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Role of deterministic and statistical length scales in size effect for quasibrittle failure at crack initiation

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ABSTRACT: An improved generalized law for combined energetic-probabilistic size effect on the nominal strength for structures failing by crack initiation from a smooth surface is used for practical purposes – the paper proposes a procedure to capture both deterministic and statistical size effects on the nominal strength of quasibrittle structures failing at crack initiation. The advantage of the proposed approach is that the necessity of time consuming statistical simulation is avoided, only deterministic nonlinear fracture mechanics FEM calculation must be performed. Results of deterministic nonlinear FEM calculation should follow deterministic-energetic formula, a superimposition with the Weibull size effect, which dominates for large sizes using the energetic-statistical formula, is possible. As the procedure does not require a numerical simulation of Monte Carlo type and uses only the results obtained by deterministic computation using any commercial FEM code (which can capture satisfactorily deterministic size effect), it can be a simple practical engineering tool.

1 INTRODUCTION AND SIZE EFFECT FORMULAE

Practical and simple approach to incorporate the statistical size effect into the design or the assessment of very large unreinforced concrete structures (such as arch dams, foundations and earth retaining structures, where the statistical size effect plays a significant role) is important. Failure load prediction can be done without simulation of Monte Carlo type utilizing the energetic-statistical size effect formula in mean sense together with deterministic results of FEM nonlinear fracture mechanics codes.

This work is based on the latest achievements of Bažant, Vořechovský and Novák (2005) who proposes a new improved law with two scaling lengths (deterministic and statistical) for combined energetic-probabilistic size effect on the nominal strength for structures failing by crack initiation from smooth surface. The role of these two lengths in the transition from energetic to statistical size effect of Weibull type is clarified. Relations to the recently developed deterministic-energetic and energetic-statistical formulas are presented. The paper by Bažant, Vořechovský and Novák (2005) also clarifies the role and interplay of two material lengths: deterministic and statistical.

The deterministic energetic size effect formula for crack initiation from smooth surface reads (e.g. Bažant and Planas 1998, Bažant and Novák 200ab, Bažant 2002):

$$\sigma_N(D) = f_r^{\infty}\left[1+\frac{rD_b}{D+l_p}\right]^{1/r} \tag{1}$$

where σ_N is the nominal strength depending on the structural size D. Parameters f_r^{∞}, D_b and r are positive constants representing the unknown empirical parameters to be determined. Parameter f_r^{∞} represents solution of the elastic-brittle strength which is reached as a nominal strength for very large structural sizes. The exponent r (a constant) controls the curvature and the slope of the law. The exponent offers a degree of freedom while having no effect on the expansion in derivation of the law (Bažant and Planas 1998, Bažant 2002). Parameter D_b has the meaning of the thickness of cracked layer. Variation of the parameter D_b moves the whole curve left or right; it represents the deterministic scaling parameter and is in principle related to grain size and drives the transition from elastic brittle (D_b=0) to quasibrittle (D_b >0) behavior.

By considering the fact that extremely small structures (smaller than D_b) must exhibit the plastic limit, a parameter l_p is introduced to control this convergence. The formula (1) represents the full size range transition from perfectly plastic behavior (when $D\to 0; D \ll l_p$) to elastic brittle behavior ($D\to\infty; D \gg D_b$) through quasibrittle behavior. Parameter l_p governs the transition to plasticity for small sizes D (the crack band model or the averaging in nonlocal models lead to a horizontal asymptote). The case of $l_p \neq 0$ shows the plastic limit for vanishing size D, which is the behavior predicted by the cohesive crack model as well as the assumption of perfectly plastic material in the crack both. For large sizes, the influence of l_p decays fast with D and therefore all the cases of $l_p \neq 0$ are asymptotically equivalent to the case $l_p = 0$ for large D.

