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STRAIN-SOFTENING MATERIALS AND FINITE ELEMENT SOLUTIONS

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ABSTRACT

Closed form and finite element solutions are reviewed for several problems with strain-softening materials. In the closed form solutions, strain-softening causes localization of the strain which is accompanied by an instantaneous vanishing of the stress. The finite element solutions agree closely with analytic solutions in many cases and exhibit a rate of convergence only slightly below that for linear problems. The main difficulty which has been identified in strain-softening constitutive models for damage is the absence of energy dissipation in the strain-softening domain, and this can be corrected by a nonlocal formulation, such as one which is reviewed here.

INTRODUCTION

In materials such as concrete or rock, failure occurs by progressive damage which is manifested by phenomena such as microcracking and void formation. In most engineering structures, the scale of these phenomena, as compared to the scale of practical finite element meshes, is usually too small to be modelled and their effect must be incorporated in the numerical analysis through a homogenized model which exhibits strain-softening.

Strain-softening, unfortunately, when incorporated in a computational model, exhibits undesirable characteristics. In static problems, finite element solutions with strain-softening often exhibit a severe dependence on element mesh size because of the inability of the mesh to adequately reproduce the localization of strain which characterizes static strain-softening. Furthermore, the solutions are physically inappropriate in that with increasing mesh refinement the energy dissipated in the strain-softening domain tends to zero [1].

It was first hoped, although in retrospect little practical evidence existed for this optimism, that in dynamic problems, strain-softening would not be as troublesome because the inertia of the continuum would alleviate the instability. Support for this can be found in the snapthrough of an arch; in this problem the load-deflection curve contains a limit point after which the force-deflection curve is negative, or softens. A static solution for the snapthrough is often very difficult, whereas a dynamic solution is relatively straightforward because the inertia of the structure alleviates some of the difficulties introduced by the negative slope in the force-deflection relation. The use of strain-softening models has become quite commonplace in dynamic concrete analysis. For example, in Marchertas et al. [3], strain-softening appeared to reproduce the salient features of dynamic concrete response even when severe failure had developed. In the community as a whole, a certain complacency evolved and little effort was devoted to examining the basic soundness of these solutions with strain-softening.

Attention was recently focused on the validity of strain-softening models by the work of Sandler and Wright [4], in which they asserted that strain-softening models are basically ill-posed because a small difference in load results in large changes in the response. Sandler's example, which will be described in more detail later, consists of a one-dimensional rod with the velocity prescribed at one end in which the material strain-softens. By increasing the load slightly, a significantly different response was obtained for the problem. Sandler and Wright also noted a strong dependence of the solution on the mesh size. They concluded that "a rate independent dynamic continuum representation of strain-softening is incapable of reproducing softening behavior in a dynamic simulation of experiments" and then proceeded to show that in this problem the introduction of viscosity eliminates the sensitivity of the response to the load. Incidentally, as will be shown later in this paper, viscous damping is not a panacea for the sensitivity observed in strain-softening solutions; in certain problems which will be described herein, sensitivity to mesh size persists even after the introduction of damping.

In an effort to develop a problem with strain-softening in which the localization does not occur on the boundary, we investigated two problems: one, which was presented in Ref. [5], consists of a linear elastic bar joined to a strain-softening bar. Solutions for this bar were obtained by the method of images and compared to finite element solutions. These results exhibited convergence with decreasing element length. A more interesting problem was subsequently constructed in which tensile waves are initiated at two ends of a bar so that strain-softening occurs at the center, Ref. [6]. It was shown that a solution exists to this problem but that the behavior of the strain-softening domain is rather unusual: the strain-softening is limited to an infinitely thin domain, in wihch the strain becomes instantaneously infinite and in which the energy dissipation is zero.

In order to remedy some of these undesirable features of strain-softening solutions, Bazant, Belytschko and co-workers [7-8] proposed a new nonlocal formulation for treating strain-softening. Nonlocal theories have been introduced by Kroner, Kunin and Krumhansl and other [9-12] and developed further by Eringen and coworkers [12-14]. The basic ingredient of a nonlocal theory is that the strains are not considered to be local quantities but reflect the state of deformation within a finite volume about any point. In this respect, the theory lends itself admirably to problems of heterogeneous media, where the representation of the microscopic detail of strain-fields and cracks is an insurmountable task. By dealing with an average of the strain over a finite domain about each point, the heterogeneity can be neglected, and the dispersion which occurs in inhomogeneous materials can be modelled without any artifices.

An obvious question which arises is why one would want to introduce this complication in order to deal with strain-softening. The reason for this is that when the constitutive equations are applied locally at points, then, as will be described here, no dissipation of energy occurs in the strain-softening process. Thus the material can fail without any permanent dissipation of energy, which is physically quite unrealistic. By introducing a nonlocal character into the constitutive law, it is possible to restrict the localization to a domain of finite size just as is observed experimentally, and to achieve a finite amount of energy dissipation in the strain-softening domain.

However, we found we could not simply extend the existing nonlocal models to account for strain-softening [15,16]. The existing nonlocal laws are not even self-adjoint, so they did not lead to symmetric stiffness matrices. This lack of symmetry was found to be quite undesirable and was corrected by introducing an averaging operation over the stresses. More important, it was found that the strain-softening could only be introduced in the non-local law in a very subtle fashion, necessitating a split of the constitutive equation into a local and nonlocal law, with the strain-softening included only in the nonlocal portion. Numerical experiments indicated that without this particular combination, numerical solutions were invariably unstable.

The nonlocal law as introduced in Refs. [7,8] offers substantial promise in providing well-posed solutions for heterogeneous materials that are subjected to damage and hence strain-softening. There are however, substantial breakthroughs that yet need to be achieved: (1) efficient implementations of nonlocal laws in the finite element method; (2) design of experimental methods for identifying the local and nonlocal portions of constitutive laws and (3) methods for reconciling the bifurcation between local damage, i.e., microcracking, and large scale fracture of a cleavage type in heterogeneous materials. However, the work reported here has shed light on the questions of numerical modelling of structures in the failure regime when strain-softening takes place and provides the basis for future work.

We have organized the material as follows: in Section 1 we describe several of the generic one-dimensional problems which can be used to examine the mathematical character of dynamic strain-softening solutions. In Section 2, some finite element solutions are presented to indicate that except in one case, the solutions indicate a certain well behavedness. In Section 3, the non-local continuum law will be described followed by conclusions in Section 4.

