

Shear Buckling of Fiber Composites and Sandwich Structures: Paradox Resolved

BY ZDENĚK P. BAŽANT¹ AND ALESSANDRO BEGHINI²

¹*McCormick School Professor and W.P. Murphy Professor of Civil Engineering and Materials Science,
Northwestern University, Evanston, Illinois 60208; z-bazant@northwestern.edu*

²*Graduate Research Assistant, Northwestern University*

Abstract: In situations where the initial stresses before buckling are not negligible compared to the tangential elastic moduli, the tangential moduli inevitably depend on the initial stresses and this dependence must be taken into account in stability analysis. The stability or bifurcation criteria have then different forms for tangential moduli associated with different choices of the finite strain measure. In view of this fact, known since 1971, it recently appeared paradoxical that, for fiber composite columns and soft-core sandwich columns, different but equally plausible assumptions yield different formulae, Engesser's and Haringx' formulae, even though the axial initial stress in the fibers (or the skins) is negligible compared to their elastic modulus and the normal initial stress in the matrix (or the core) is negligible compared to the shear modulus of the matrix (or the core). This apparent paradox, debated in recent symposia on composites, is now explained by variational energy analysis. It is shown that the shear stiffness of a sandwich column generally depends on the axial force carried by the skins, which is not negligible compared to the shear stiffness if the column is short. The Engesser-type, Haringx-type and other possible formulae associated with different finite strain measures are all, in principle, equivalent, although a different shear stiffness of the core, depending linearly on the applied axial load, must be used for each. The Haringx-type formula, however, is most convenient because it represents the only case in which the shear modulus of the core can be taken as independent of the axial force in the skins and equal to the shear modulus measured in a simple shear test (e.g., a torsional test). An extension of the analysis further shows that Haringx's formula is preferable for a highly orthotropic composite because a constant shear modulus of the soft matrix can be used for calculating the shear stiffness of the column. Besides, the numerical implications associated with these theoretical considerations are reported in terms of Finite Element analysis of the buckling load of a sample column.

Introduction

The load capacity of sandwich structures has been studied for over half a century and major advances have been achieved. However, accurate and unambiguous predictions still cannot be made on the basis of the existing theories. With the advent of the use of composites in large structures, such as the hulls, decks, bulkheads, masts and antenna covers for very large ships, the problem recently gained in importance and the remaining problems need to be solved.

Sandwich shell failures due to fracture of the skins, cores and interfaces are often combined with the loss of stability, and therefore the problems of buckling, fracture and damage cannot be completely separated. In the field of elastic stability analysis, there still exists one fundamental unresolved problem which impinges on all the problems of failure—namely the role of shear of highly deformable cores. This problem is particularly acute for sandwich shells with stiff fiber composite laminate skins and very light polymeric cores because the skin-to-core elastic moduli ratio can be (in the case of Divinycell 100 foam) as high as 2000.

The conference presentation will outline and discuss a recent variational analysis of the problem based on different finite strain measures (Bažant 2003), and will also feature some supporting results of finite element studies (which are still in progress at the time of writing).

Apparent Paradox in Shear-Beam Theories for Sandwich Buckling

Four decades ago, there used to be polemics among the proponents of different three-dimensional

stability formulations associated variationally with different finite strain measures (see, e.g., the preface of Biot's, 1965, book), different objective stress rates, and different incremental differential equations of equilibrium (proposed by Hadamard, Biot, Trefftz, Truesdell, Pearson, Hill, Biezeno, Hencky, Neuber, Jaumann, Southwell, Cotter, Rivlin, Engesser, Haringx, etc.—see [9] (p. 732 and chapter 11) and [3]. These polemics were settled by the demonstration [4] that all these formulations are equivalent because the tangential elastic moduli of the material cannot be taken the same but must rather have different values in each formulation. It was also concluded that these differences matter if initial stresses at the critical state of buckling are not negligible compared to the elastic moduli ([9] Sec. 11.4).