The large-size asymptote of the deterministic energetic size effect formula (1) is horizontal: $\sigma_N(D)/f_r^{\infty}=1$, see fig. 1a). But this is not in agreement with the results of nonlocal Weibull theory as applied to modulus of rupture (Bažant and Novák 2000b), in which the large-size asymptote in the logarithmic plot has the slope $-n/m$ corresponding to the power law of the classical Weibull statistical theory (Weibull, 1939). In view of this fact, there is a need to superpose the energetic and statistical theories. Such superimposition is important, for example, for analyzing the size effect in vertical bending fracture of arch dams, foundation plinths or retaining walls.

A formula in which the statistical part of size effect is superposed on the energetic part was derived by Bažant and Novák (2000). Further it was generalized to satisfy the requirement for a horizontal asymptote for vanishing D (Bažant 2003, 2004). The statistical characteristic length l_s needed for this generalization was simply assumed to be equal to D_b, but the study of an analogous problem for glass fibers by Vořechovský and Chudoba (2005) indicated that l_s may differ from D_b and a mathematical derivation of a realistic l_s value was given by Bažant, Vorechovský and Novák (2005). It thus transpired that the statistical part of size effect in structures with stationary strength random field has a large-size asymptote in the classical Weibull form (straight line of slope $-n/m$ in a double-log plot), while the left (small size) asymptote is horizontal. The value of the horizontal asymptote for $D \to 0$ is the mean strength of the random field, and in Weibull understanding it is the mean strength measured for the reference length being equal to the autocorrelation length l_s, see Vořechovský and Chudoba (2005). So, by introduction of the random strength field, we introduce the length scale given by l_s. Upon incorporating this result (i.e., the statistical part) into formula (1) we get the final law (Bažant, Vořechovský and Novák, 2005):

$$\sigma_N = f_r^{\infty}\left[\left(\frac{L_0}{L_0 + D}\right)^{rn/m} + \frac{rD_b}{l_p + D}\right]^{1/r} \qquad (2)$$

This formula (which coincides with the law derived by Bažant, 2004, except for the value of L_0) exhibits the following features:

- The small-size left asymptotic is the deterministic plastic response (with parameter l_p controlling the transition to small sizes).
- Large-size asymptote is the Weibull power law (i.e., the statistical size effect, which is a straight line of slope $-n/m$ in the double-logarithmic plot of nominal strength versus size)
- Two scaling lengths: deterministic (D_b) and statistical (L_0) are present. The mean size effect is partitioned into deterministic and statistical parts, each of which has its own length scale. Parameter D_b controls the transition from elastic-brittle to quasibrittle failure, and L_0 controls the transitional zone from constant nominal strength to local Weibull via the random strength field. Note that the autocorrelation length l_s has direct connection to our statistical length L_0. This correspondence is explained in detail for glass fibers in Vořechovský and Chudoba (2005) and for concrete in Bažant, Vorechovský and Novák (2005).

The summations in the denominators of (2) prevent both the statistical and deterministic parts from growing to infinity for small D. Although this is not important for practice, because the small-size plastic behavior is reached only for theoretical sizes smaller than feasible (e.g., smaller than one aggregate size), it is important theoretically in order to allow the use of asymptotic matching.

Note that, for $m \to \infty$, Eq. (2) degenerates to the deterministic formula (1). The same applies if $L_0 \to \infty$. The interplay of these two scaling lengths using the ratio L_0/D_b is demonstrated by Bažant, Vořechovský and Novák (2005). The question arises at to what is the meaning of the ratio L_0/D_b? Since scaling lengths are in concrete controlled mainly by the grain sizes, we expect $L_0 \approx D_b$, and so the simpler law with $L_0 = D_b$ should be an excellent performer in most practical cases.

2 SUPERIMPOSITION OF FEM DETERMINISTIC-ENERGETIC AND STATISTICAL SIZE EFFECTS

As already mentioned, deterministic modeling with NLFEM can capture only deterministic size effect. Therefore, a procedure for superposing the statistical size effect is needed. Such a procedure can be formulated as follows:

1) Suppose that the modeled structure has characteristic dimension D_t. The natural first step is to create a finite element computational model for this real size, as realistic as possible (in terms of meshing, boundary conditions, material etc.). With this code, one predicts the nominal strength of the structure (based on the failure load, corresponding to the peak load of load-deflection diagram) for size D_t. Because the statistical part of size effect is neglected, the

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strength is, in this (first) step, overestimated. The larger the structure, the greater the overestimation. The result is a point in the size effect plot, the solid circle in Fig. 1(a).

