Applications of Statistics and Probability in Civil Engineering – Faber, Köhler & Nishijima (eds)
© 2011 Taylor & Francis Group, London, ISBN 978-0-415-66986-3

Size effect on probability distribution of fatigue lifetime of quasibrittle structures

Jia-Liang Le
Department of Civil Engineering, University of Minnesota, Minneapolis, USA

Zdeněk P. Bažant
Department of Civil and Environmental Engineering, Northwestern University, Evanston, USA

ABSTRACT: The design of various engineering structures, such as buildings, infrastructure, aircraft, ships, as well as microelectronic components and medical implants, must ensure an extremely low probability of failure (such as 10^{-6}) during their service lifetime. Such a low probability is beyond the means of histogram testing. Therefore, one must rely on some physically based probabilistic model for the statistics of structural lifetime. This study focuses on the structures consisting of quasibrittle materials, which are brittle materials with in homogeneities that are not negligible compared to structure size (exemplified by concrete, fiber composites, tough ceramics, rocks, sea ice, bone, wood, and many more at micro- or nano-scale). This paper presents a new theory of the lifetime distribution of quasibrittle structures failing at the initiation of a macro crack from one representative volume element of material under cyclic fatigue. The formulation of this theory begins with the derivation of the probability distribution of critical stress amplitude by assuming that the number of cycles and the stress ratio are prescribed. The Paris law is then used to relate the probability distribution of critical stress amplitude to the probability distribution of fatigue lifetime. The theory naturally yields a power-law relation for the stress-life curve (S-N curve), which agrees well with Basquin's law. The theory indicates that quasi-brittle structures must exhibit a marked size effect on the mean structural lifetime under cyclic fatigue and consequently a strong size effect on the S-N curve. It is shown that the theory matches the experimentally ob-served systematic deviations of lifetime histograms of various engineering and dental ceramics from the Weibull distribution.

1 INTRODUCTION

For many structures, such as aircraft, ships, bridges and biomedical implants, the fatigue lifetime is an important aspect of design. However, when a long life-time is required, it is next to impossible to obtain the lifetime histogram purely experimentally, by waiting until the structure or material specimen fails. Therefore, one must rely on a realistic theory of failure probability that can be calibrated and verified indirectly through its predictions other than the histograms of fatigue lifetime. The same applies to the strength limit for failure probability 10^{-6}.

This study is focused on structures consisting of quasibrittle materials, which are heterogenous materials with brittle constituents and material inhomogeneities that are not negligible compared to structure size or cross section dimension and, consequently, develop a non- negligible fracture process zone (FPZ). They are exemplified by concrete as the archetypical case, rocks, coarse-grained and toughened ceramics, dental ceramics, fiber composites, fiber—reinforced concretes, rocks, masonry, mortar, stiff cohesive soils, grouted soils, rigid foams, sea ice, consolidated snow, wood, paper, carton and bone, as well as many high-tech, bio- and bio-inspired materials and most materials on the micro- and nano scales.

The non-negligible size of FPZ inevitably causes size dependent failure behavior. The smallest possible structures fail in a quasi-plastic manner and very large ones in a brittle manner (Bažant 2004). Previous studies (Bažant 2004, Bažant & Pang 2006, Bažant & Pang 2007, Bažant, Le, & Bažant 2009, Le, Bažant, & Bazant 2009) showed that, due to this size dependence, the type of cumulative distribution function (cdf) of monotonic strength of quasibrittle structures, as well as their static (or creep) lifetime, varies with the size of structure, and also its geometry. It is thus logical to expect the probability distribution of fatigue lifetime to be size dependent. To demonstrate it and develop the appropriate theory is the objective of this paper.

Attention will here be limited to a broad class of structures of the so-called positive geometry. They are those that fail (under controlled load) right at the initiation of a macrocrack from a damaged representative volume of material (RVE), which occurs when the derivative of the stress intensity factor with respect to crack length is initially positive. This class of structures is statistically equivalent to a chain of RVEs, where the RVE is defined as the smallest material volume whose failure triggers the failure of entire structure.

For very large structures, for which the RVE size, n_s, is negligible compared to the structure size, the failure is perfectly brittle. Since the number, n_s, of RVEs in the chain can be considered as infinite, the probability distribution of fatigue lifetime must then be the two-parameter Weibull distribution (Weibull 1939). The reason is that the left tail is a power-law, as justified by recent theoretical arguments based on the activation energy of bond breakage (Bažant & Pang 2006, Bažant & Pang 2007, Bažant, Le, & Bazant 2009, Le & Bažant 2010b). The defining characteristic of quasibrittle structures is that the FPZ is not small enough, or n_s is not large enough, to make the Weibull distribution applicable, as shown in the previous studies of statistics of monotonic strength and creep lifetime (Bažant, Le & Bazant 2009, Le, Bažant, & Bazant 2009).