DYNAMIC STRAIN-SOFTENING SOLUTIONS

The problem by Sandler and Wright [4] is shown in Fig. 1. The essence of their argument was that the solutions are very sensitive to the constant v_o, which gives the maximum prescribed velocity at the left hand boundary, for certain values and that the solution changes markedly and so does not appear to converge as the mesh is refined. Although the Sandler-Wright stress-strain law is nice from the viewpoint that it provides a very continuous relationship between stress and strain in the loading domain, it is not amenable to any attempts at a closed form solution by d'Alembert methods because of the dispersive character of the wave solution even in the loading range. For this reason, we have limited our studies to piecewise linear prescribed velocities or stresses and stress-strain laws of the type shown in Fig. 2. Note that the stress goes to zero as the strain becomes large on the strain-softening sides of the law (usually the tensile side).

The analytic solutions to this problem are developed next. The salient characteristic of the analytic solution is the appearance of an infinite strain on the boundary once the strain ε_o is exceeded. The construction of the solution for this case will follow that described by Bazant and Belytschko [6] for a similar problem. As will be seen, when strain-softening occurs, then the strain immediately localizes and reaches infinity within a time interval that approaches zero. Therefore, the solution can be generated by adding an image wave which cancels the incident wave so that the strain-softening point is instantaneously converted to a free boundary.

The governing equations can be stated as follows

$$\sigma,_x = \rho u,_{tt} \tag{1.1}$$

$$\sigma,_t = E(\varepsilon)u,_{xt} = E(\varepsilon)\varepsilon,_t \tag{1.2}$$

where σ and ε are the stress and strain, u the displacement and subscripts preceded by commas denote differentiation; ρ is the density and E the tangent modulus. We will consider two types of boundary conditions on the left-hand side, $x = 0$:

$$\text{velocity condition:} \qquad u_{,t}(0, t) = -v_o \langle t \rangle \tag{1.3}$$

$$\text{traction condition:} \qquad \sigma(0, t) = \sigma_o \langle t \rangle \tag{1.4}$$

where the symbol $\langle f \rangle$ designates $fH(f)$, where H is the Heaviside step function. The velocity boundary condition will be considered first. The right-hand boundary is assumed to be sufficiently far so that the rod can be considered semi-infinite.

Note that prior to the onset of strain-softening, the problem is governed by the standard one-dimensional wave equation

$$u_{,xx} = \frac{1}{c^2} u_{,tt} \tag{1.5}$$

where

$$c^2 = \frac{E}{\rho} \tag{1.6}$$

Once the strain-softening regime of the material is attained, then at those points the governing equation is

$$\bar{c}^2 u_{,xx} + u_{,tt} = 0 \tag{1.7}$$

$$\bar{c}^2 = -\frac{E}{\rho} \tag{1.8}$$

and $\bar{c}$ vanishes once ε_p is attained. Equation (1.7) is elliptic in space-time, which is quite peculiar in that no information can be transmitted from a point which is strain-softening to adjacent points. Hadamard [17] commented on this in 1903 and he claimed that the negative character of the square of the wave-speed precluded its applicability to real materials since the wave speed would then be imaginary. However, the case of $c = 0$ has been treated extensively by Taylor [18], who noted that for perfectly-plastic solids the deformation is localized at the point of impact. Wu and Freund [19] have recently presented a lucid description of these localization phenomena and investigated the effects of strain-rate sensitivity and heat transfer on the localization. However, the analyses were limited to the case where in the limit $c = \bar{c} = 0$.

We will here consider the strain-softening situation using the concepts developed in [6]. The present situation differs from [6] only in that the stress wave is a ramp rather than a step-wave, but it will be found that all of the singularities associated with a step input remain.

The procedure of constructing a solution consists of three steps:

1) it is shown that the boundary between the strain-softening and elastic domain cannot move, so the strain-softening domain is limited to a point;

2) this is shown to imply the strain and strain-rate in the strain-softening points must be infinite;

3) since the strain-rate is infinite, for the class of materials considered here in which $\sigma \to 0$ as $\varepsilon \to \infty$, the stress can instantaneously be considered to vanish at the points which strain-soften.

The last conclusion enables the solution to be easily constructed by d'Alembert methods by simply adding a wave to satisfy this zero stress condition.

For the prescribed velocity problem, let t_1 be the time when the left-hand end, $x = 0$, reaches ε_p and begins to strain-soften; t_1 is given by

$$t_1 = \frac{2c\varepsilon_p}{v_o} \tag{1.9}$$

and the solution prior to the onset of strain-softening is given by

$$u = -\frac{v_o}{2} < (t - \frac{x}{c})^2 > \tag{1.10}$$

$$\varepsilon = \frac{v_o}{c} < t - \frac{x}{c} > \tag{1.11}$$

Strain-softening first occurs at $x = 0$. We now show that the boundary between the elastic and the softening interface cannot move. For this purpose, the usual formula for velocity V of discontinuities is used (a development is given in [6])

$$\sigma^+ - \sigma^- = \rho\, V^2(\varepsilon^+ - \varepsilon^-) \tag{1.12}$$

where the superscript + and - designate the state variables to the right and left of the discontinuity, respectively. If the material is strain-softening behind the interface and not yet before it, it follows that $\varepsilon^- > \varepsilon^+$ and $\sigma^+ > \sigma^-$. Substituting these inequalities into Eq. (1.12), it follows that V^2 must be negative or zero; since the former assumption would yield an imaginary velocity for the discontinuity, only $V = 0$ is tenable, and it can already be concluded that

$$\sigma^+ = \sigma^- \tag{1.13}$$

To show that the strain and strain-rates must be infinite at a point which strain-softens, a solution is constructed in the strain-softening domain, which is considered to be $0 < x < s$ where $s \to 0$. It can be seen that

$$u = u^* + [a\,(t - t_1) + \varepsilon_p]x \tag{1.14a}$$

$$u^* = -\frac{v_o}{2} < (t_1 - \frac{x}{c})^2 > \tag{1.14b}$$

satisfies the governing equation in the strain-softening domain, (1.7). This solution, (1.14) is now matched to a solution in the elastic domain

$$u = -\frac{v_o}{2} < (t - \frac{x}{c})^2 > + f(\xi)\, H\,(t - t_1) \tag{1.15}$$

$$\xi = t - \frac{x}{c} - t_1 \tag{1.16}$$

where the second term is a wave emanating in the strain-softening region which will be used to match the displacements and stress-conditions across the interface. Note that from Eq. (1.14a), it follows that

$$\varepsilon^- = \frac{\partial u}{\partial x} = a(t - t_1) + \varepsilon_p \tag{1.17}$$

