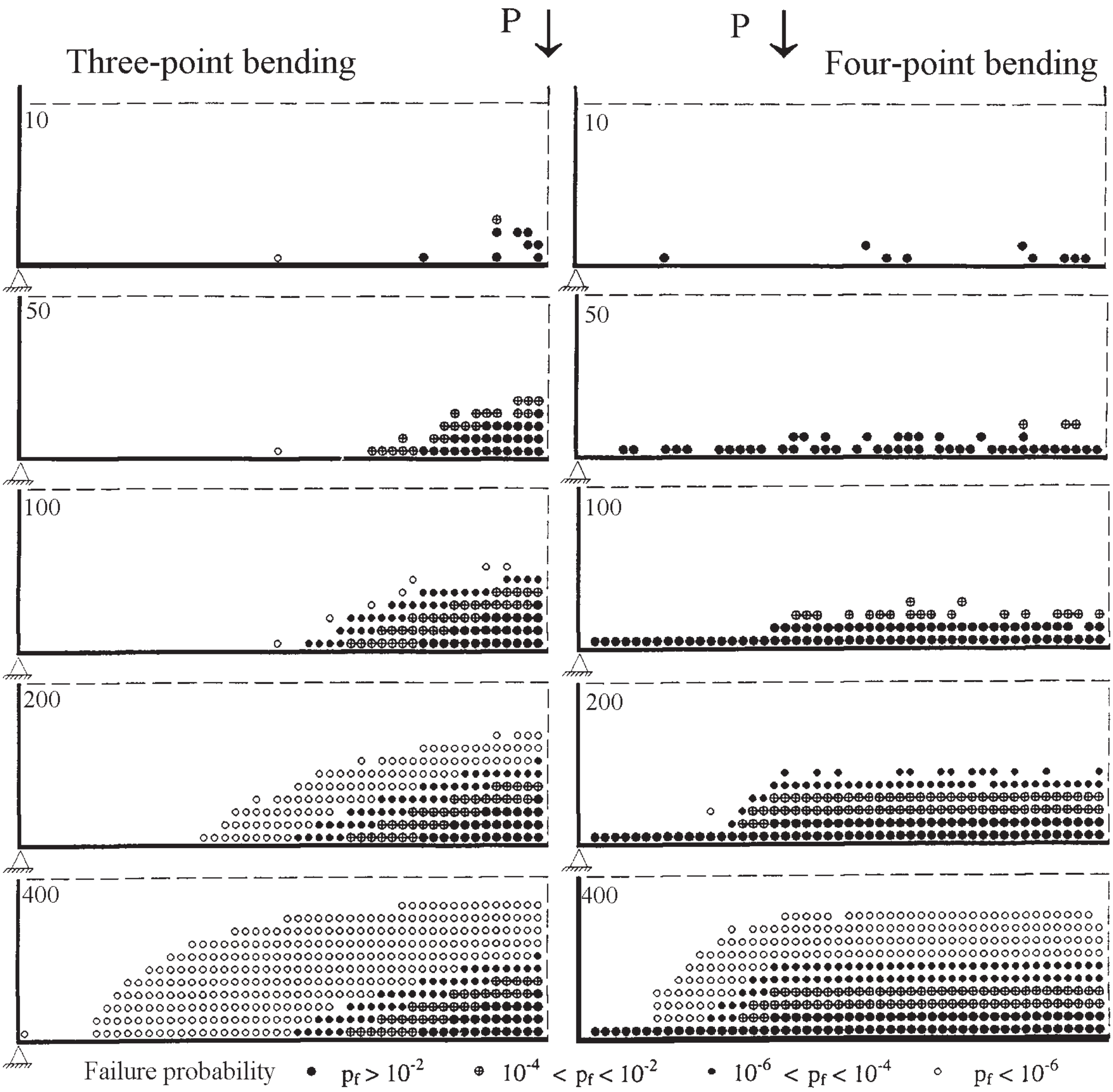

Figure 1: Monte Carlo simulation of failures according to spatial distribution of failure probability.