This paper will present a derivation of the probability distribution of fatigue strength, defined as the critical stress amplitude for a given number of cycles and a given minimum-to-maximum stress ratio. The probability distribution of fatigue lifetime will then be deduced from the cdf of fatigue strength and the law of fatigue crack growth.

2 STATISTICS OF FATIGUE STRENGTH ON THE NANOSCALE

A simple and clear physical basis for the probability of fracture growth exists only on the atomic scale. The jumps of the front of an interatomic crack represent a quasi-steady process because, even at the rate of impact, the interatomic bonds break at roughly the rate of one per 105 thermal atomic vibrations. Consequently, on the atomic scale, the crack jump probability must be the same as the crack jump frequency. So, we begin by analyzing a nanoscale element.

Here we consider the structure to be subjected to a cyclic load, which can be characterized by two quantities: the stress amplitude $\Delta\sigma = \sigma_{max} - \sigma_{min}$ and the stress ratio $R = \sigma_{min}/\sigma_{max}$. The corresponding stress history for a nanoscale element is hard to determine, especially for the first few cycles during which the residual stress field builds up rapidly. However, when focused on the high cycle fatigue, the first few cycles are not of particular interest. After only a few cycles, the stress profile for the nanoscale element stabilizes. The stress amplitude on the nanoscale $\Delta\tau = \tau_{max} - \tau_{min}$ and the nanoscale stress ratio $R_\tau = \tau_{min} - \tau_{max}$ can thus be related to the stress amplitude $\Delta\sigma$ and the stress ratio R on the macroscale: $\Delta\tau = c_1\Delta\sigma$ and $R_\tau = c_2 R$. Parameters c_1 and c_1 are empirical but could conceivably be determined through a detailed micro-mechanical analysis of the build-up of residual stresses.

The frequency of breakage of particle bonds in a disordered nano-element, or of atomic bonds in an atomic lattice block, can be determined from Kramers' formula (Risken 1989) for the first-passage time in the transition between two states (before and after the bond breakage):

$$f_1 \approx \mu_T e^{-Q_0/kT} V_a \tau^2 / E_1 kT \tag{1}$$

where Q_0 = the dominant activation energy barrier on the free energy potential surface, k = Boltzmann constant, T = absolute temperature, $\mu_T = kT/h$, $h = 6.626 \times 10^{-34}$ Js = Planck constant = (energy of a photon)/(frequency of its electromagnetic wave), V_a = activation volume, and E_1 = elastic modulus of nano-structure.

Assuming that each crack jump is an independent process, the frequency of reaching the critical crack length at which the nano-element fails is the sum of the net frequencies of forward jumps over all these barriers. For the cyclic stress at the nanoscale, $\tau = \tau(t)$, it may be assumed that the energy bias due to applied stress depends only on the current stress, but not on the stress history (Krausz & Krausz 1988). Therefore, for a given number of cycles N_0, the frequency of occurrence of a failure event is given by:

$$f_a \propto \int_0^{N_0 t_c} \frac{\tau^2(t)}{E_1 kT} \mathrm{d}t \tag{2}$$

$$\propto f(R_\tau)\Delta\tau^2 \tag{3}$$

where function $f(R_\tau)$ depends on the stress history. Since a quasi-steady state can be realistically assumed, the failure probability is proportional to the frequency of failure events. Therefore, the failure probability of the nano-element is:

$$P_f \propto f(R_\tau)\Delta\tau^2 = f(R_\tau)(c_1\Delta\sigma)^2 \tag{4}$$

Eq. 4 shows that the distribution of fatigue strength of a nano-element follows a power law with zero threshold.

3 MULTISCALE TRANSITION OF STATISTICS OF FATIGUE STRENGTH

To relate the probability distributions of fatigue strength at nano- and macro-scales, a certain approximate statistical multiscale transition framework is required. In the previous studies (Bažant & Pang 2006, Bažant & Pang 2007, the multiscale transition of strength distribution is represented by a hierarchical model (Fig. 4e of Bažant & Pang (2007)), which consists of series couplings (the chain model) and parallel couplings (the fiber bundle model). Physically, the parallel coupling represents the load re-distribution mechanisms at different scales as well as the condition of compatibility between one scale and its sub-scale. The series model represents (in the sense of the weakest-link model) the localization of damage at each scale.