Although the differences between various stability criteria are insignificant for most buckling problems, because the initial stresses are negligible compared to the tangential moduli, there are some exceptions. A very important one is the buckling of sandwich plates with a very soft core and buckling of fiber composites with a highly orthotropic fiber reinforcement and a very soft matrix. In sandwich plates, which are very sensitive to buckling (Goodier and Hsu 1954, Plantema 1966, Allen 1969, Kovářík and Šlapák 1973, Michiharu 1976, Chong et al. 1979, Frostig and Baruch 1993, etc.), the initial axial stress in the skins of a sandwich column is negligible compared to the elastic modulus of the skins, and the initial axial stress in the foam core is zero. Consequently, it may at first seem that the shear stiffness of the core should not depend on the axial force in the skins, which would imply that there should be no differences among the critical local formulae associated with different finite strain measures.

Consequently, it came as a surprise that the Engesser-type (Engesser 1889, 1891) buckling formula for sandwich columns, which is associated with the Doyle-Ericksen finite strain tensor of order $m = 2$, gave for short sandwich columns much smaller critical loads than the Haringx-type (Haringx 1942, 1948–1949) formula, which is associated with the Doyle-Ericksen tensor of order $m = -2$. Using equal shear stiffness values for both formulae, Kardomateas (2000, 2001), Huang and Kardomateas (2000), Simitzes and Shen (2000), etc., showed that the Haringx-type buckling formula gave results closer to the experiments on sandwich columns and also to three-dimensional finite element simulations. The differences between the two buckling formulas, illustrated in Fig. 1, have been analyzed in detail in a paper just published [7, 8] and will now be reviewed. The discussions of these differences began about sixty years ago (e.g., Timoshenko and Gere 1963, Bažant 1971, 1992, 1993, Ziegler 1982, Reissner 1972, 1982, Simo and Kelly 1984, Simo et al. 1984, Gjelsvik 1991, Wang and Alwis 1992, Attard 2002, Bažant 1992, 1993). However, no consensus on the theory has yet emerged [5, 6], although the experiments on helical springs (Haringx 1948–1949), elastomeric bearings (Buckle et al. 2002) and latticed columns (Gjelsvik 1991), stressed in the linear range of material behavior, clearly favor Haringx's formula.

First let us recall the class of Doyle-Ericksen finite strain tensors $\epsilon = (U^m - I)/m$ (where $m =$ real parameter, $I =$ unit tensor, and $U =$ right-stretch tensor). These tensors, which include virtually all the strain measures ever used, have the second-order approximation:

$$\epsilon_{ij}^{(m)} = e_{ij} + \frac{1}{2}u_{k,i}u_{k,j} - \alpha e_{ki}e_{kj}, \quad e_{ki} = \frac{1}{2}(u_{k,i} + u_{i,k}), \quad \alpha = 1 - \frac{1}{2}m \quad (1)$$

(Bažant 1971); $e_{ij} =$ small (linearized) strain tensor and the subscripts refer to Cartesian coordinates x_i , $i = 1, 2, 3$. The stability criteria expressed in terms of any of these strain measures are mutually equivalent if the tangential moduli associated with different m -values satisfy Bažant's (1971) relation:

$$C_{ijkm}^{(m)} = C_{ijkm} + \frac{1}{4}(2 - m)(S_{ik}\delta_{jm} + S_{jk}\delta_{im} + S_{im}\delta_{jk} + S_{jm}\delta_{ik}) \quad (2)$$

(see also [9], p. 727); $C_{ijkm} =$ tangential moduli associated with Green's Lagrangian strain ($m = 2$), and $S_{ij} =$ current stress (Cauchy stress).

Engesser (1889, 1891) and Haringx (1942) presented different formulae for the first critical load in buckling of columns with significant shear deformations (Fig. 2a,b). They read:

$$P_{cr} = \frac{P_E}{1 + (P_E/GA)} \quad (\text{Engesser}) \quad (3)$$

$$P_{cr} = \frac{GA}{2} \left(\sqrt{1 + \frac{4P_E}{GA}} - 1 \right) \quad (\text{Haringx}) \quad (4)$$

$$\text{where } P_E = (\pi^2/l^2)EI \quad (5)$$

Here E, G = elastic Young's and shear moduli, P_E = Euler's critical load, l = effective buckling length, and EI, GA = bending stiffness and shear stiffness of cross section. The discrepancy between these two formulae, regarded before 1971 as a paradox, was shown [4, 9] to be caused by a dependence of the tangential shear modulus $C_{1212} = G$ on the axial stress $S_{11} = -P/A$, which is different for different choices of the finite strain measure, i.e., for different m . Engesser's formula corresponds to Green's Lagrangian strain tensor ($m = 2$), and Haringx's formula to Lagrangian Almansi strain tensor ($m = -2$), with the shear moduli related according to (2) as

$$G^{(2)} = G^{(-2)} + P/A \quad (6)$$

(a negligible difference in the E -values is ignored). The difference in shear moduli in (6), of course, becomes significant only if the axial stress $S_{11} = -P/A$ is not negligible compared to G . Such a situation arises for the continuum approximation of built-up (lattice) columns or for highly orthotropic fiber composite columns.

