

# 1 DAMAGE NONLOCALITY DUE TO MICROCRACK INTERACTIONS: STATISTICAL DETERMINATION OF CRACK INFLUENCE FUNCTION

(Invited Lecture)

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## Abstract

After a brief review of the nonlocal continuum concepts for strain-softening distributed damage due to microcracking, the present lecture describes in detail a recently proposed formulation which is derived by micromechanics analysis of crack interactions. In this formulation, the inelastic stress increments must satisfy a Fredholm integral equation whose kernel is a continuum crack influence function. This function depends on the relative crack orientations, is tensorial, and decays for large distances  $r$  as  $r^{-2}$  (in two dimensions). A statistical determination of the continuum crack influence function is proposed. It consists in averaging the discrete crack influence function over all possible locations of the source crack. Numerical values and diagrams of the typical crack influence function are given. The proposed formulation appears to be a more rational and more realistic model for localization problems of cracking damage in continuous bodies.

Keywords: Damage mechanics, fracture mechanics, crack propagation, finite element analysis, plasticity, strain-softening, localization of damage.

## 1 Introduction

As is now generally agreed, finite element analysis of distributed softening damage cannot be based on a classical, that is, local, constitutive model. Such a model introduces incorrect excessive localizations, spurious size effect, and spurious mesh sensitivity in finite element computations. To overcome these problems one must supplement to the constitutive model some sort of the so-called localization limiter. One effective type of the localization limiter is the nonlocal continuum [1, 2, 7].

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The nonlocal continuum is a concept which was first introduced for different purposes, namely to model the small scale effects of lattice structures or other inhomogeneities in elastic solids ([26, 27, 41] and many others, as reviewed in [2]). An effective type of the nonlocal damage concept, in which the local damage or fracturing strain that figures in the incremental stress-strain relation is replaced by its spatial average, was proposed in [45] and [12] (also [5, 10, 11, 13]).

Introduction of nonlocality into the concept of damage was initially justified by computational arguments—particularly the necessity of preventing localization of strain softening damage to zone of zero volume. A mathematical justification based on physics of the material behavior has been lacking. Intuitively, it has been felt that the need for nonlocality of continuum damage description has something to do with the progressive development of large zones of distributed microcracking, which typically precede sharp macrofractures in quasibrittle materials such as concrete, mortars, rock, toughened ceramics, various types of composites, ice, etc. Some simplified arguments based on a system of microcracks have been shown to result into some form of nonlocality [3, 4], however, they ignored interactions between growing microcracks. These are certain to be significant, as revealed by studies of many researchers, especially Kachanov [36, 37, 39] (also Pijaudier-Cabot and Bažant, [46]; and Bažant and Tabbara, [13]).

Considerable attention has recently been devoted to a special case of nonlocal continuum models for strain softening, in which the nonlocality is introduced through gradients of total strain or damage strain (or plastic strain). These models can be regarded as the first terms of the Taylor series expansion of the nonlocal spatial integral [1, 8]. Attention has also been given to micropolar or Cosserat-type modifications of plasticity [22-25, 43, 47, 48]. Again, however, these gradient type models have so far been justified only by the mathematical need to regularize the boundary value problem, and no convincing physical justification based on micromechanics has been given.

Apart from the problem of continuum modeling of damage, micromechanics of crack systems which are the physical source of damage has been studied extensively for many years [14, 16, 20, 30, 33-37, 39, 40, 42]. However, these studies have focused on the fundamental problem of determining the effective elastic moduli of a solid containing various types of systems of microcracks. Such analysis requires assuming the solid to be in a macroscopically statistically homogeneous (uniform) state. This precludes revealing the properties that govern localization, the principal characteristic of which is the macroscopic statistical nonuniformity of the field of microcracks. Powerful methods have been developed for the problem of determining the effective macroscopic elastic moduli, for example Hill's self-consistent model, methods of periodic cells, method of composite cylinders or composite spheres, variationally based bounds such as Hashin-Shtrickman bounds, various statistical models for macrohomogenous crack arrays, etc.