3.1 *Chain model*

Consider a chain of elements (or links) subjected to cyclic loading with a prescribed number of cycles and stress ratio. The fatigue strength $\Delta\sigma_c$ of the chain, i.e., the critical stress amplitude that leads to failure, is determined by the smallest fatigue strength of all the elements. Since the chain survives if and only if all its elements survive, one can calculate the survival probability of the chain, $1-P_f$, from the joint probability theorem. Assuming that the random fatigue strengths of the elements are statistically independent, we can write the failure probability of the chain with n_c elements as follows:

$$P_{f,chain}(\Delta\sigma_c) = 1 - \Pi_{i=1}^{n_c}[1 - P_i(\Delta\sigma_c)] \quad (5)$$

Based on this equation and by the same method as used by Bažant & Pang (2007) for static loads, it is easy to prove for cyclic loading two essential asymptotic properties of the chain model:

1. If the cdf's of fatigue strengths of all the elements have a power-law tail of exponent p, then the cdf of fatigue strength of the whole chain has also a power-law tail and its exponent is also p; and
2. when n_c is large enough, the cdf of fatigue strength of the chain approaches the Weibull distribution: $P_f = 1 - e^{-n_c(\Delta\sigma_c/s_0)^p}$, where s_0 is a scaling constant.

3.2 *Bundle model*

The bundle model consists parallel elements (often called fibers) spanning two rigid plates. After one element fails, the load will be redistributed among the surviving elements. When a certain portion of the elements fails, the bundle reaches the maximum load F (and fails if the load is controlled). It fails totally if and only if all the elements fail ($F \rightarrow 0$). The failure statistics of the bundle has been extensively investigated for the static strength (e.g. (Daniels 1945, Phoenix 1978, Smith 1982, Bažant & Pang 2007, Le & Bažant 2010b)).

In this study, we are interested in the cdf of the fatigue strength $\Delta\sigma_b$ of the bundle for a prescribed stress ratio R and a given number of cycles N_0. We will analyze some asymptotic properties of this cdf by considering a bundle with two elements having random strength and the same cross section, although the generalization to any number of elements in the bundle is straightforward.

Consider a bundle under cyclic loading with a prescribed stress ratio R. For a given stress ratio R and any number of cycles, the fatigue strength $\Delta\sigma_i$ (i = 1,2) of the elements is assumed to be known. The elements are numbered so that $\Delta\sigma_1 < \sigma_2$. Fig. 1a-c shows the loading histories of both the bundle and its two elements. The bundle reaches its strength limit and fails at the N_0th cycle. After the first N_1 cycles, the first element fails and the second element carries the entire load for the rest of N_0–N_1 cycles.

The first element is subjected to a cyclic load with stress amplitude $\Delta\sigma_b$ and stress ratio R. Under cyclic load, some subcritical crack inside the element grows from its original length a_0 to a critical length a_c at which the first element fails. The growth rate of the subcritical crack can be described by the Paris law (Paris & Erdogan 1963):

$$\frac{da}{dN} = Ae^{Q_0/kT}\Delta K^n \quad (6)$$

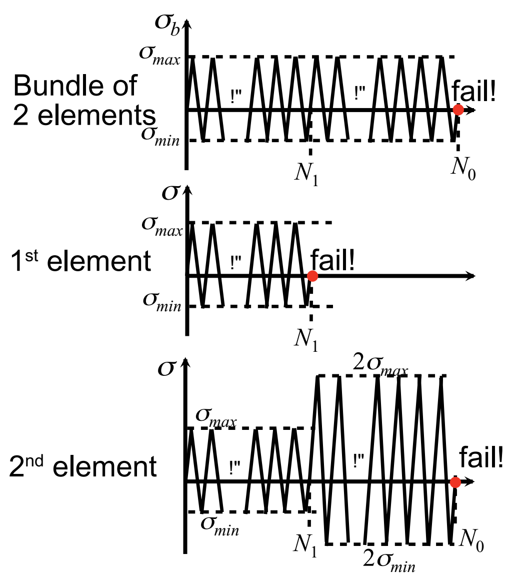

Figure 1. Loading histories of bundle and its elements.

where ΔK = amplitude of the stress intensity factor. A recent study (Le & Bažant 2010a) showed that the Paris law can be physically justified by considering that the macro-scale energy dissipation of the fracture process zone is equal to the energy dissipation of all the nano-cracks within that FPZ. The exponent of Paris law increases from 2 at the nanoscale, to some larger number at the macroscale.