For elastic sandwich columns, the motivation of this study, a new paradox has recently been noticed, as a consequence of the numerical and experimental studies of Huang and Kardomateas (2000), Kardomateas (2000, 2001b), Simitse and Shen (2000) and Gjelsvik (1991). Let L = length of sandwich column, l = effective length, and P = axial force. The core has thickness h and shear modulus G . The skins have axial elastic modulus E (Fig. 2a) and thickness t , $t \ll h$ and $E \gg E$ -modulus of the core, and so the entire axial force and bending moment are carried by the skins, while the entire shear force is carried by the core. Therefore, $EI = Ebt(h+t)^2/2 + Ebt^3/6 \approx Ebt^2/2 =$ bending stiffness of the sandwich ($t \ll h$), and $GA = Gbh =$ shear stiffness of the sandwich, b being the cross section width. With these notations,

$$P_{cr} = \frac{P_E}{1 + (P_E/Gbh)} \quad (\text{Engesser type}) \quad (7)$$

$$P_{cr} = \frac{Gbh}{2} \left[\sqrt{1 + \frac{4P_E}{Gbh}} - 1 \right] \quad (\text{Haringx type}) \quad (8)$$

where $G = G_{core}$ = shear modulus of the core, and $P_E = (\pi^2/l^2)Ebt^2/2 =$ Euler load.

In similarity to (6), it may be checked that, if the replacement

$$G_{core} \leftarrow G_{core} - \frac{2t}{h} \sigma_{skins} \quad (9)$$

with $\sigma_{skins} = -P_{cr}/2bt$ is made in the Engesser-type formula (7), the Haringx-type formula (8) results (Bažant 2001). This replacement, however, appears paradoxical; the shear modulus in the *core* should not depend on the axial stress in the *skins*. Furthermore, since the axial stress in the core is negligible compared to the shear modulus of core, it appears paradoxical, in view of (2), that the G -moduli associated with different strain measures need to be used. We thus have a new kind of paradox. The resolution of this paradox, presented in [7, 8], will now be briefly reviewed.

Finite Strain Variational Analysis

In the sandwich beam theory, the skins and the core are constrained by the hypothesis of planar (though non-normal) cross sections. Keeping it in mind, one may adapt the general variational analysis of column buckling, expressing the incremental potential energy of the column accurately up to the second order in displacement gradients [4, 9].

We introduce Cartesian coordinates x_i ($i = 1, 2, 3$); Fig. 2a. The incremental displacements from the initial undeflected configuration of the column carrying axial load P are u_i ; $u_3 = w(x)$ = small lateral deflection, and $u_1 = u(x, y, z)$ = small axial displacement; ψ a small rotation of the cross section (Fig.2c,d). The shear angle $\gamma = \theta - \psi$ (Fig. 2c,d) where $\theta = w' =$ slope of the deflection curve. The second-order incremental potential energy $\delta^2\mathcal{W}$ for small deflections $w(x)$ and small axial displacements $u(x)$ is

$$\begin{aligned} \delta^2\mathcal{W} = & \int_0^L \int_A \left[S^0(y, z)(\epsilon_{11}^{(m)} - e_{11}) + \frac{1}{2}E^{(m)}(y, z)e_{11}^2 + \frac{1}{2}G^{(m)}(y, z)\gamma^2 \right] dA dx \\ & + \int_A \int_0^L \frac{1}{2}E^{(m)}(y, 0)(u_0/L)^2 dA dx \end{aligned} \quad (10)$$

([9], chpt. 11); $y = x_2$ and $z = x_3 =$ coordinates of the cross section whose area is A ; $S^0(y, z) =$ initial axial normal stress; $E^{(m)}(y, z)$, $G^{(m)}(y, z) =$ tangential elastic moduli.