If the two displacement solutions, Eqs. (1.14a) and (1.15) are now matched across the interface x = s, then

$$u^* + [a(t - t_1) + \varepsilon_p]s = -\frac{v_o}{2} < (t - \frac{s}{c})^2 > + f(\bar{\xi}) \tag{1.18a}$$

$$\bar{\xi} = t - \frac{s}{c} - t_1 \tag{1.18b}$$

Eliminating a from Eqs. (1.17) and (1.18) yields

$$\varepsilon^- = \frac{1}{s}\left[f(\bar{\xi}) - \frac{v_o}{2} < t - \frac{s}{c} > - u^*\right] \tag{1.19}$$

It can be seen that as $s \to 0$, $\varepsilon^- \to 0$ instantaneously, which through Eq. (1.13) implies $\sigma^+ = 0$. The function of $f(\xi)$ is then found by this condition. Using the displacement field of Eq. (1.15) and letting σ^+, and hence ε^+, vanish, we find

$$f'(\xi) = \varepsilon_p c\, H(t - t_1 - \frac{x}{c}) + v_o < t - t_1 - \frac{x}{c} > \tag{1.20a}$$

$$f(\xi) = \varepsilon_p c < t - t_1 - \frac{x}{c} > + \frac{v_o}{2} < (t - t_1 - \frac{x}{2})^2 > \tag{1.20b}$$

Hence the complete solutions is

$$u = -\frac{v_o}{2} < t - \frac{x}{c} > + \varepsilon_p c < t - t_1 - \frac{x}{c} > + \frac{v_o}{2} < (t - t_1 - \frac{x}{c})^2 > \tag{1.21a}$$

$$u_{,t} = -\frac{v_o}{2} H(t - \frac{x}{c}) + \varepsilon_p c\, H(t - t_1 - \frac{x}{c}) + v_o < t - t_1 - \frac{x}{c} > \tag{1.21b}$$

This solution will subsequently be compared to finite element solutions.

The solution for the traction condition, Eq. (1.4), can be found by replacing v_o by $\sigma_o c/E$. However, in the stress boundary form of this problem Eq. (1.4) the introduction of the image at the strain-softening point poses a difficulty since the first point to strain-soften is on the boundary to begin with. Thus, in one sense it can be said that this boundary must satisfy two different boundary conditions: Eq. (1.4) and $\sigma = 0$.

This contradiction can only be reconciled by requiring the second type of boundary condition (that the stress vanishes) to take precedence. This notion of a boundary condition depending on the result of the solution is not totally unexpected in an analysis of a continuum which fails. For example, in a buckling problem with unstable postbuckling behavior, the prescribed stress would also be limited by the capacity of the structure. Yet the situation in the buckling problem is not completely analogous: in a dynamic buckling problem, any stress may be prescribed and the excess stress will generate accelerations, which depend on the magnitude of the stress, whereas in this problem, the solution is completely independent of the value of the prescribed stress once the failure stress is exceeded. Nevertheless, this model does appear to represent a physically meaningful situation: the behavior of a rod in which the material can sustain a limited tensile strain before it fails, and the solution appears reasonable.

From a mathematical viewpoint, the character of the solution procedure presents some other dilemmas, First of all, we consider Δs to be a segment in developments of Eq. (1.19), but it is only a point. Second, since the strain softening portion is localized in a one dimensional solution to a point and analogously, in a two dimensional solution, to a line, within conventional theories for partial differential equations, the body would no longer be considered to be a single body: instead the effect of strain-softening is to subdivide the initial body by introducing interior boundaries. Although mathematical theories for such partial differential equations are not known to us at the present time, there is no reason to arbitrarily exclude such phenomenological models.

Another difficulty posed by this model is that the energy dissipated in the formulation of the strain-softening region is not finite but instead vanishes. This can be seen from the fact that the only irreversible energy loss in the material shown in Fig. 2 occurs in the strain softening domain. Because the strain-softening domain in a one dimensional problem becomes a point, and since the energy dissipated per unit length is finite, the total energy dissipated vanishes. This in fact is a more serious difficulty than the mathematical difficulties, for the strain-softening constitutive equation is often intended to represent microcracking, which is a dissipative process. It will be seen that in spite of the mathematical questions, the behavior of finite element solutions is not altogether pathological.

Other Remarks:

1) The solution is puzzling when the ramp loads Eqs. (1.3) and (1.4) are replaced by step functions. According to the present analysis, if $\sigma_o > \epsilon_p E$, then the boundary point should reach strain-softening instantaneously and no wave should reach the interior.

2) The solution does not depend on the specific functional dependence of stress on strain in the strain-softening portion, provided that the stress vanishes as the strain becomes large.

One conceptual difficulty of the Sandler-Wright problem is that strain softening occurs only at the boundary, which confuses the role of the boundary condition and the strain-softening. For this reason we have attempted to construct problems in which the strain-softening occurs within the domain of the problem.

The strictly one dimensional problems of this type are shown in Fig. 3 and 4. The first consists of an elastic rod joined to a rod with a strain-softening material [5]. We will not give the closed-form solution but only explain its major features. If the applied stress is sufficiently large, then strain-softening is initiated at the interface between the two materials. The strain localizes at this point, and as in the previous problem, the stress vanishes instantaneously at the interface. The solution can thus be viewed as a case in which a body separates into two.

The second problem, given in [6], consists entirely of a strain-softening material. Equal and opposite velocities v_o (or forces) are applied to the two ends of the bar, so that tensile waves are generated at the two ends. These propagate to the center; when they meet at the center, the value of the stress there becomes twice the applied stress, so strain-softening is possible at this midpoint even though it did not occur at the boundaries.