However, these techniques, representing the homogenization techniques for random inhomogeneities, are not applicable to the development of a continuum model for damage localization. In the homogeneous state, various important interactions between microcracks or other effects cancel each other, but they become essential in the case of spatially nonuniform, statistically nonhomogeneous

deformation. Therefore the homogenization techniques cannot be applied to the present problem. A different approach is needed. Such an approach has been proposed in a recent conference paper [5] and was formulated in detail in a report [13] and a journal article [6]. Application of this approach to localization into a planar band within an infinite layer or infinite space is studied in a forthcoming journal article [32], on which a preliminary conference presentation was given [9]. The effect of microdefect interactions on damage localizations has also recently been taken into account in a different formulation with some similarities to the present one in the work of Okui, Horii and Akuyama [44].

The purpose of the present workshop lecture is two-fold: To review the basic aspects of this new approach to nonlocal continuum damage, and to present a new statistical formulation of the crack influence function which characterizes the nonlocality due to microcracking. The basic idea of this statistical definition has already been mentioned in the addendum to [6].

The problem of continuum smearing of damage may be illustrated by Fig. 1, showing the plot of macro-continuum stress  $\sigma$  and strain  $\epsilon$  in the post-peak strain softening range. The aforementioned classical homogenization techniques, pursued by many authors, provide the value of the secant elastic moduli, characterizing the slope of the line  $\overline{014}$  in this figure. These moduli or the slope of this line are determined under the assumption that the microcracks do not grow during the load increment and remain statistically uniformly distributed.

A much more difficult problem, to which little attention has been devoted [13], is to determine the slope of line  $\overline{12}$  which corresponds to the case when the microcracks are growing and remaining at the critical state of fracture propagation during the loading increment, but under the restriction that the microcracking remain statistically homogeneous (uniform). A still more difficult, and fully realistic, problem is to determine the effect of localization of microcracking during the load increment. This effect is to change the response slope  $\overline{12}$  to the response slope  $\overline{15}$ , which can be less steep or steeper than the slope  $\overline{12}$ . The slope  $\overline{12}$  represents the local constitutive law because in a macroscopically homogeneously deformed solid the interactions between the microcracks cancel each other. These interactions are essential for determining the slope  $\overline{15}$  for the nonlocal response and are the focus of the present formulation.

## 2 Review of new nonlocal damage model

The physical cause of post-peak strain softening is the gradual spread of distributed microcracking. Accordingly, consider an increment of prescribed loads or boundary displacements for an elastic solid that contains, at the beginning of the load step, many microcracks numbered as  $\mu = 1, \dots, N$ . On the macroscale, the microcracks are considered to be smeared, as required by a continuum model. Exploiting the principle of superposition, we may decompose the loading step into two substeps, as follows.

In the first substep, the cracks (already opened) are imagined temporarily

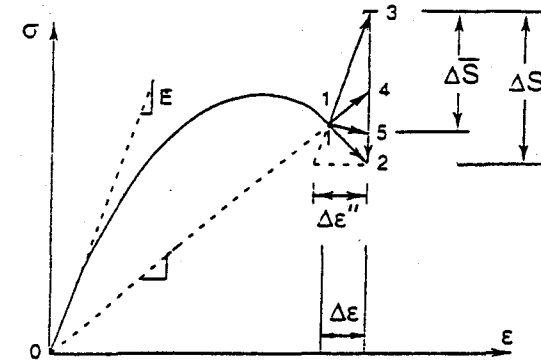


Figure 1: Strain increment and the corresponding local and nonlocal inelastic stress increments

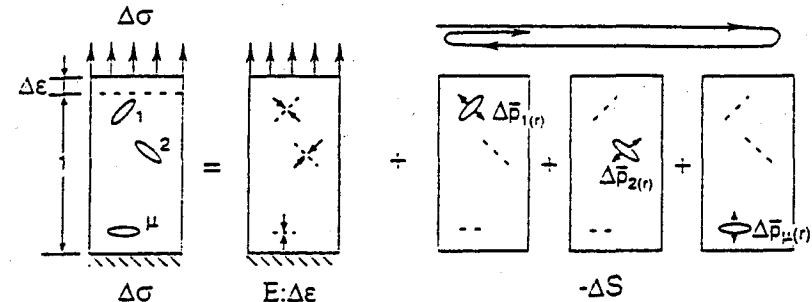


Figure 2: Superposition method for a body with many cracks

“frozen” (or “filled with a glue”), that is, they can neither grow and open wider nor close and shorten. Also, no new cracks can nucleate. The stress increments, caused by strain increments  $\Delta\epsilon$  and transmitted across the temporarily frozen (or glued) cracks (Fig. 2a), are then simply given by  $E : \Delta\epsilon$ . This is represented by the line segment  $\overline{13}$  (Fig. 1) having the slope of the initial elastic modulus  $E$ . In the second substep, the prescribed boundary displacements and loads are held constant, the cracks are “unfrozen” (or “unglued”), and the stresses transmitted across the cracks are relaxed, which is equivalent to applying pressures (surface tractions) on the crack faces (Fig. 2). In response to this pressure, the cracks are now allowed to open wider and grow (remaining in the critical state, according to the crack propagation criterion), or to close and shorten. Also, new cracks are now allowed to nucleate.