Imposing the conditions of plane cross sections and setting $\alpha = 1 - \frac{1}{2}m$, one can obtain from (10) the following expression:

$$\delta^2\mathcal{W} = \frac{1}{2} \int_0^L \left\{ R^{(m)}\psi'^2 + [H^{(m)} + \frac{1}{4}(2 - m)P] (w' - \psi)^2 - Pw'^2 \right\} dx \quad (11)$$

Here $R^{(m)} = E^{(m)}\frac{1}{2}bth^2 =$ bending stiffness, $H^{(m)} = G^{(m)}bh =$ shear stiffness of the cross section. The necessary condition of stability loss and bifurcation is that the first variation of the second-order work $\delta^2\mathcal{W}$ during any kinematically admissible deflection variations $\delta w(x)$ and $\delta u(x)$ must vanish (Trefftz condition). This condition leads to a system of two ordinary linear homogeneous differential equations for $w(x)$ and $\psi(x)$, with coefficients depending on P . It is found (Bažant 2001) that a non-zero solution exists if and only if:

$$\frac{1}{4}(2 - m)P^2 + [H^{(m)} + \frac{1}{4}(2 + m)P_E^{(m)}]P - H^{(m)}P_E^{(m)} = 0 \quad (12)$$

where $P_E =$ Euler load $= P_E^{(m)} = \pi^2 R^{(m)} / L^2$. This quadratic equation has for $m = 2$ and $m = -2$ the following solutions, which are analogous to Engesser's and Haringx's formulae, respectively;

$$\text{for } m = 2: P_{cr} = \frac{P_E^{(2)}}{1 + (P_E^{(2)}/H^{(2)})} \quad (13)$$

$$\text{for } m = -2: P_{cr} = \frac{H^{(-2)}}{2} \left[\sqrt{1 + \frac{4P_E^{(-2)}}{H^{(-2)}}} - 1 \right] \quad (14)$$

[7, 8]. It has been shown ([4, 9]) that the case $m = 2$ is associated by work with Truesdell's objective stress rate, and the case $m = -2$ with Cotter and Rivlin's (convected) objective stress rate (or Lie derivative of Kirchhoff stress).

Further it is possible to obtain from (12) an infinite number of sandwich buckling formulae, each associated with any chosen value of m . Curiously, however, no investigators has proposed critical load formulae associated with other m values, although many investigators (e.g., Biot 1965; or Biezeno, Hencky, Neuber, Jaumann, Southwell, Oldroyd, Truesdell, Cotter, Rivlin—see [9], chapter 11) introduced formulations for objective stress rates, three-dimensional stability criteria, surface buckling, internal buckling, and incremental differential equations of equilibrium associated with $m = 1, 0$ and -1 .

Paradox Resolution: Definition of Shear Stiffness for Stressed Sandwich

In similarity to (6), one may expect the shear stiffnesses for the Engesser's and Haringx's formulae to be related as $H^{(2)} = H^{(-2)} + Ph/2t$. When this relation is substituted into (13) and the resulting

equation is solved for $P = P_{cr}$, (14) indeed ensues. However, unlike homogeneous columns weak in shear, the foregoing transformation cannot be physically justified on the basis of the general transformation of tangential moduli in (2), nor its special case in (6), because the axial stress S^0 in the core is negligible.

Why should the shear modulus of the core be adjusted according to the axial stress in the skins? This seems to be a paradox. To resolve it, we must examine the definition of the shear stiffness H of a sandwich.

Imagine a homogeneous pure shear deformation of an element Δx of the sandwich column; $u_{1,1} = u_{1,3} = e_{11} = 0$, $u_{3,1} = \gamma$, $e_{13} = e_{31} = \gamma/2$. Based on (10), the second-order incremental potential energy of the element is found to be:

$$\delta^2 \mathcal{W} = \Delta x \int_A \left[-\frac{P}{2bt} \left(\frac{1}{2} u_{k,1} u_{k,1} - \alpha e_{k1} e_{k1} \right) + \frac{1}{2} G_m \gamma^2 \right] dA \quad (15)$$