The solution is given in [6] for prescribed velocities that are step-functions in time. As in the previous case, localization occurs at the midpoint where the strain becomes infinite. The solution is symmetric about the midpoint, $x = 0$ and is given in [6]; for the left half ($x < 0$ and $0 < t < 2L/c$)

$$u = v_o < t - \frac{x + L}{c} > - v_o < t - \frac{L - x}{c} > \tag{1.22a}$$

$$\varepsilon = \frac{v_o}{c} [H(t - \frac{x + L}{c}) - H(t - \frac{L - x}{c}) + 4 < ct - L > \delta(x)] \tag{1.22b}$$

where $\delta(x)$ is the Dirac delta function.

Another problem we have considered is a sphere loaded on its exterior surface, see Fig. 5. This problem is not easily physically realizable with a tensile load; however, it is physically meaningful with a compressive load where strain-softening also occurs in some materials, (although the stress usually does not vanish as the dilatation becomes large).

The interesting feature of this problem is that when the load is a ramp-function in time, prior to the onset of strain-softening at an interior surface, a stress wave can have passed through this surface. Since the wave which is beyond the strain-softening surface is amplified as it passes to the center, the formation of additional strain-softening surfaces is possible. As a result, this problem has considerably more structure than the one-dimensional problems.

FINITE ELEMENT SOLUTIONS

Finite element solutions for the Sandler-Wright problem, Fig. 1, with the material law given in Fig. 2 are shown in Fig. 6. Solutions were obtained with meshes of 50, 100 and 200 elements. Linear displacement, constant strain elements and lumped mass matrices were used. Time integration was performed with the central difference method.

The finite element solutions are compared with the analytic solution given in the previous section. It can be seen that the agreement is quite good and improves with mesh refinement, although the instantaneous drop in the velocity which is a result of the strain-localization cannot be reproduced well even with the finest mesh.

The rate of convergence is shown in Fig. 7. Here the error e is defined by

$$e^2 = \int_o^{t_o} \int_o^L (v^{FEM} - v^{ANA})^2 \, dx \, dt$$

As can be seen from Fig. 7, the rate of convergence is approximately $h^{1.46}$ for the velocity. This is only slightly less than the theoretical value of h^2 expected for linear solutions by these methods, so the sensitivity to meshing which Sandler and Wright pointed out is not evident.

Figure 8 shows the finite element solutions for the spherical wave problem given in Fig. 5 with strain-softening. In this solution, the strain-softening diagram in Fig. 2 pertains to the relation between pressure and dilatation. The following constants were used; bulk modulus $K = 1.0$, density $\rho = 1$, shear modulus $G = 1.E-6$, $\varepsilon_y = 1.0$, $\varepsilon_f = 5.0$. Damping was added so that for the coarsest mesh, the maximum element frequency was damped at 40% of critical. A unit step function is prescribed for the radial stress on the outside surface.

Although the classical nonlocal theory directly uses the stress σ in the momentum equation, Eq. (1.1), the resulting form is not self-adjoint [15,16]. This leads to the existence of spurious, zero-energy modes of deformation for certain weight functions $w(x)$: deformations which are associated with vanishing strains $\bar{\varepsilon}$ and hence do not generate any stresses. These spurious modes have been found for constant weight functions $w(x)$.

To remedy this difficulty the stress σ is processed through an operator identical to (3.1)

$$\bar{\sigma}(x) = \int_{x-\ell/2}^{x+\ell/2} \sigma(x + s)\, w(s) ds \qquad (3.3)$$

and the resulting stress is used in the equation of motion, Eq. (1.1). Once Eq. (3.3) is added to the process, spurious modes are eliminated even for constant weight functions, $w(x)$.

Even with a self-adjoint form of the nonlocal laws, solutions are unstable for strain-softening materials for constant weights $w(x)$. So far, only by combining a local and nonlocal law has stability been achieved [16, 7]. By superimposing two distinct field systems, one local and without strain-softening, the nonlocal one with strain-softening, stability is achieved in a model which exhibits a negative slope for a finite segment. This type of composite local, nonlocal model has been termed an imbricate continuum by Bazant [8] because it can be represented by an overlapping mesh of finite elements.

The governing equations for this model can be summarized as follows

$\sigma_{,t} = E\, \bar{\varepsilon}_{,t}$	E can be negative	(3.4)
$\tau_{,t} = E\, \varepsilon_{,t}$	$E > 0$	(3.5)
Eq. (3.3) : $\sigma \rightarrow \bar{\sigma}$		(3.6)
$S = (1 - \gamma)\, \bar{\sigma} + \gamma\, \tau$	$0 < \gamma < 1$	(3.7)
$S_{,x} = \rho u_{,tt}$		(3.8)

Figure 9 gives an indication of the rate of convergence for the nonlocal model with strain-softening for the problem in Fig. 1. Here $\gamma = 0.1$, and ℓ is 0.2 of the total length of the bar. Results are shown for the cases where the number of elements N = 5, 15, 45, and 95 at six different times. It can be seen that for more than 15 elements, there is little change in the distribution of the mean strain $\bar{\varepsilon}$ at various times. The local strain converges less rapidly but is not ill-behaved. By contrast, in the local formulation, the strain becomes larger and larger at the midpoint as the mesh is refined.

The solutions presented in Fig. 9 are taken from Ref. [7] and were obtained by taking a discrete form of the nonlocal continuum, which consists of several overlapping series of elements. This process has not yet been attempted in multi-dimensional problems.

CONCLUSIONS

The following are the major conclusions of the work summarized here:

1. Analytic solutions can be established for certain simple problems which include strain-softening materials. The solutions exhibit singular strain distributions but the rate of convergence of finite element solutions is quite rapid.

2. In the spherical wave problem, numerical solutions of strain-softening models exhibit severe dependence on element mesh size. This is particularly true of field variables inside the surface of initial strain-softening. Nonlocal models provide rapidly converging solutions to this problem.

3. A major difficulty of local laws with strain-softening is that the energy dissipation vanishes. Thus, the failure process is not accompanied by energy dissipation, which is physically unrealistic.

4. Nonlocal laws provide a means for obtaining rapid convergence and finite energy dissipation in failure. However, the technology for efficiently implementing these techniques in large-scale, multi-dimensional problems remains to be developed.