3 COMPARISON WITH EXISTING TEST DATA

The present theory has been compared with most important data sets found in the literature (Bažant & Novák 2000a,b,c). Details on extensive data and comparative calculations can be found in referenced literature. Here only selected comparisons are included from many results.

3.1 *Estimation of cumulative probability distribution function*

The Weibull-type integral makes it possible to estimate the failure probabilities corresponding to different load levels. Covering the full range of probabilities, one can estimate the probability distribution function for the modulus of rupture. The proper load levels are such that the entire range of the cumulative probability distribution function from 0 to 1 could be covered almost regularly. Thus it is efficient to use the idea of the stratified sampling called Latin hypercube sampling (McKay et al. 1979, Novák et al. 1998).

The probability distribution functions of the ratio of modulus of rupture to strength are plotted in Figure 2 for different sizes. The sample size $N = 16$ has been chosen for calculations - 16 different probabilities which are taken as the input into the nonlocal Weibull model. As expected, the steepness increases with increasing size, which means that the scatter decreases with the size. This agrees with the well-known fact that the statistical correlation of strength imposed by averaging has a major influence only for small sizes. Such trends for the distribution functions were already in general sketched by Shinozuka (1972).

An important source of statistical information are Koide et al.'s (1998) tests of 279 plain concrete beams in four-point bending, aimed at determining the influence of the beam length L on the flexural strength of beams. Koide's are excellent data which allow comparing the cumulative probability distribution function (CPDF) of the maximum bending moment M_{max} at failure, over its full range. The data points in Figure 3 show the empirical cumulative probability density functions for one selected span (Koide's series C). A good agreement with Koide et al.'s data has been achieved. The calculations indicate a decrease of the flexural strength as the span increases. Notice the similar trends in Figure 3 (limited sizes), and more generally in Figure 2 (an extremely broad range of sizes).

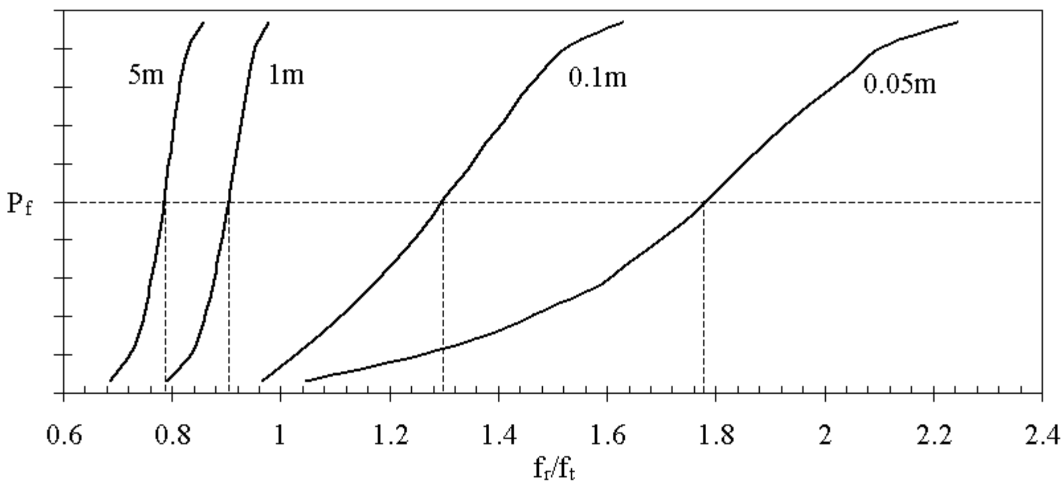

Figure 2: Cumulative probability distributions of modulus of rupture for different sizes.