Upon rearrangements, the incremental potential energy density per unit height of column ($\Delta x = 1$) can be put in the form [7]:

$$\delta^2 \mathcal{W} = bh \Delta x \left(G^{(m)} - \frac{2+m}{4} \frac{P}{bh} \right) \frac{\gamma^2}{2} \quad (16)$$

and in particular, for $m = 2$ (Engesser type) and $m = -2$ (Haringx type),

$$\delta^2 \mathcal{W} = \begin{cases} bh \Delta x & [G^{(2)} - (P/bh)] \gamma^2 / 2 & \text{(Engesser type } G) \\ bh \Delta x & G^{(-2)} \gamma^2 / 2 & \text{(Haringx type } G) \end{cases} \quad (17)$$

Since the foam in an axially loaded sandwich column carries no appreciable axial stresses, we should use that $G^{(m)}$ definition for which the shear stiffness of the core requires no correction for the effect of the axial force P carried by the skins. As we see, that is the latter, Haringx-type, expression (for $m = -2$). In that, and only that, case, the shear modulus $G^{(-2)}$ is equal to that obtained in a pure shear test without normal stress, for example, in the simple torsion test of a thin-wall tube made of the foam. The use of Engesser-type formula is of course equivalent but the shear modulus of the core must be corrected for the effect of the axial forces $F = P/2$ carried by the skins. Contrary to past practice, it is not admissible to use in Engesser's formula the G value measured in a pure shear test.

To facilitate deeper insight, Fig. 3 shows two kinds of shear deformation of a sandwich element ($\Delta x = 1$). In the first kind (Fig. 3a), the shearing of the element is accompanied by a second-order axial extension of the skins, equal to $1 - \cos \gamma \approx \gamma^2/2$ (per unit height). Therefore, one must take into account the work of the initial forces F on this extension, which is $(2F\gamma^2/2)bh$ or $-bhS^0(-\gamma^2/2)$ (per unit height, $\Delta x = 1$). This work must be added to the work of the shear stresses, $(G\gamma^2/2)bh$, in order to obtain the complete second-order work expression.

In the second kind of shear deformation (Fig. 3b), the initial forces F do no work. So, the incremental second-order work expressions for these two kinds of shear deformation, respectively, are as follows:

$$\delta^2 \mathcal{W} = \begin{cases} bh (G^{(2)} + S^0) \gamma^2 / 2 & \text{(case a)} \\ bh G^{(-2)} \gamma^2 / 2 & \text{(case b)} \end{cases} \quad (18)$$

[7, 8]. Now note that these two cases (Fig. 3) give the same incremental second-order work if $G^{(2)} = G^{(-2)} - S^0$ or $G^{(2)} = G^{(-2)} + 2F/bh$. This agrees with (6).

So we may conclude that a constant shear modulus G , equal to the shear modulus obtained in a simple shear test of the foam, can be used only in the Haringx-type formula ($m = -2$).

Differential Equations of Equilibrium Associated with Engesser's and Haringx's Theories

Alternatively, it is possible to derive Engesser's and Haringx's critical load formulae from the differential equations of equilibrium (p.738 in [9]). Fig. 2(c,d) shows two kinds of cross sections of a sandwich

column in a deflected position: (a) the cross section that is normal to the deflected column axis, on which the shear force due to axial load is $Q = Pw'$ and (b) the cross section that was normal to the column axis in the initial undeflected state, on which the shear force due to axial load is $\bar{Q} = P\psi$. For a simply supported (hinged) column, the bending moment is $M = -Pw$ in both cases. The force-deformation relations are $M = Ebt h^2 \psi' / 2$ and Q or $\bar{Q} = Gbh\gamma = Gbh(w' - \psi)$ in case a or b, respectively. Eliminating M, γ, ψ and Q or \bar{Q} , one gets the corresponding two forms of a linear homogeneous differential equation for $w(x)$, of which the first is found to lead to Engesser's formula (3) and the latter to Haringx's formula (4). Thus it is concluded that Engesser's formula ($m = 2$) is obtained when the shear deformation γ is assumed to be caused by the shear force acting on the cross section that is normal to the deflected axis of column, and Haringx's formula ($m = -2$) when γ is assumed to be caused by the shear force acting on the rotated cross section that was normal to the beam axis in the initial state [7, 8].