By separating the variables and integrating the Paris law from the original crack length to the length a_c, one obtains:

$$\Delta\sigma_b^{n_e} N_1 = e^{Q_0/kT} I_1 \tag{7}$$

where n_e = exponent of the Paris law for one element, $I_1 = A^{-1} l_{e1}^{1-n_e/2} \int_{\alpha_0}^{a_c} k_{e1}^{-n_e}(\alpha) \mathrm{d}\alpha$, l_{e1} = characteristic size of the first element, $\alpha = a/l_e$ = relative crack length and k_{e1} = dimensionless stress intensity factor of the first element. It is clear that, for a particular element, I_1 must be a constant for different cyclic loads as long as the stress ratio is kept constant. Therefore, one can easily obtain the critical number of cycles N_1 of the first element in terms of its fatigue strength:

$$N_1 = N_0 \Delta\sigma_1^{n_e} / \Delta\sigma_b^{n_e} \tag{8}$$

The second element experiences the same load history as the first element does for the first N_1 cycles. After the first element fails, the stress in the second element doubles, i.e., the stress amplitude becomes $2\Delta\sigma_b$, because both elements have the same elastic stiffness and the same deformation. However, the stress ratio in the second element still remains to be R. The second element eventually fails at N_0th cycle (Fig. 1c). Therefore, one can integrate the Paris law for the second element taking into account its increased stress amplitude:

$$\Delta\sigma_b^{n_e} N_1 + \Delta(2\sigma_b)^{n_e}(N_0 - N_1) = e^{Q_0/kT} I_2 \tag{9}$$

where $I_2 = A^{-1} l_{e2}^{1-n_e/2} \int_{a_0}^{a_c} k_{e2}^{-n_e}(\alpha) \mathrm{d}\alpha$; l_{e2} = characteristic size of the second element, and k_{e2} = dimensionless stress intensity factor of the second element. Similar to the analysis for the first element, one can replace $e^{Q_0/kT} I_2$ of the second element by $\Delta\sigma_2^{n_e} N_0$. Therefore,

$$\Delta\sigma_b^{n_e} N_1 + \Delta(2\sigma_b)^{n_e}(N_0 - N_1) = \Delta\sigma_2^{n_e} N_0 \tag{10}$$

Substituting Eq. 8 into Eq. 10, one can express the fatigue strength of the bundle as a function of the fatigue strengths of each element:

$$\Delta\sigma_b = [\Delta\sigma_1^{n_e}(1 - 1/2^{n_e}) + \Delta\sigma_2^{n_e}/2^{n_e}]^{1/n_e} \tag{11}$$

If the fatigue strength of the bundle does not exceed a certain value S, then the fatigue strengths of elements are bounded by the region $\Omega_2(S)$ (described by Eq. 11). Assuming that the fatigue strengths of two elements are independent random variables, then the cdf of fatigue strength of the bundle is given by:

$$G_2(S) = 2\int_{\Omega_2(S)} f_1(\Delta\sigma_1) f_2(\Delta\sigma_2) \mathrm{d}\Delta\sigma_1 \mathrm{d}\Delta\sigma_2 \tag{12}$$

where f_i = probability density function (pdf) of the fatigue strength of the ith element (i = 1,2).

The foregoing analysis can be readily extended to a bundle with n_b elements. Eq. 11 can be generalized as:

$$\Delta\sigma_b = \left\{ \sum_{i=1}^{n_b} [\beta_i(n_e)\Delta\sigma_i]^{n_e} \right\}^{1/n_e} \tag{13}$$

where $\beta_i(n_e) = [(n_b - i + 1)^{n_e}/n_b{}^{n_e} - (n_b - i)^{n_e}/n_b{}^{n_e}]^{1/n_e}$. One can easily show that $\beta_i = 1/n_b$ for $n_e = 1$, and $\beta_i = (n_b - i + 1)/n_b$ for $n_e \to \infty$. The cdf of fatigue strength of the bundle can then be written as:

$$G_{n_b}(S) = n_b! \int_{\Omega_{n_b}(S)} \prod_{i=1}^{n_b} f_i(\Delta\sigma_i) \mathrm{d}\Delta\sigma_1 \Delta \mathrm{d}\sigma_2 ... \mathrm{d}\Delta\sigma_{n_b} \tag{14}$$

here $\Omega_{n_b}(S)$ is the feasible region of stresses in all the elements, which is defined by the following inequalities:

$$\left[\sum_{i=1}^{n_b} \beta_i(n_e) \Delta\sigma_i^{n_e} \right]^{1/n_e} \le S \tag{15}$$

$$\Delta\sigma_1 \le \Delta\sigma_1 ... \le \Delta\sigma_{n_b - 1} \le \Delta\sigma_{n_b} \tag{16}$$

Two important asymptotic properties of the cdf of fatigue strength of the bundle are of particular interests. The first is the type of cdf of fatigue strength of large bundles. Consider the following two extreme values of n_e:

1. When $n_e = 1$, the fatigue strength of the bundle is simply the sum of the fatigue strengths of all the elements. This is the same as the mathematical representation of the cdf of strength of a plastic bundle, in which each element deforms at constant stress after its strength limit is reached. By virtue of the Central Limit Theorem, the cdf of fatigue strength must follow the Gaussian distribution except for its far left tail.
2. When $n_e \to \infty$, the fatigue strength of the bundle may be written as:

$$\Delta\sigma_b = \max\left[\Delta\sigma_1, \frac{n_b - 1}{n_b}\Delta\sigma_2, ..., \frac{1}{n_b}\Delta\sigma_{n_b}\right] \quad (17)$$

where $\Delta\sigma_1, \Delta\sigma_2, ...\Delta\sigma_{n_b}$ are the fatigue strengths of the elements ordered by the sequence of their breaking, i.e., according to increasing strength. This is the same as the mathematical formulation of the cdf of strength of a brittle bundle. The strength distribution of a brittle bundle can be described by the recursive equation of Daniels (1945), who further showed that the strength distribution of brittle bundles approaches the Gaussian distribution as the number of elements tends to infinity.

Therefore, one may expect that the cdf of fatigue strength of large bundles should approach the Gaussian distribution for any value of $n_e \geq 1$.

Another important property is the tail of the cdf of fatigue strength of the bundle. Let us assume that the fatigue strength of each element has a cdf with a power-law tail, i.e., $P_i(\Delta\sigma) = (\Delta\sigma/s_0)^{p_i}$. Considering the transformation $y_i = \Delta\sigma_i/S$, we can rewrite Eq. 14 as

$$G_{n_b}(S) = n_b! S^{\sum_i p_i} \int_{\Omega_{n_b}(1)} \left(\prod_{i=1}^{n_b} \frac{p_i y_i^{p_i - 1}}{s_0^{p_i}} \right) \mathrm{d}y_1 ... \mathrm{d}y_{n_b} \quad (18)$$

where $\Omega_{n_b}(1)$ is the corresponding feasible region of the normalized fatigue strength. Thus it is proven that, if the fatigue strength of each element has a cdf with a power-law tail, then the cdf of fatigue strength of the bundle will also have a power-law tail, and the power-law exponent will be the sum of the exponents of the power-law tails of the cdf's of fatigue strength of all the elements in the bundle. As shown in previous work (Bažant and Pang 2007, Le and Bažant 2010), this property of the tail probability distribution also holds for the cdf of monotonic strength of bundles consisting of elements with arbitrary load-sharing rules.

3.3 *Probability distribution of fatigue strength of one RVE*

Since the chain models for fatigue strength and monotonic strength share the same equation, the formulations of the bundle models for the fatigue strength and the monotonic strength ought to be similar. Because of the hierarchical model (Fig. 4e in Bažant & Pang (2007)) to calculate the cdf of fatigue strength of one RVE, one may expect that the cdf of fatigue strength of one RVE is similar to the cdf of static strength of one RVE. Based on the previous studies of the statistics of static strength of one RVE (Bažant & Pang 2007, Le & Bažant 2010b), the cdf of fatigue strength of one RVE can thus be approximately described by the Gaussian distribution with a Weibull tail grafted on the left at the probability of about 10^{-4}–10^{-3}. Mathematically this is similar to Eq. 52 in (Bažant & Pang 2007).

4 PROBABILITY DISTRIBUTION OF FATIGUE LIFETIME

Now consider the tests of fatigue strength and fatigue lifetime conducted on the same RVE. In the fatigue strength test, the RVE is subjected to a cyclic load with a prescribed number of cycles N_0 and a given stress ratio R, and the critical load amplitude (i.e., the fatigue strength ΔP_m), at which the RVE fails, is recorded. In the fatigue lifetime test, the load amplitude ΔP_0 and the stress ratio R are prescribed, and what is recorded is the critical number of cycles N_f at which the RVE fails.

An RVE fails under cyclic load when the dominant subcritical crack grows from its original length a_0 to a certain critical length a_c. The growth rate of this subcritical crack follows the Paris law (Eq. 6). By separation of variables,

$$\Delta\sigma^n N = \int_{\alpha_0}^{\alpha_c} \frac{\mathrm{d}\alpha}{A k^n(\alpha) l_0^{n-1}} \quad (19)$$

where $\Delta\sigma = (P_{max} - P_{min})/bl_0 =$ nominal stress amplitude, $\alpha = a/l_0 =$ dimensionless crack size, $k(\alpha) =$ dimensionless stress intensity factor of the RVE, and $l_0 =$ RVE size. Applying Eq. 19 to the tests of both fatigue strength and fatigue lifetime, one can relate the fatigue strength for the given number of cycles to the fatigue lifetime N_f for the given load amplitude:

$$\Delta\sigma_f = \Delta\sigma_0 (N_f/N_0)^{1/n} \quad (20)$$

Substituting Eq. 20 into the cdf of fatigue strength, one obtains the probability distribution of fatigue lifetime of one RVE:

for $N_f < N_{gr}$:

$$P_1(N_f) = 1 - \exp[-(N_f/s_N)^{\bar{m}}] \quad (21)$$

for N_f \$ N_{gr}:

$$P_1(N_f) = P_{gr} + \frac{r_f}{\delta_G\sqrt{2\pi}} \int_{\gamma N_{gr}^{1/n}}^{\gamma N_f^{1/n}} e^{-(N' - \mu_G)^2/2\delta_G^2} \mathrm{d}N' \quad (22)$$

where $\gamma = \Delta\sigma_0 N_0^{1/n}$, $N_{gr} = (\Delta\sigma_{gr}/\Delta\sigma_0)^n s_0^{1/n} N_0$, $s_N = N_0 \Delta\sigma_0^{-n}$, and $\bar{m} = m/n$. Similar to the cdf of fatigue strength, the probability distribution of fatigue lifetime follows the Weibull distribution, which has a power-law tail. The core of cdf of fatigue lifetime, which is expressed by Eq. 22, does not follow the Gaussian distribution.

For structures that fail at the initiation of a macro-crack from one RVE, the RVE must be defined as the smallest material volume whose failure triggers the failure of the structure. Statistically, such structures can be modeled as a chain of RVEs. According to the joint probability theorem and the assumption that the fatigue lifetimes of RVEs are independent random variables, the cdf of structure lifetime under a prescribed cyclic load can be written as:

$$P_f(N_f, \Delta\sigma_0) = 1 - \prod_{i=1}^{n_s} \left[1 - P_1(N_f, \Delta\sigma_0 s(x_i))\right] \tag{23}$$

where n_s = number of RVEs in the structure, $\Delta\sigma_0 S(x_i)$ = amplitude of maximum principal stress at the center of the ith RVE, $\Delta\sigma_0$ = amplitude of maximum principal stress in the structure, and $S(x_i)$ = dimensionless stress field (such that max $S(x_i)$ = 1). For sufficiently large structures, the tail part of the lifetime cdf of one RVE determines the failure of the entire structure. The cdf of fatigue lifetime of large-size structures follows the Weibull distribution. This distribution corresponds to the perfectly brittle failure behavior, to which the extreme value statistics (or infinite weakest-link model) apply.

5 OPTIMUM FITS OF FATIGUE LIFETIME HISTOGRAMS

Experimental studies of statistics of fatigue lifetime have been pursued for decades. The two-parameter Weibull distribution has been widely used to fit the observed histograms (Studarta, Filser, Kochera & Gauckler 2007, Hoshide 1995), but significant deviations have consistently been found.

Fig. 2 presents the optimum fits of lifetime histograms of various quasibrittle structures, such as engineering and dental ceramics, by both the two-parameter Weibull distribution and the present theory. The experiments are summarized as follows: a)-c) Structural Alumina ceramics (99% Al_2O_3): round bar specimens were tested under fully reversed cyclic load by using a rotating bending machine (Sakai & Fujitani 1989, Sakai & Hoshide 1995). Three stress levels were used in the experiment and 20 specimens were tested for each stress-level. d) Dental ceramic composites: Glass infiltrated Al_2O_3-ZrO_2 with feldspathic glass (Inc-VM7) (Fig. 2a) and yttria-

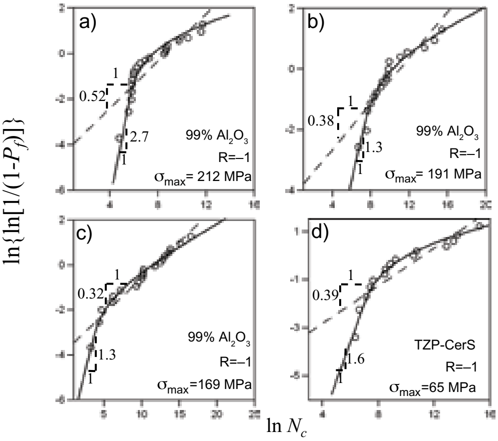

Figure 2. Optimum fitting of lifetime histograms.

stabilized ZrO_2 with feldspathic glass (TZP-CerS) (Fig. 2b). For each material, 30 specimens with size 4 mm × 5 mm × 50 mm were tested under fully reversed cyclic bending (Staudarata, Filser, Kochera, & Gauckler 2007).

As seen in Fig. 2, the lifetime histograms do not follow a straight line on the Weibull scale. Instead, they consist of two parts separated by a short kink. The lower part of the histogram follows a straight line whereas the upper part of the histogram diverges to the right from the straight line. Clearly, the two-parameter Weibull distribution cannot fit such histograms closely. On the other hand, the present theory gives an excellent fit for both parts of the histogram.