2) Then we scale the geometry of computational model up and down, in order to obtain a set of similar structures with characteristic sizes D_i, i=1,..., N. Based on numerical experience, at least 4 sizes but better about 10, in order to cover the transitional range of size effect fully, spanning from very small to very large sizes. Then we calculate the nominal strength for each size, σ_i , i=1,..., N. Note that for two very large sizes the nominal strength values reaching into the horizontal asymptote should be almost identical (if not, crack initiation is not the failure mechanism and other phenomena, such as stress redistribution, play a significant role and the present procedure cannot be applied). The computational model must of course be mesh-objective in presence of softening (as, e.g., the crack band model or nonlocal continuum damage model). To ensure that the phenomenon of stress redistribution (causing a deterministic size effect) would be correctly captured, well tested models are recommended for strength prediction. Special attention should be paid to the selection of the constitutive law and localization limiter. The result of this step is a set of points (circles) in the size effect plot, as shown in Fig. 1(a).

3) The next step is to fit optimally the deterministic-energetic formula (1) to the set of N pairs $\{D_i, \sigma_i\}$ i=1,..., t, ..., N. The result of this step is a set of values of four parameters: f_r^∞, D_b, r and l_p. The parameter l_p can be excluded from fitting because it can be identified a priori by plastic analysis (this is fully described by Bažant, Vořechovský and Novák, 2005). Parameter f_r^∞ can also be excluded because this limit can be estimated from nonlinear FEM analysis as the value to which the nominal strength converges with increasing size. So, for very large sizes, we can prescribe $\sigma_N / f_r^\infty=1$ as the asymptotic limit. The result of this step is illustrated by the curve fitted to the set of points in Fig. 1(a).

4) There are three remaining parameters which are needed for the statistical-energetic formula (2): n,m and L_0. Parameter n is the number of spatial dimensions of scaling (n=1,2 or 3). Parameter m represents the Weibull modulus of FPZ, characterizing the Weibull distribution of random strength. A recent study of Bažant and Novák (2000a) revealed that, for concrete and mortar, the asymptotic value of Weibull modulus $m\approx24$ rather than 12, the value widely accepted before. The ratio n/m therefore represents the slope of MSEC in the size effect plot for $D\to\infty$. This means that the nominal strength decreases, for two-dimensional (2D) similarity (n=2) and extreme sizes, as the -1/12 power of the structure size. Note, that for different materials the asymptotic value of Weibull modulus is different. The results of these 4 steps are shown for illustration in Fig 1(a).