The foregoing equilibrium derivation, however, does not show that the values of shear stiffness in both formulae must be different. Especially, it does not show that only the shear stiffness in the direction of the rotated cross section can be kept constant.

For further interesting implications for buckling of highly orthotropic fiber composites, built-up lattice columns, layered elastomeric bearings and spiral springs, see [7, 8].

Finite Element Analysis of a Sandwich Column

To support the theoretical considerations reported above, finite element analysis of the critical load of a sandwich column is carried out. In the computation, a sample column of length $L = 10$ m is considered, with skin thickness $t = 0.045$ m, core thickness $h = 0.91$ m and unit width. Different values of the Young modulus E for the skin, varying in the range between $E = 10$ GPa and $E = 105$ GPa, are considered, and the Poisson ratio ν is 0.26 (constant), while the core is characterized by $E = 75$ MPa, $\nu = 0.25$ and $G = 30$ MPa in all calculation. The consideration of very different E values brings to light the effect of the ratio E/G in the computation. The analysis is conducted in two dimensions, using isoparametric 4-node elements, and the skin and the core are treated as Saint Venant-Kirchhoff materials. The analysis obtained with a standard finite element code is shown by the points marked by star signs in Fig. 4. The results are in agreement with the prediction of the critical load given by Engesser's formula because the updating algorithm considers the material moduli tensor as constant with respect to the Lagrangian coordinate. However, if the moduli are updated on the basis of the stress in the skin as shown in (9), the computation shows a perfect agreement with Haringx's formula (see points marked by the plus signs in Fig. 4). This is equivalent to consider an updating algorithm involving a constant Eulerian tangent stiffness tensor and a step-by-step integration based on the Lie derivative of the Kirchhoff stress (which is work-conjugate with the Lagrangian Almansi finite strain tensor referred to the current state, $m = -2$). The spatial tangent stiffness tensor represents the push-forward to the current configuration through the deformation gradient of the 4th order Lagrangian tangent stiffness tensor. Therefore, the tangent constitutive model (in the sense of a consistent algorithmic tangent for a finite step) for the updating can be written as:

$$L_v(\boldsymbol{\tau}) = \mathbf{c} : \mathbf{d} \quad (19)$$

where $L_v(\bullet)$ is the Lie derivative, $\boldsymbol{\tau}$ is the Kirchhoff stress, \mathbf{c} is the Eulerian tangent stiffness tensor and \mathbf{d} is the rate of deformation tensor.

The critical loads given by Haringx's formula, Engesser's formula and Euler's formula are computed considering the exact stiffness for a skin of finite thickness and not the approximation for $t \ll h$ considered above. This last point is particularly important and attention must be paid to verifying the applicability of the approximation for the problem considered.

Acknowledgment:

Financial support under Grant ONR-N00014-91-J-1109 from the Office of Naval Research to Northwestern University (monitored by Dr. Yapa D.S. Rajapakse and directed by Bazant) is gratefully acknowledged.