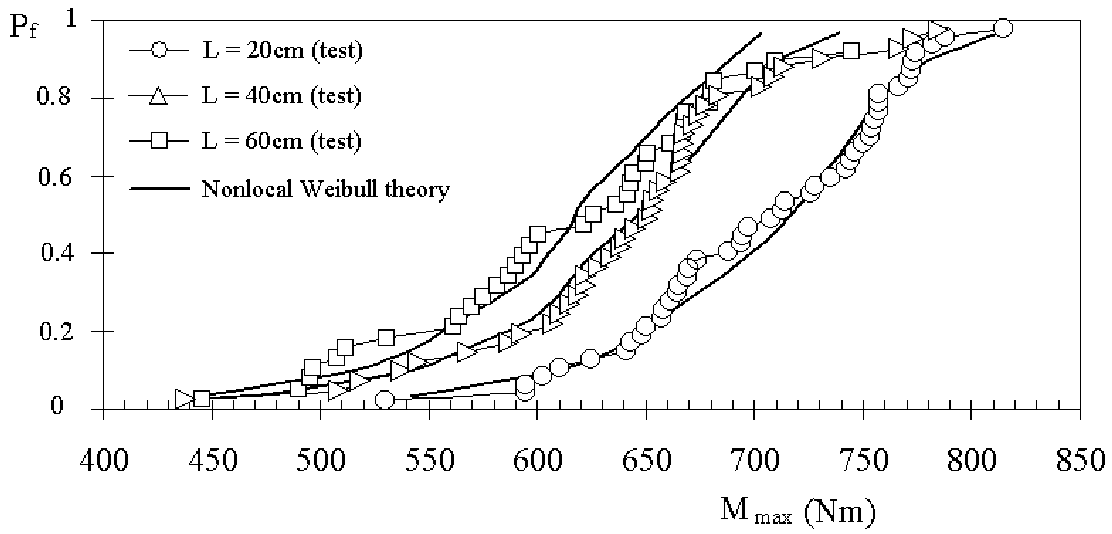

Figure 3: Comparison of CPDF of maximum bending moment from Koide's (1998), series C, of four-point bending tests and from the probabilistic nonlocal theory.

3.2 *Indirect comparison: Energetic-statistical formula*

The size effect on the modulus of rupture has been shown to follow the energetic-statistical formula (Bažant & Novák 2000c):

$$f_r = f_{r,\infty}\left[\left(\frac{D_b}{D}\right)^{rn/m} + \frac{rD_b}{D}\right]^{1/r} \tag{3}$$

where $f_{r,\infty}$, D_b, r and m are positive constants, representing unknown empirical parameters, and n is the number of dimensions in geometric similarity. Data fitting with the new formula (3) reveals that, for concrete and mortar, the Weibull modulus $m \approx 24$ rather than 12, the value widely accepted so far (Bažant & Novák 2000c). Fitting of the formula to the main test data sets available in the literature showed an excellent agreement with a rather small coefficient of variation of errors of the formula compared to the test data. The result is shown in Figure 4. The corresponding coefficient of variation is ω = 0.023 and the optimum values of the parameters are $f_{r,\infty}$ = 3.68 MPa, D_b = 15.53 mm and r = 1.14.

Furthermore, the new formula was verified numerically also by the nonlocal Weibull theory. The result of nonlinear fitting of formula (3) using the nonlocal solutions of failure probability (medians of modulus of rupture) of the beam is presented in Figure 5. The corresponding parameters are $f_{r,\infty}$ = 3.76 MPa, D_b = 48.66 mm and r= 1.28. As it can be seen, both curves are very close and this favorable comparison supports (but of course does not prove) the correctness of the present energetic-statistical size effect formula (3), as well as the nonlocal Weibull material model.

3.3 *Small failure probabilities*

One advantage of the present approach is that small failure probabilities can be estimated without an increase of computational time (as is typical for Monte-Carlo based approaches in classical reliability engineering). The same approach (Weibull integral) is used for estimation of the median ($P_f = 0.5$) and e.g. for $P_f = 10^{-9}$. A broad range of failure probabilities is shown, as an illustration, in Figure 6, for data of Lindner and Sprague (1956). Naturally, for small failure probabilities, the curves approach the Weibull type of size effect.