Under the assumption that no cracks grow or close (nor new cracks nucleate), the unfreezing (or ungluing) at prescribed increments of loads or boundary displacements that cause macro-strain increment  $\Delta\epsilon$  would engender the stress drop  $\overline{34}$  down to point 4 on the secant line  $\overline{01}$  (Fig. 1). The change of state of the solid would then be calculated by applying the opposite of this stress drop onto the crack surfaces. However, when the cracks propagate (and new cracks

nucleate), a larger stress drop defined by the local strain-softening constitutive law and represented by the segment  $\Delta S = \overline{32}$  in Fig. 1 takes place. Thus, the normal surface tractions  $\Delta p_\mu = \mathbf{n}_\mu \Delta S_\mu \mathbf{n}_\mu$ , representing the normal component of tensor  $\Delta S_\mu$ , must be considered in the second substep as loads  $\Delta p_\mu$  that are applied onto the crack surfaces (Fig. 1), the unit normals of which are denoted as  $\mathbf{n}_\mu$ .

Let us now introduce two simplifying hypotheses: (1) Although the stress transmitted across each temporarily frozen crack varies along the crack, we consider only its average, i.e.,  $\Delta p_\mu$  is constant along each crack. This approximation, which is crucial for our formulation, was introduced by Kachanov [37, 38]. He discovered by numerical calculations that the error is negligible except for the rare case when the distance between two crack tips is at least an order of magnitude less than their size. (2) We consider only Mode I crack openings, i.e. neglect the shear modes (modes II and III).

An effective kind of superposition method is that used by Kachanov [37, 38], which was also used in [19-21, 28, 31], and in a displacement version was introduced already by Collins [18]. In this kind of superposition, one needs to have the solution of the given body for the case of only *one* crack, with all the other cracks considered frozen (Fig. 2). The cost to pay for this advantage is that the pressures to be applied at the cracks are unknown in advance and must be solved.

By virtue of Kachanov's approximation, we apply the superposition method to the average crack pressures only. The opening and the stress intensity factor of crack  $\mu$  ( $\mu = 1, 2, \dots$ ) are approximately characterized by the uniform (or average) crack pressure  $\Delta \bar{p}_\mu$  that acts on a *single* crack within the given solid that has elastic moduli  $E$  and contains no other crack. This pressure is solved from the superposition relation:

$$\Delta \bar{p}_\mu = \langle \Delta p_\mu \rangle + \sum_{\nu=1}^N \Lambda_{\mu\nu} \Delta \bar{p}_\nu \quad \mu = 1, \dots, N \quad (1)$$

Here  $\langle \dots \rangle$  represents the averaging operator over the crack length;  $\Lambda_{\mu\nu}$  are the crack influence coefficients representing the average pressure at the frozen crack  $\mu$  caused by a unit uniform pressure applied on unfrozen crack  $\nu$ , with all the other cracks being frozen; and  $\Lambda_{\mu\mu} = 0$  because the summation in (1) must skip  $\nu = \mu$ . The reason for the notation  $\Delta \bar{p}_\mu$  with an overbar instead of the operator  $\langle \dots \rangle$  is that the unknown crack pressure is uniform and thus its distribution over the crack area never needs to be calculated and no averaging operation actually needs to be carried out. From (1), we obtain

$$\Delta(\mathbf{n}_\mu \bar{S}_\mu \mathbf{n}_\mu) = \langle \Delta(\mathbf{n}_\mu S_\mu \mathbf{n}_\mu) \rangle + \sum_{\nu=1}^N \Lambda_{\mu\nu} \Delta(\mathbf{n}_\nu \bar{S}_\nu \mathbf{n}_\nu) \quad (2)$$

The values of  $\Delta \bar{S}_\mu$  are graphically represented in Fig. 1 by the segment  $\Delta \bar{S} = \overline{35}$ . Due to crack interactions, this segment can be smaller or larger than segment  $\overline{32}$ .