6 SIZE EFFECT ON STRESS-LIFE CURVE

The foregoing analysis (Eq. 23) shows that the cdf of fatigue lifetime depends on the structure size as well as the geometry (which is introduced through the stress distribution). Naturally, the mean fatigue lifetime $\bar{N}_f$, too, must depend on the structure size and geometry. According to the weakest-link model, $\bar{N}_f = \int_0^\infty \prod_{i=1}^{n_s} [1 - P_1(\Delta\sigma_0 s(x_i), N')] \mathrm{d}N'$.

An analytical expression for N_f seems impossible. However, similar to previous analysis of the size effect on the mean strength and the creep lifetime (Bažant, Le, & Bazant 2009), one may use the approximation:

$$\bar{N}_f = \left[\frac{C_a}{D} + \left(\frac{C_b}{D} \right)^{\psi/\bar{m}} \right]^{1/\psi} \tag{24}$$

where $\bar{m}$ = Weibull modulus of fatigue lifetime. The values of C_a, C_b and ψ ensue by matching three asymptotic conditions: $[\bar{N}_f]_{D\to l_0}$, $[d\bar{N}_f/dD]_{D\to l_0}$, and $[\bar{N}_f D^{1/\bar{m}}]_{D\to\infty}$.

In the derivation of the cdf of fatigue strength of one RVE, the Paris law was integrated to obtain a simple equation that relates the fatigue lifetime and the applied stress amplitude (Eq. 19). Consider now that two cyclic load histories with the same stress ratio but different stress amplitudes ($\Delta\sigma_{01}$ and $\Delta\sigma_{02}$) are applied to the same RVE. Based on Eq. 19, one finds that the fatigue lifetimes of RVE for these two load histories are related by: $N_2 = \frac{\Delta\sigma_{01}^n}{\Delta\sigma_{02}^n} N_1$. Similarly, consider further that two cyclic load histories that give the same nominal stress ratio ($\sigma_{1,max}/\sigma_{1min} = \sigma_{2,max}/\sigma_{2,min}$) but different nominal stress amplitudes ($\Delta\sigma_1$ and $\Delta\sigma_2$) are applied to the same structure. Since the stress in each RVE is proportional to the nominal stress, the ratio of stress amplitudes on each RVE for these two load cases is $\Delta\sigma_1/\Delta\sigma_2$.

For the first loading history, in which the nominal stress amplitude is $\Delta\sigma_1$, the failure probability of the whole structure is $P_f = 1 - \prod_{i=1}^{\nu}[1 - P_1(N_f)]$. The failure probability of the structure under the second load history, in which the nominal stress amplitude is $\Delta\sigma_2$, can be written as: $P_f = 1 - \prod_{i=1}^{n_s}\{1 - P_1[(\Delta\sigma_1^n/\Delta\sigma_2^n)N_f]\}$. Therefore, the mean fatigue lifetimes for these two load histories are related by $\Delta\sigma_1^n \bar{N}_{1c} = \Delta\sigma_2^n \bar{N}_{2c}$. This leads to a general relation between the mean fatigue lifetime and the nominal stress amplitude:

$$N_f \Delta\sigma_0^n = C \tag{25}$$

where C = constant. This is the well-known power law form for the stress-life (S-N) curve (Basquin 1910) for the fatigue loading, which is supported by numerous test data on quasibrittle materials such as ceramics (Suresh 1998, Kawakubo 1995) and cortical bones (Turner, Wang, & Burr 2001).

Because of the size effect on the mean fatigue lifetime (Eq. 24), constant C in Eq. 25 must depend on the structure size and geometry:

$$N_f \Delta\sigma_0^n = C = \Delta\sigma_0^n \left[\frac{C_a}{D} + \left(\frac{C_b}{D} \right)^{\varphi\bar{m}} \right]^{1/\varphi} \tag{26}$$

Eq. 26 implies that, in a bi-logarithmic plot, the S-N curve must shift horizontally to the left as the structure size increases, as shown in Fig. 3. Eq. 26 is particularly important for the design process since it allows the mean lifetime of full-scale structures under a relatively low stress amplitude to be determined from the laboratory tests on prototypes under a relatively high stress amplitude.

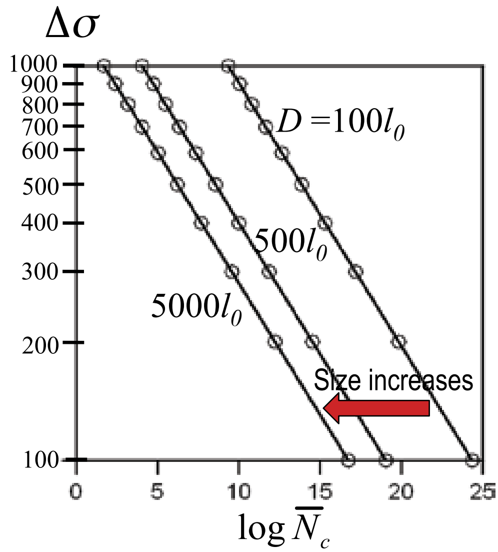

Figure 3. Size effect on S-N curve.