References

- [1] Allen, H.G. (1969). *Analysis and Design of Sandwich Panels*. Pergamon Press, Oxford.
- [2] Attard, M.A. (1992). Draft of a manuscript on "Finite strain beam theory", University of New South Wales, Australia (private communication to Bažant).
- [3] Bažant, Z.P. (1968). "Conditions of deformation instability of a continuum and their application to thick slabs and a half space" (in Czech, with English summary), *Stavebnický Časopis (SAV, Bratislava)*, 16, 48–64.
- [4] Bažant, Z.P. (1971). "A correlation study of incremental deformations and stability of continuous bodies." *Journal of Applied Mechanics*, Trans. ASME, 38, 919–928.
- [5] Bažant, Z.P. (1992). Discussion of "Stability of built-up columns" by A. Gjelsvik, *ASCE J. of Engineering Mechanics*, 118(6), 1279–1281.
- [6] Bažant, Z.P. (1993). Discussion of "Use of engineering strain and Trefftz theory in buckling of columns" by C.M. Wang and W.A.M. Alwis, *ASCE J. of Engineering Mechanics*, 119(12), 2536–2537.
- [7] Bažant, Z.P. (2001). "Shear Buckling of Sandwich, Fiber-Composite and Lattice Columns, Bearings and Helical Springs: Paradox Resolved." Theoretical and Applied Mechanics Report No. 01-12/C402s, McCormick School of Engineering and Applied Science, Northwestern University, Evanston.
- [8] Bažant, Z.P. (2003). "Shear buckling of sandwich, fiber-composite and lattice columns, bearings and helical springs: paradox resolved." *J. of Appl. Mech. ASME* 70 (Jan.), 75–83.
- [9] Bažant, Z.P., and Cedolin, L. (1991). *Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories*, Oxford University Press, New York; and republication with updates, Dover Publ., New York 2003.
- [10] Biot, M.A. (1965). *Mechanics of Incremental Deformations*. J. Wiley & Sons, New York.
- [11] Buckle, I., Nagarajaiah, S., and Ferrell, K. (2002). "Stability of Elastomeric Isolation Bearings: Experimental Study." *J. of Structural Engineering ASCE* 128 (1), 3–11.
- [12] Chong, K.P., Wang, K.A., and Griffith, G.R. (1979). "Analysis of continuous sandwich panels in building systems." *Building and Environment* 44.
- [13] Engesser, F. (1889a). "Die Knickfestigkeit gerader Stäbe." *Zentralblatt des Bauverwaltung*. 11, 483.
- [14] Engesser, F. (1889b). "Die Knickfestigkeit gerader Stäbe." *Z. Architekten und Ing. Verein zu Hannover* 35, p. 455.
- [15] Engesser, F. (1891). "Die Knickfestigkeit gerader Stäbe." *Zentralblatt der Bauverwaltung* 11, 483–486.
- [16] Frostig, Y., and Baruch, M. (1993). "Buckling of simply supported sandwich beams with transversely flexible core—a high order theory." *ASME J. of Eng. Mech.* 119 (5), 955–972.
- [17] Goodier, J.N. and Hsu, C.S. (1954). "Nonsinusoidal buckling modes of sandwich plates." *J. of Aeronautical Sciences*, 525–532.
- [18] Gjelsvik, A. (1991). "Stability of built-up columns." *ASCE J. of Engrg. Mechanics* 117 (6), 1331–1345.
- [19] Haringx, J.A. (1942). "On the buckling and lateral rigidity of helical springs." *Proc., Konink. Ned. Akad. Wetenschap.* 45, p. 533
- [20] Haringx, J.A. *Phillips Research Reports*, Vol.3 (1948) and Vol. 4 (1949), Phillips Research Laboratories, Eindhoven.
- [21] Huang, H., and Kardomateas, G.A. (2000). "Buckling and initial postbuckling behavior of sandwich beams including transverse shear." *AIAA Journal*—submitted too.
- [22] Kardomateas, G.A. (2000). *ASME International Congress*, Orlando 2000.
- [23] Kardomateas, G.A. (2001a). "Elasticity solutions for sandwich orthotropic cylindrical shell under external pressure, internal pressure and axial force." *AIAA Journal* 39 (4), 713–719.
- [24] Kardomateas, G.A. (2001b). "Three-dimensional elasticity solutions for the buckling of sandwich columns." AMD-TOC, *ASME Intern. Mechanical Engrg. Congress* (held in New York), 1–6.
- [25] Kovařík, V., and Šlapák, P. (1973). *Stability and Vibrations of Sandwich Plates* (in Czech). Academia, Prague.
- [26] Michiharu, O. (1976). "Antisymmetric and symmetric buckling of sandwich columns under compressive loads." *Trans., Jap. Soc. of Aeronautical and Space Science* 19, 163–178.