Let us now introduce another simplification: In each loading step, the influence of the microcracks at point  $\xi$  of the macro-continuum upon the microcracks

at point  $\mathbf{x}$  of the macro-continuum is determined only by the dominant microcrack orientation. This orientation is normal to the unit vector  $\mathbf{n}_\mu$  of the maximum principal inelastic macro-stress tensor  $\Delta \bar{S}^{(1)}$  at the location of microcrack  $\mu$ . We use the definition:  $\Delta \bar{S}_\mu^{(1)} = \Delta(\mathbf{n}_\mu \bar{S}_\mu \mathbf{n}_\mu) = [\mathbf{n}_\mu \bar{S}_\mu \mathbf{n}_\mu]_{\text{new}} - [\mathbf{n}_\mu \bar{S}_\mu \mathbf{n}_\mu]_{\text{old}}$ . The subscripts 'new' and 'old' denote the values at the beginning and end of the loading step, respectively. According to this simplification, the dominant crack orientation generally rotates from one loading step to the next. Eq. (2) may now be written as:

$$\Delta \bar{S}_\mu^{(1)} - \sum_{\nu=1}^N \Lambda_{\mu\nu} \Delta \bar{S}_\nu^{(1)} = \langle \Delta S_\mu^{(1)} \rangle \quad (\mu = 1, 2, \dots, N) \quad (3)$$

Now comes the most difficult step. We need to determine the nonlocal field equation for the macroscopic continuum that represents the continuum counterpart of (3). The homogenization theories as known are inapplicable, because they apply only to macroscopically uniform fields, whereas the nonuniformity of the macroscopic field is the most important aspect in the case of localization problems.

The following simple concept has been proposed in [13, 5]: *The desired continuum field equation must be such that its discrete approximation can be written in the form of the matrix crack interaction relation (3).* This concept leads to the following field equation for the continuum approximation of microcrack interactions:

$$\Delta \bar{S}^{(1)}(\mathbf{x}) - \int_V \Lambda(\mathbf{x}, \xi) \Delta \bar{S}^{(1)}(\xi) dV(\xi) = \langle \Delta S^{(1)}(\mathbf{x}) \rangle \quad (4)$$

Indeed, approximation of the integral by a sum over the continuum variable values at the crack centers yields (3). Here we introduced  $\Lambda(\mathbf{x}_\mu, \xi_\nu) = \mathcal{E}(\Lambda_{\mu\nu})/V_c =$  crack influence function;  $V_c$  is a constant that may be interpreted roughly as the volume per crack, and  $\mathcal{E}$  is a certain statistical averaging operator. Some suitable form of such statistical averaging is implied in the macro-continuum smoothing and is inevitable because in a random crack array the characteristics of the individual cracks must be expected to exhibit enormous random scatter.

It must be admitted that the sum in (3) is a somewhat unorthodox approximation of the integral from (4) because the values of the continuum variable are not sampled at certain predetermined points such as the chosen mesh nodes but are distributed at random, that is, at the centers of the random microcracks. A rigorous mathematical theory for such an approximation seems to be lacking at present. Another point to note is that (3) is only one of various possible discrete approximations of (4).

When (4) is approximated by finite elements, it is again converted to a matrix form similar to (3). However, the sum then runs over the integration points of the finite elements. This means the crack pressures (or openings) that are translated into the inelastic stress increments are only sampled at these integration points, in the sense of their density, instead of being represented individually as in (3). Obviously, such a sampling can preserve only the long-range interactions of the

cracks and the short-range averaging. The individual short-range crack interactions will be modified. This is a statistical problem of great conceptual difficulty which we will try to tackle in a simplified manner later in this paper.

For macroscopic continuum smearing, the averaging operator (...) over the crack length now needs reinterpretation. Because of the randomness of the microcrack distribution, the macro-continuum variable at point  $\mathbf{x}$  should represent the spatial average of the effects of all the possible microcrack realizations within a neighborhood of point  $\mathbf{x}$  whose size is roughly equal to the spacing  $\ell$  of the dominant microcracks (which is in concrete approximately determined by the spacing of the largest aggregates); hence,

$$\langle \Delta S^{(1)}(\mathbf{x}) \rangle = \int_V \Delta S^{(1)}(\boldsymbol{\xi}) \Phi(\mathbf{x}, \boldsymbol{\xi}) dV(\boldsymbol{\xi}) \quad (5)$$

where  $\Phi$  represents the same bell-shaped weight function as used in previous nonlocal continuum models (e.g., [45]). This function, which is scalar and is also taken as isotropic (same for every direction), should vanish or almost vanish everywhere outside a domain of diameter roughly equal to  $\ell$ .