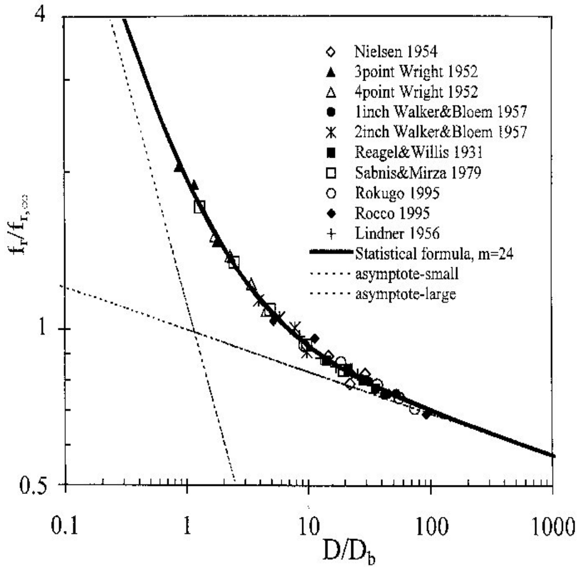

Figure 4: Optimum fit to existing test data.

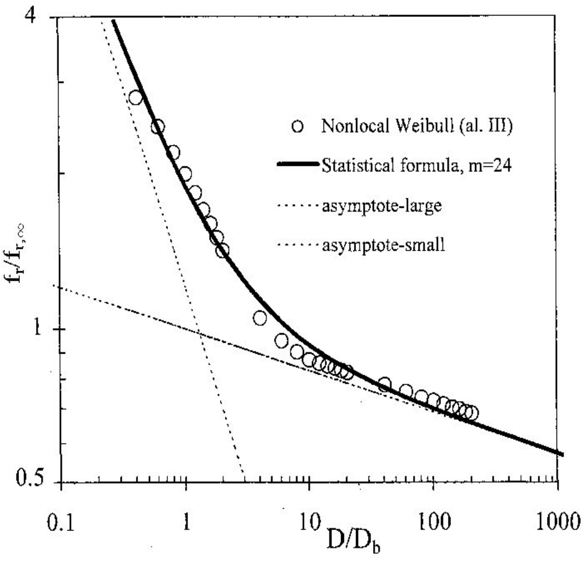

Figure 5: Optimum fit to the nonlocal Weibull theory.

4 COMPARISON WITH STOCHASTIC FINITE ELEMENT MODELS

Applications of the theory of random fields to the finite element method have led during the last fifteen years to the development of the stochastic finite element method (SFEM) (Schuëller 1998). One advantage of SFEM is that any number of variables or random fields can be used to simulate the uncertainties of material, environmental and geometric parameters. In the present nonlocal Weibull approach, the reliability problem is reduced to one dominant random variable (strength). The ran-