- [27] Plantema, F.J. (1966). *Sandwich Construction: The Bending and Buckling of Sandwich Beams, Plates and Shells*. J. Wiley & Sons, New York.
- [28] Reissner, E. (1972). "On one-dimensional finite-strain beam theory: The plane problem." *J. of Applied Mathematics and Physics* 23, 795–804.
- [29] Reissner, E. (1982). "Some remarks on the problem of column buckling." *Ingenieur-Archiv* 52, 115–119.
- [30] Simitses, G.J., and Shen, L. (2000). "Static and dynamic buckling of sandwich columns." *Mechanics of Sandwich Structures*, AD-Vol. 62 / AMD-Vol. 245 (Y.D.S. Rajapakse et al., ed.), Am. Soc. of Mech. Engrs., New York, pp. 41–50 (presented at ASME Orlando).
- [31] Simo, J.C., Hjelmstad, K.D., and Taylor, R.L. (1984). "Numerical formulation of elasto-viscoplastic response of beams accounting for the effect of shear." *Computer Methods in Applied Mechanics and Engineering*. 42, 301–330.
- [32] Simo, J.C., and Kelly, J.M. (1984). "The analysis of multilayer elastomeric bedarings." *ASME J. of Applied Mechanics*. 51, 256–262.
- [33] Timoshenko, S.P., and Gere, J.M. (1961). *Theory of elastic stability*. McGraw–Hill Co., New York (pages 135, 142).
- [34] Wang, C.M., and Alwis, W.A.M. (1992) "Use of engineering strain and Trefftz theory in buckling of columns". *ASCE J. of Engineering Mechanics* 118 (10).
- [35] Ziegler, F. (1982). "Arguments for and against Engesser's buckling formulas." *Ingenieur-Archiv* 52, 105–113.

List of Figures

1	Difference between Engesser's and Haringx's formulas for a sandwich with $E_{skin}/G_{core} \approx 1400$	9
2	Sandwich column in (a) initial state and (b) deflected state; (c,d) cross section rotation, shear angle and shear force due to axial load.	9
3	Shear deformation of an element of sandwich column under initial axial forces $F = P/2$; (a) with second-order axial extension $\gamma^2/2$, and (b) at no axial extension.	10
4	Comparison of the FEM analysis with the Engesser's and Haring's predictions.	10

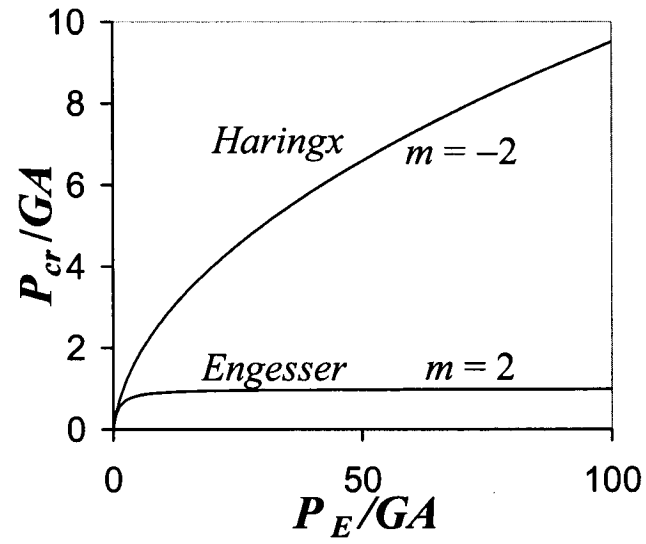


Figure 1: Difference between Engesser's and Haringx's formulas for a sandwich with $E_{skin}/G_{core} \approx 1400$.

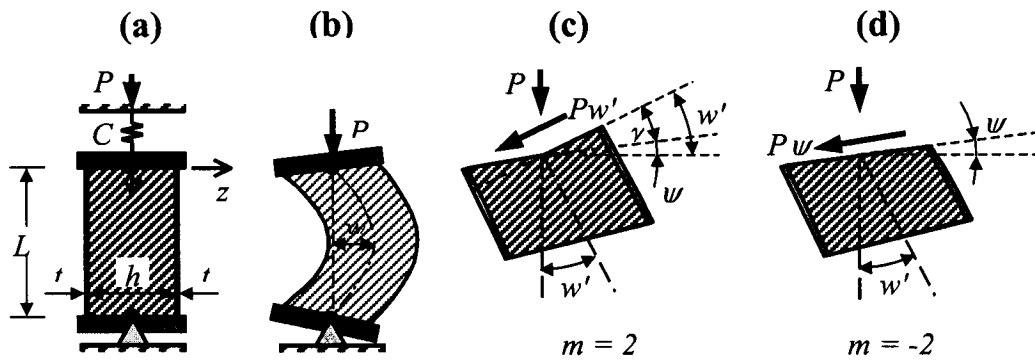


Figure 2: Sandwich column in (a) initial state and (b) deflected state; (c,d) cross section rotation, shear angle and shear force due to axial load.