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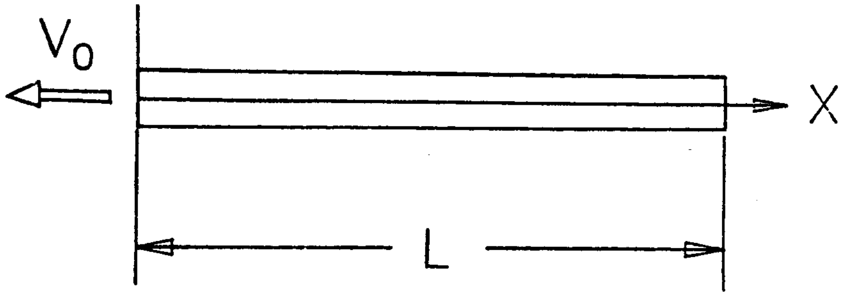

Figure 1. Sandler-Wright Problem [4]; stress-strain law in [4] is
$\sigma = E_0 \varepsilon \exp(-\varepsilon/\varepsilon_0)$, $\varepsilon_0 = 0.002$;
$u_{,t}(0, t) = v_0 \, [1 - \cos(\pi t/t_0)]/2$ for $t \le t_0$,
$u_{,t}(0, t) = v_0$ for $t > t_0$, $t_0 = 2 \times 10^{-4}$ sec.

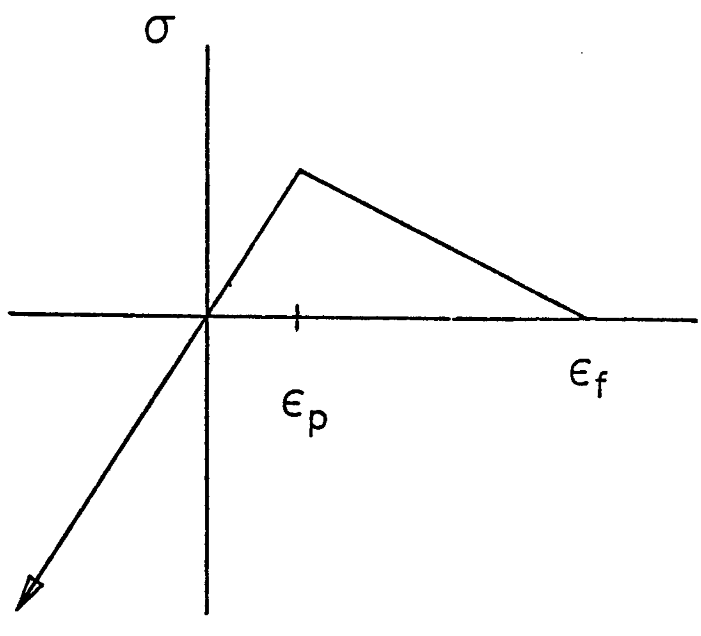

Figure 2. Stress-strain law with strain-softening and nomenclature.

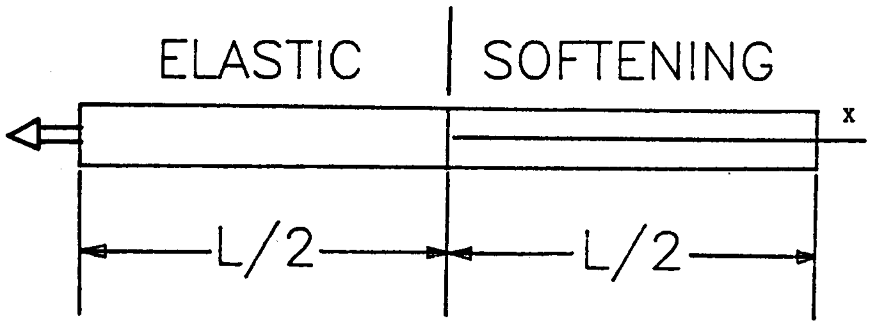

Figure 3. Problem with strain-softening at interface between elastic and softening domain [5].

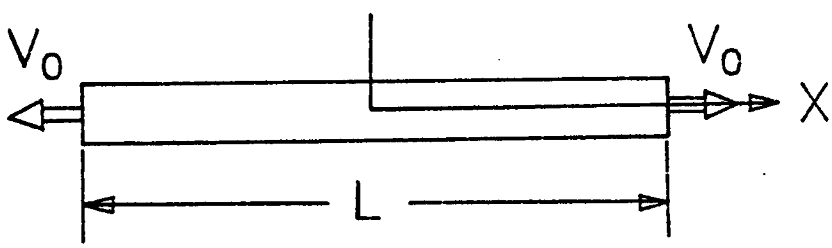

Figure 4. One dimensional problem in which strain-softening occurs at x = 0 [6].

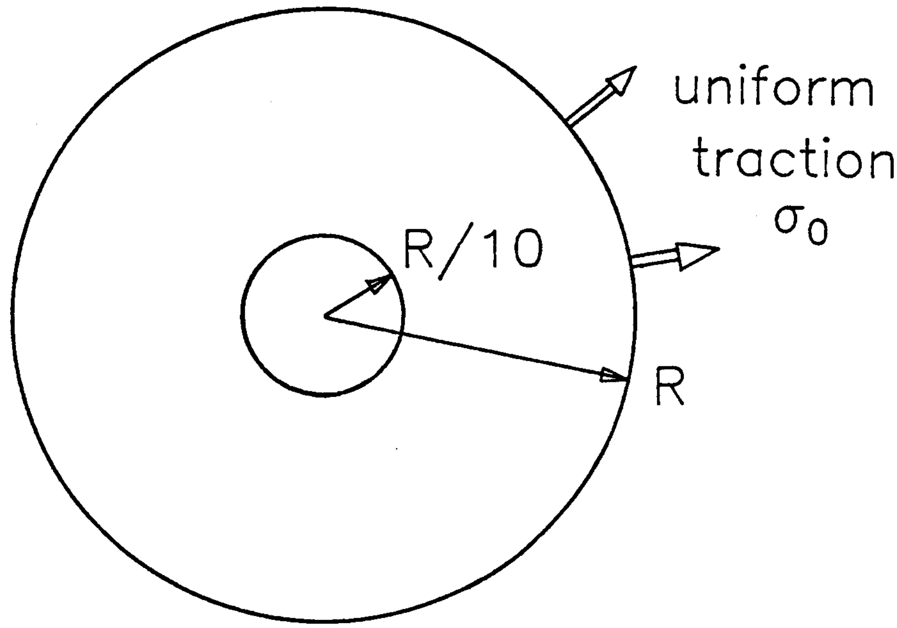

Figure 5. Spherical wave problem.