Eq. (4) represents a Fredholm integral equation for the unknown  $\Delta \bar{S}^{(1)}(\mathbf{x})$ , which corresponds in Fig. 1b to the segment  $\bar{3}\bar{5}$ . The inelastic strain increment tensors  $\Delta S^{(1)}(\mathbf{x})$  on the right-hand side, which correspond in Fig. 1b to the segment  $\bar{3}\bar{2}$ , are calculated from the strain increments using the given local constitutive law (for example the microplane model or continuum damage theory).

Eq. (4) supplements the total incremental stress-strain relation

$$\Delta \sigma = C_u : \Delta \epsilon - \Delta \bar{S} \quad (6)$$

where  $\Delta \sigma$ ,  $\Delta \epsilon$  are the increments of the stress and strain tensor, and  $C_u$  is the fourth-rank tangential stiffness tensor for unloading of the material. Also  $\Delta \sigma = C_l : \Delta \epsilon$  and  $\Delta \bar{S} = (C_u - C_l) : \Delta \epsilon$  where  $C_l$  denotes the fourth-rank tangential stiffness tensor for loading, whose matrix is not positive definite in the case of strain-softening.

It should be noted that unloading criteria (as well as conditions such as the continuity condition) have nothing to do with nonlocality. They appear only in the constitutive law, which is local. This is a major advantage compared to the previous nonlocal models.

To obtain the complete formulation of the boundary value problem, Eqs. (5) and (6) must be supplemented by the strain-displacement relations, the differential equations of equilibrium for  $\Delta \sigma$  and the boundary conditions for stresses or displacements.

Equation (5) has further been generalized to the case where the dominant microcracks occur in all three principal stress directions [5, 6, 13].

### 3 Crack influence function and its statistical determination

The basic characteristics of the new formulation is the crack influence function  $\Lambda$ , whose rate of decay is determined by a certain characteristic length  $\ell$ . This

function represents the stress field due to pressurizing a single crack in the given elastic structure, all other cracks being absent. In practice, the structure is always finite, and thus the values of  $\Lambda_{\mu\nu}$  should in principle be calculated taking into account the geometry of the structure. However, the crack is often very small compared to the dimensions of the structure. Then, as an approximation, one can use the stress field for a single crack in an infinite body, which is well known and calculated easily (this is of course not possible for cracks very near the boundary of the structure).

The cracks in structures are distributed randomly and their number is vast. Thus, on the macro-continuum level, function  $\Lambda$  cannot characterize the stress fields of the individual cracks. Rather, it should characterize the stress field of a representative crack obtained by a suitable statistical averaging of the random situation on the microstructure level.

A method of rigorous mathematical formulation of the macroscopic continuum crack influence function  $\Lambda$  was briefly proposed in the addendum to [6] and will now be developed in detail.

The crack that is pressurized by unit pressure, as specified in the definition of  $\Lambda$ , will be called the source crack. The crack in the structure on which the influence is to be found will be called the target crack. For the purpose of calculations, the target crack is of course closed and glued, as if it were not present in the solid, and the stresses transmitted across the target crack are calculated assuming the body to be continuous. Function  $\Lambda(0, \boldsymbol{\xi})$  represents the influence of a source crack centered at  $\mathbf{x} = 0$  on a target crack centered at  $\boldsymbol{\xi}$ .

At the given macro-continuum point, there may or may not be a crack in the microstructure. Function  $\Lambda$  corresponding to that point must reflect the smeared statistical properties of all the possible microcracks occurring near that point. To do this, we must idealize the random crack arrangements in some suitable manner.