7 CONCLUSIONS

This study shows that the type of probability distribution of fatigue lifetime depends on structure size and geometry. Consequently, the stress-life curve (S-N curve) is also size-dependent. This has serious implications for the design and safety assessments of lifetime of large concrete structures, as well as large composite aircraft frames and ship hulls, microelectronic devices, bone implants, etc.

REFERENCES

Basquin, O.H. (1910). The exponential law of endurance tests. *Proc. A Mater., ASTEA 10*, 625–630.

Bažant, Z.P. (2004). Scaling theory of quaisbrittle structural failure. *Proc. Nat'l. Acad. Sci., USA 101*, 13397–13399.

Bažant, Z.P., J.-L. Le, & M.Z. Bazant (2009). Scaling of strength and lifetime distributions of quasibrittle structures based on atomistic fracture mechanics. *Proc. Nat'l. Acad. Sci., USA 106*, 11484–11489.

Bažant, Z.P. & S.D. Pang (2006). Mechanics based statistics of failure risk of quasibrittle structures and size effect on safety factors. *Proc. Nat'l. Acad. Sci., USA 103*, 9434–9439.

Bažant, Z.P. & S.D. Pang (2007). Activation energy based extreme value statistics and size effect in brittle and quasibrittle fracture. *J. Mech. Phys. Solids 55*, 91–134.

Daniels, H.E. (1945). The statistical theory of the strength of bundles and threads. *Proc. R. Soc. London A. 183*, 405–435.

Hoshide, T. (1995). Proof testing and subsequent fatigue characteristics in ceramics. In *Cyclic Fatigue in Ceramics, Elsevier Science B. V. and The Society of Materials Science*, Japan, pp. 207–232.

Kawakubo, T. (1995). Fatigue crack growth mechanics in ceramics. In *Cyclic Fatigue in Ceramics, Elsevier Science B. V. and The Society of Materials Science*, Japan, pp. 123–137.

Krausz, A. & K. Krausz (1988). *Fracture Kinetics of Crack Growth*. Netherlands: Kluwer Academic Publisher.

Le, J.-L. & Z.P. Bažant (2010a). Scaling of probability distribution of fatigue lifetime of quasibrittle structures. Technical report, Northwestern University, Evanston, Illinois.

Le, J.-L. & Z.P. Bažant (2010b). Unified nano-mechanics based probabilistic theory of strength, scaling, static crack growth and lifetime of quasibrittle structures. Technical report, Northwestern University, Evanston, Illinois.

Le, J.-L., Z.P. Bažant, & M.Z. Bazant (2009). Crack growth law and its consequences on lifetime distributions of quasibrittle structures. *J. Phys. D: Applied Physics 42*, 214008.

Paris, P.C. & F. Erdogan (1963). A critical analysis of crack propagation law. *J. Basic Engrg. 85*, 528–534.

Phoenix, S.L. (1978). The asymptotic time to failure of a mechanical system of parallel members. *SIAM J. Appl. Maths. 34*, 227–246.

Risken, H. (1989). *The Fokker-Plank Equation*. Berlin: Springer Verlag.

Sakai, T. & K. Fujitani (1989). A statistical aspect on fatigue behavior of alumina ceramics in rotating bending. *Engrg. Frac. Mech. 34*, 653–664.

Sakai, T. & T. Hoshide (1995). Statistical aspect on fatigue fracture of structural ceramics under cyclic loads. In *Cyclic Fatigue in Ceramics, Elsevier Science B. V. and The Society of Materials Science*, Japan, pp. 189–206.

Smith, R. (1982). The asymptotic distribution of the strength of a series-parallel system with equal load sharing. *Ann Probab. 10*, 137–171.

Studarta, A.R., F. Filser, P. Kochera, & L.J. Gauckler (2007). Fatigue of zirconia under cyclic loading in water and its implications for the design of dental bridges. *Dent. Mater. 23*, 106–114.

Suresh, S. (1998). *Fatigue of Materials*. Cambridge: Cambridge University Press.

Turner, C., T. Wang, & D. Burr (2001). Shear strength and fatigue properties of human cortical bone determined from pure shear tests. *Calcif. Tissue Int. 69*, 373–378.

Weibull, W. (1939). The phenomenon of rupture in solids. *Proc., Royal Swedish Inst. Eng. Res. 153*, 1–55.