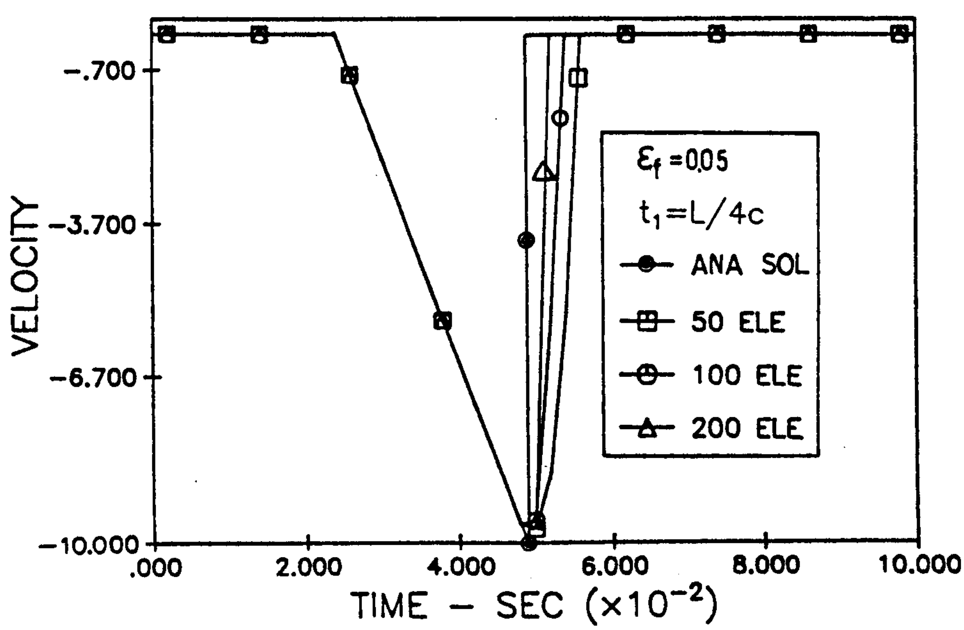

Figure 6. Velocity distribution for problem in Fig. 1 at x = L/4; ϵ_p = 0.01, ϵ_f = 0.05, L = 100, c = 10^3.

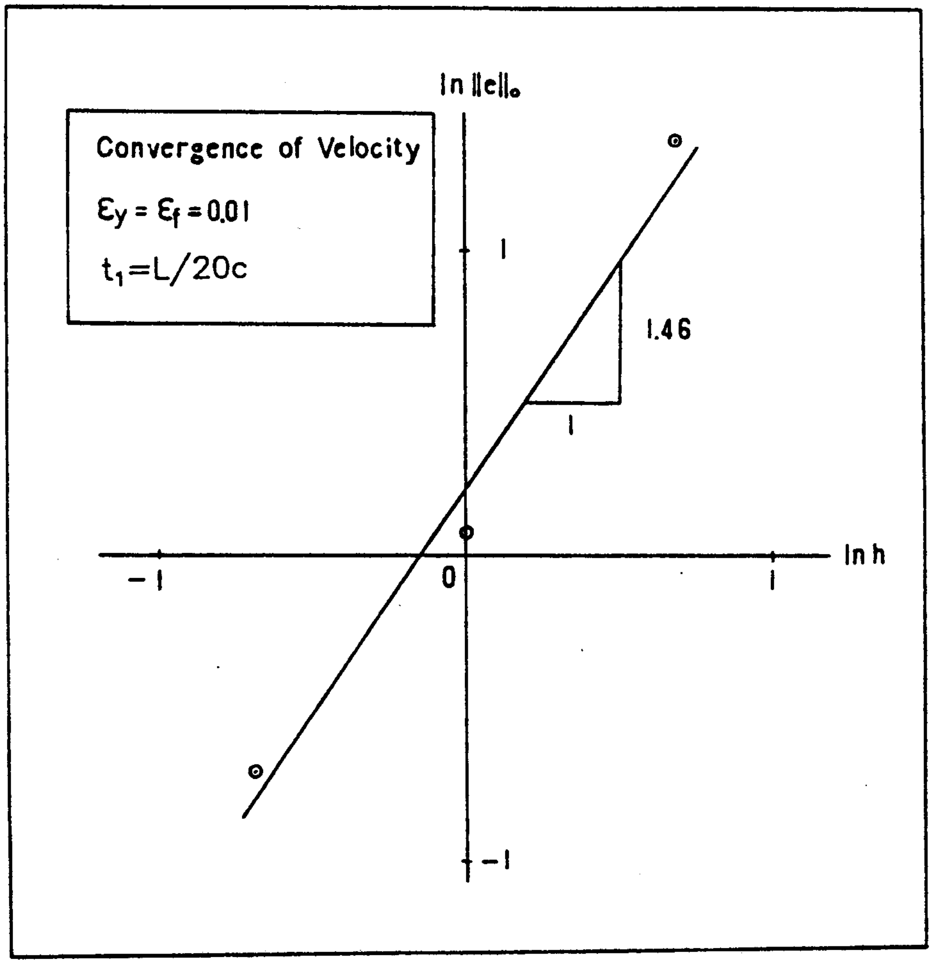

Figure 7. Rate of convergence of velocity for the Sandler-Wright problem, Fig. 1.