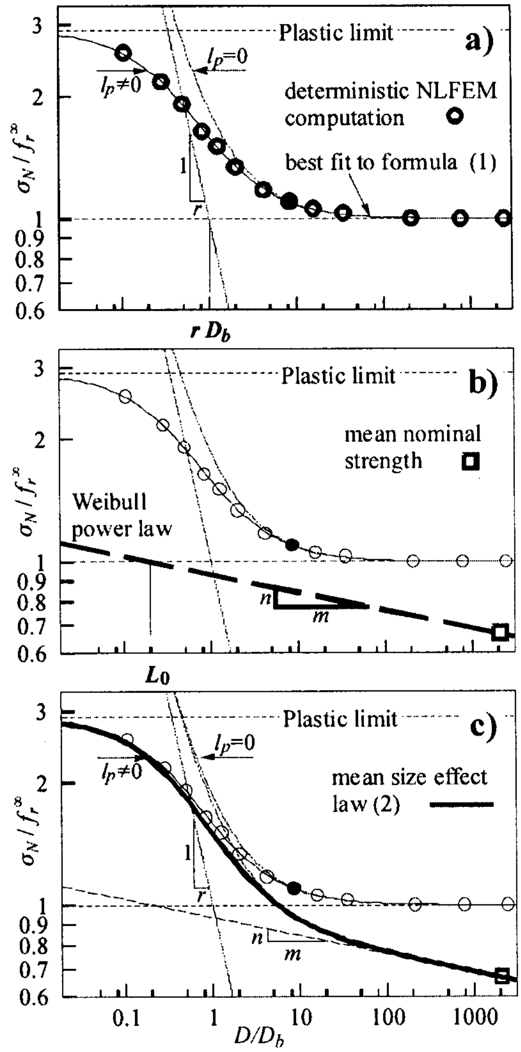

Figure 1: Illustration of the superimposition steps. a) Steps 1-4 resulting in deterministic fit; b) step 5 – determination of parameter L_0; c) final formula and nominal strength prediction for the real structural size

Parameter L_0 is now the only remaining parameter to be determined. As it characterizes the statistical length scale, it might seem that one would need to incorporate statistical software into a nonlinear finite element code. But there is a much simpler alternative, based on evaluating the local Weibull integral over the elastic stress field of the structure. The estimation statistical length scale l_s is the first step. A good estimate is $l_s \approx D_b$. Since a choice about a scatter of FPZ strength must be made (Weibull modulus controlling the power of size effect for large sizes), one can compute the elastic field for a large-size structure having, and from the Weibull integral obtain the corresponding mean nominal strength. Once the point of mean strength of such large structure (a point in the size effect plot with coordinates D_{stat}, σ_{stat}) is known, one can pass through that point a straight line of slope $-n/m$, representing the Weibull asymptote. Graphically, the intersection of the statistical (Weibull) asymptote with the deterministic strength f_r^∞ for infinite structure size (a horizontal asymptote) gives the statistical scaling length on the D-axis; Fig.1(b). The numerical solution to L_0 can be written as:

$$L_0 = D_{stat}\left(\frac{\sigma_{stat}}{f_r^\infty}\right)^{n/m} \tag{3}$$

and, therefore, this parameter does not need to be fitted, rather an analytical expression can be used. Note that the large size strength (mean strength σ_{stat}) can be computed by Weibull integral, where the choice of reference volume V_0 and Weibull modulus (scatter) must be made (this is described, in detail, e.g. by Bažant and Planas, 1998):

$$P_f = 1-\exp\left(-\int_V \left\langle \frac{\sigma(x)}{s_0}\right\rangle^m \frac{\mathrm{d}V(x)}{V_0}\right) \tag{4}$$

where V is the volume (area, length) of the structure depending on its dimension (n); s_0 is the Weibull scaling parameter and V_0 is an elementary volume of the material for which the Weibull distribution has parameters m and s_0. The function σ_x is the maximum principal stress at a point of coordinate vector x. One can avoid the computation of nonlocal integral (and determination of the load leading to P_f, which corresponds to the mean load) by means of numerical simulation of Monte Carlo type. In such case, it is recommended to use directly the stability postulate of extreme values for discretization of random blocks and their association with scaled PDF. This approach has been used in the numerical example and is described in detail by Novák, Bažant and Vořechovský (2003).

5) Once all the parameters of the statistical-energetic formula have been determined, the nominal strength can be calculated for any size. Using the real size D_t of the structure, one can predict the corresponding nominal strength $\sigma_{N,t}$ using Eq. (2). This prediction will be generally different (and lower) than the initial deterministic prediction; Fig. 1(c). The larger structure, the larger the difference is. The formula will provide us the strength prediction for the mean strength. Additionally, a scatter of strength can be determined just by using the fundamental assumption of Weibull distribution. For the distribution we know two parameters; the shape parameter, m, is prescribed initially, and the scale parameters, s, can be calculated easily from the predicted mean and the Weibull modulus (shape parameter).

3 SUMMARY AND CONCLUSIONS

The paper presents an analytical formula for the nominal mean strength prediction in crack initiation problems. The paper suggests a practical procedure for superposition of deterministic and statistical size effects at crack initiation. It requires only a few finite element analyses using scaled structure sizes and a simple evaluation of Weibull statistics from the elastic field in a large size structure. The prediction can be carried out without any Monte Carlo simulation, with can be quire tedious.

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