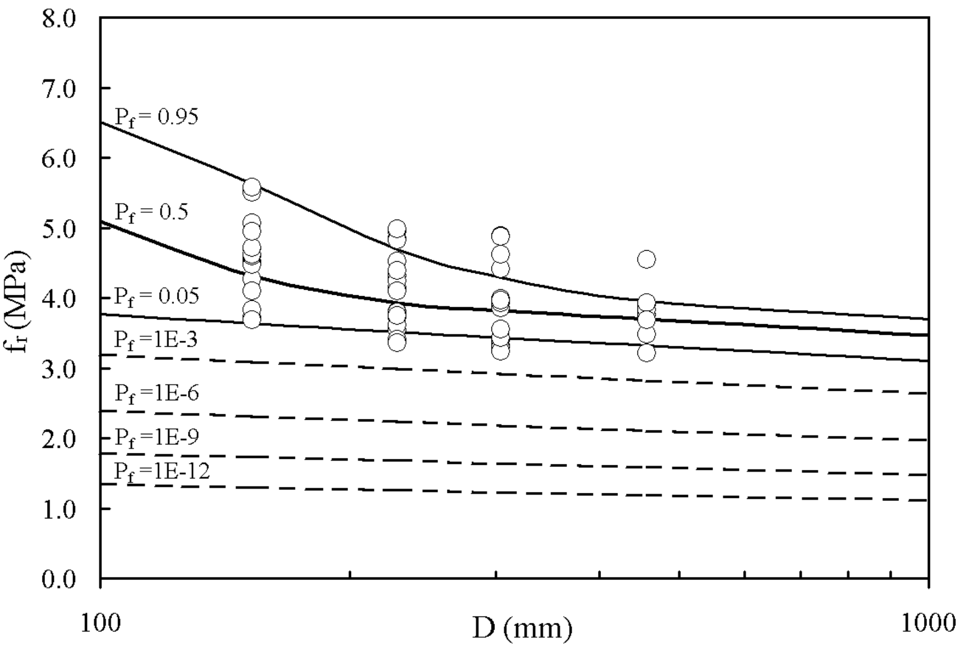

Figure 6: Failure probabilities vs. size for data of Lindner and Sprague (1956).

domness of elastic moduli and other parameters of a nonlinear strain softening constitutive law can nevertheless be taken into account by standard Monte Carlo simulation.

The nonlocal Weibull theory is conceptually transparent and simple. In calculating the failure load probability, SFEM is considerably more complicated than the present nonlocal Weibull approach and, despite its many achievements, cannot yet handle really complex structures because of the tremendous amount of computational effort required.

The essential random field characteristics required as the input to SFEM , particularly the correlation length, are very difficult to determine in a rational manner, and have generally been estimated heuristically based on intuitive judgement. In the nonlocal Weibull theory, on the other hand, the parameter of spatial correlation is the characteristic length, which is the same as that in the deterministic nonlocal damage theory and has an intimate relationship to the heterogeneity of the material (it may be taken as several times the maximum aggregate size in concrete and may also be related to the fracture energy and strength of the material).

According to the best writers' knowledge, the Weibull-type size effect has not yet been reproduced by SFEM. The decisive parameter in SFEM is the correlation length which prescribes spatial correlation over the structure. The correlation length modifies the size effect curve in the region where this parameter is smaller than the element size. There is a clear relationship—the larger the correlation length, the stronger the spatial correlation of strength along the structure, and consequently the smaller the decrease of nominal strength of the structure with its increasing size. Problems occur in trying to obtain the extreme value asymptote using the random field approach. Approximately, the requirement is that the ratio of the correlation length to the element size should not drop bellow one. This poses a major obstacle to using SFEM for describing the size effect, especially for large structure sizes.

Some advances in this topic were achieved by several authors, e.g. Carmeliet & Hens (1994). But these authors usually confine their studies to the region of reasonable sizes. The ratio of the correlation length to the element size implies some limitations. To obtain the extreme value asymptote using the random field approach, the number of discretization points (e.g. nodes in finite element method) should increase proportionally as the structure size increases. In other words, keeping the same element size for different sizes of the structure is preferable to the alternative of keeping same number of elements. This requirement for size effect studies using SFEM can be crucial or even impossible to adopt: The number of elements can become extremely large!

In the nonlocal Weibull theory there is no such limitation: For any mesh $dV(x)$ ($dx\,dy$ for 2D problems), the Weibull integral is calculated through algebraic sum, and there is always a correct increase of the failure probability with an increasing structure size for a certain load (which leads to a size effect of Weibull type for very large sizes).

5 CONCLUSIONS

1. In the nonlocal generalization of Weibull theory the failure probability of a small material element is a function of the nonlocal (spatially averaged) continuum variables rather than the local stress. This generalization can be applied to unnotched specimens, and in particular to the test of the modulus of rupture (flexural strength).

2. A new generalized formula (3) that amalgamates the energetic and statistical size effects for failures at crack initiation has been developed. Its correctness is supported by good agreement with structural analysis according to the statistical nonlocal material model.

3. The present models agree well with the test data sets found in the literature.

4. The main benefit of the present theory is the possibility to predict the full probability distribution of structural strength, and in particular the modulus of rupture.

5. Compared to the existing stochastic finite element approaches, a great simplification is achieved by the fact that the nonlocal structural analysis with strain softening can be conducted deterministically because the probability analysis is separated from the stress analysis, similar to the classical Weibull theory.

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