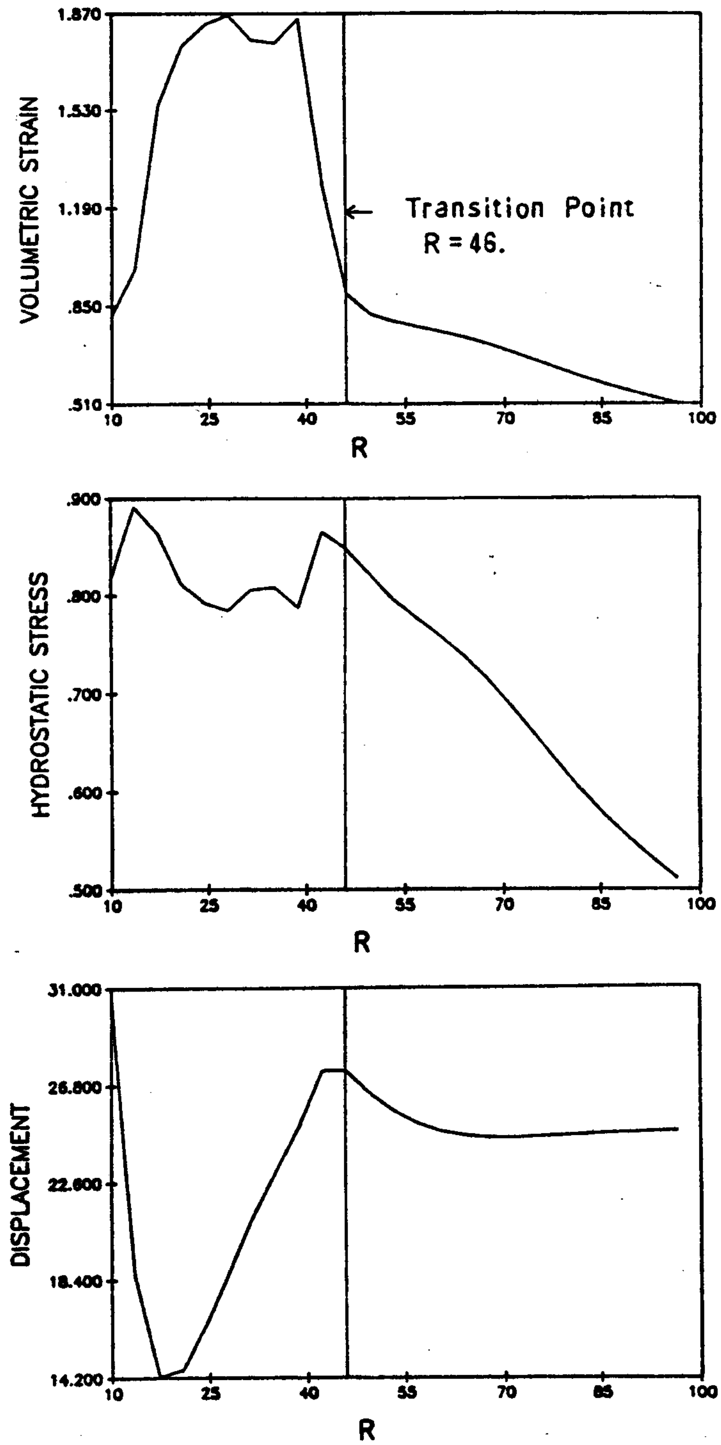

Figure 8a.

Results of spherical wave problem with viscous damping for three meshes.

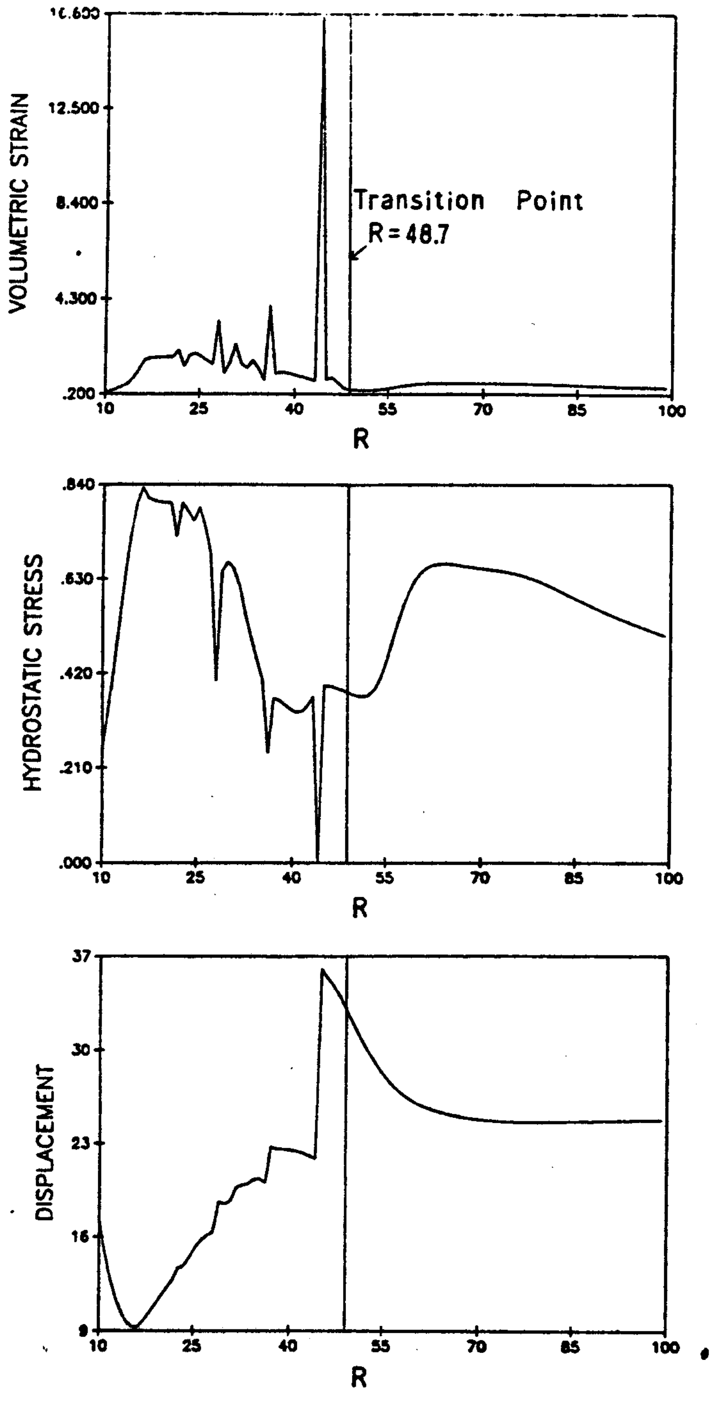

Figure 8b. Results of spherical wave problem with viscous damping for three meshes

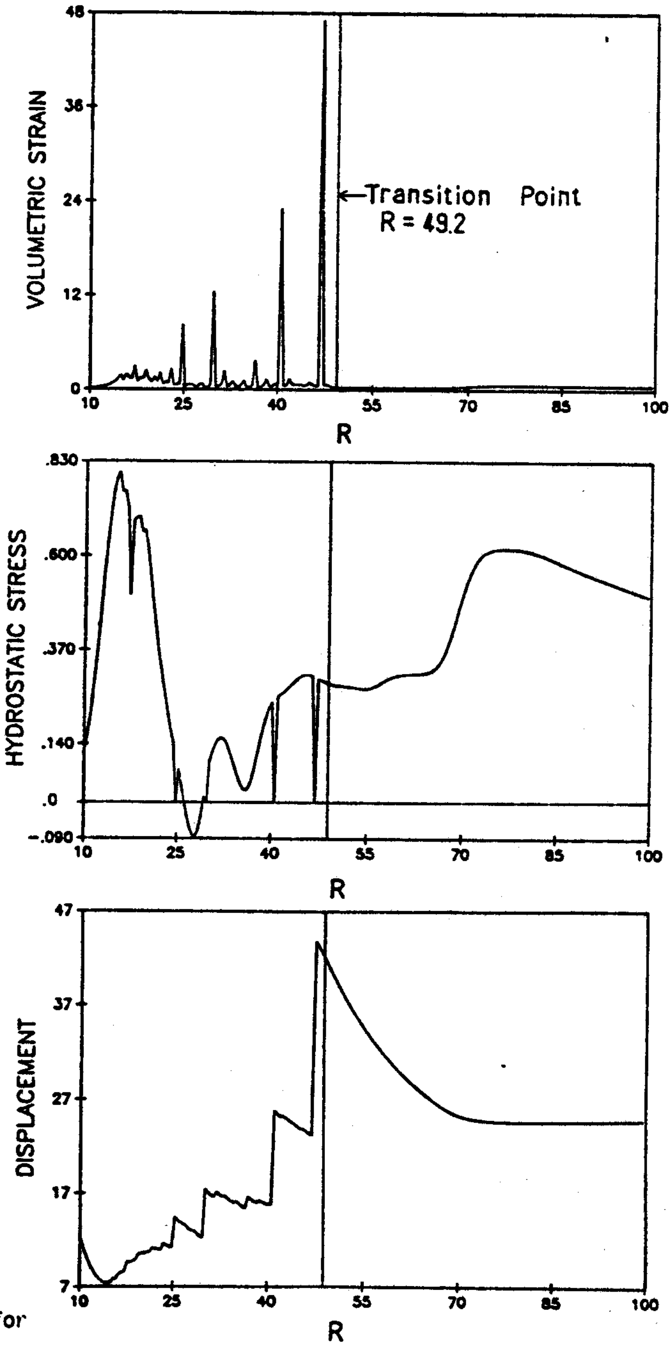

Figure 8c. Results of spherical wave problem with viscous damping for three meshes.

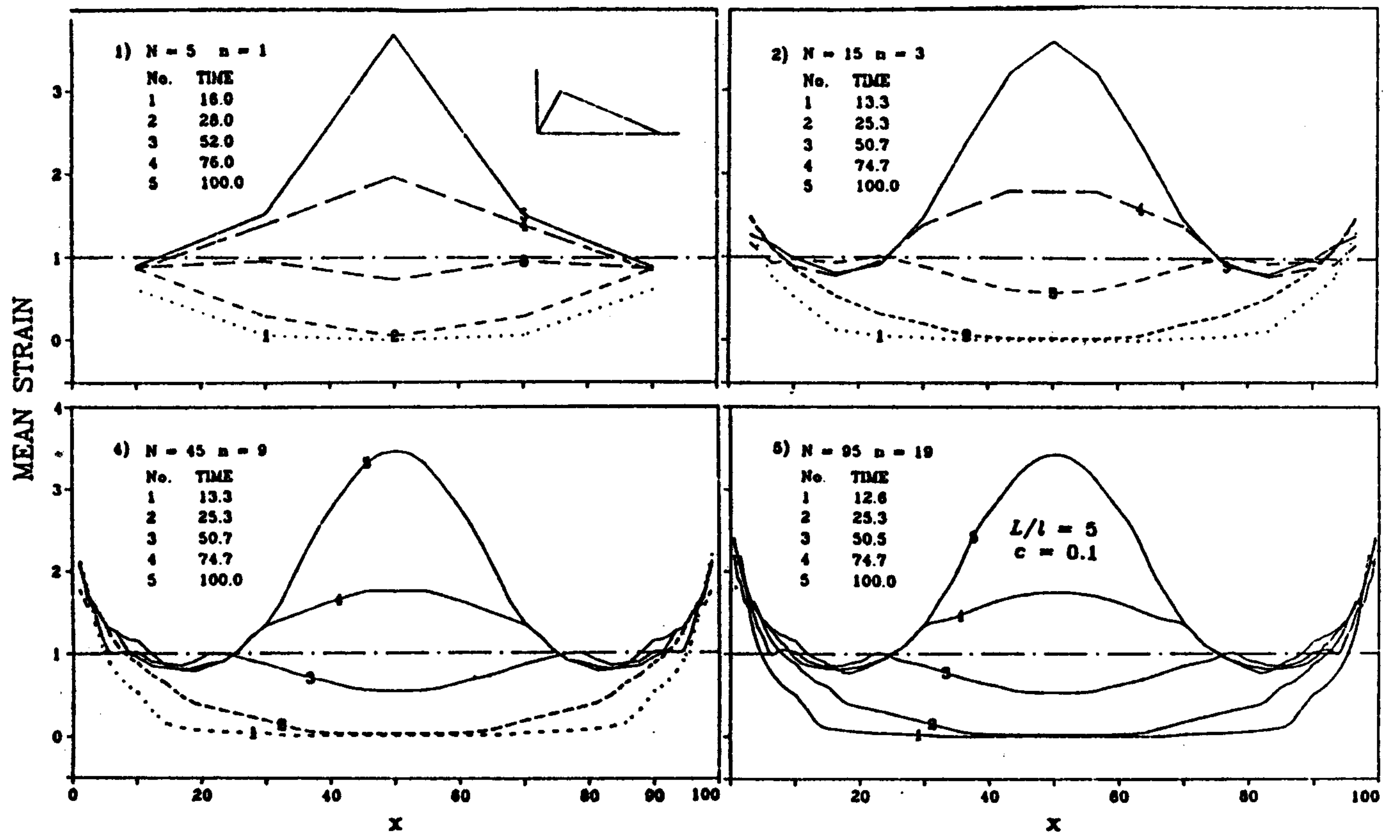

Figure 9. Results for problem in Fig. 4 with nonlocal strain-softening continuum.