We will suppose that the center of the source crack can occur randomly anywhere within a square of size  $s$  centered at point  $\mathbf{x} = 0$ ; see Fig. 3a, where various possible cracks are shown by the dashed curves, but only one of these, the crack showed by the solid lines, is actually realized. The value of  $s$  is imagined to represent the typical spacing of the dominant cracks. In a material such as concrete, approximately  $s \approx m d_a$  where  $d_a$  = spacing of the largest aggregate pieces and  $m$  = coefficient larger than 1 but close to 1 ( $m$  would equal  $d_a$  if the aggregates were arranged at the ideal cubic packing and if there were no mortar layers within the contact zones). To simplify the statistical structure of the system of dominant cracks, one may imagine the material to be subdivided by a square mesh of size  $s$  as shown in Fig. 3b, with one and only one crack center occurring within each square of the mesh. This is of course a simplification of reality because a square mesh introduces a certain directional bias (as is well known from finite element analysis of fracture). It would be more realistic to assume that the possible zone of occurrence of the center of each crack is not a square but has a random shape and area about  $s \times s$ , and that all these areas are arranged randomly. But this would be too difficult for statistical purposes, and probably unimportant with respect to the other simplifications of the model.

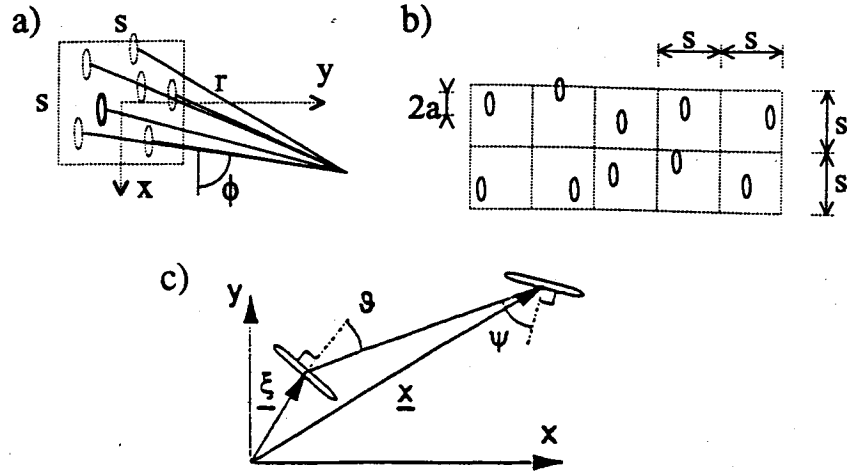


Figure 3: (a) Various possible random locations of the source crack influencing a target crack, (b) dominant cracks appearing in regions of size  $s$  that determine their typical spacing, and (c) general orientations of the source crack and the target crack relative to the influence ray

Let us now center coordinates  $x$  and  $y$  in the center of the square  $s \times s$ , as shown in Fig. 3a, and consider the influence of a source crack within this square on a target crack at coordinates  $\xi \equiv (\xi, \eta)$ . The macroscopic crack influence function should describe the influence of any possible source crack within the given square in the average smeared macroscopic sense. Therefore,  $\Lambda(\mathbf{0}, \xi)$  is defined as the mathematical expectation  $\mathcal{E}$  with regard to all the possible random realizations of the source crack center within the given square  $s \times s$ , that is

$$\Lambda(\mathbf{0}, \xi) = \mathcal{E} [\sigma^{(1)}(\xi - x, \eta - y)] \quad (7)$$

The vector  $(\xi - x, \eta - y) = \mathbf{r}$  = vector from the center  $\mathbf{x} \equiv (x, y)$  of a source crack to the center  $\xi \equiv (\xi, \eta)$  of the target crack. In detail,

$$\Lambda(\mathbf{0}, \xi) = \frac{1}{s^2} \int_{-s/2}^{s/2} \int_{-s/2}^{s/2} w(x, y) \sigma^{(1)}(\xi - x, \eta - y) dx dy \quad (8)$$

Here  $\sigma^{(1)}$  is the stress in the direction perpendicular to the target crack caused by applying unit pressure on the faces of the source crack, and the integrals represent the statistical averaging over the square  $s \times s$ . We have inserted in this expression certain specified weights  $w(x, y)$ . At first one might think that uniform weights  $w$  might be appropriate, but that would not be realistic near the boundaries of the square because a crack cannot intersect a crack centered in the adjacent square, and in practice would not even lie too close to it. Rigorously, one would have to consider the joint probability of the occurrences of the crack center locations in the adjacent squares, but this would be too complicated. We prefer to simply reduce the probability of occurrence of the source crack as the

boundary of the square is approached. For numerical computations we choose a bell-shaped function in both the  $x$  and  $y$  directions, given as

$$w(x, y) = w_0 \left[ 1 - \left( \frac{2x}{s} \right)^2 \right]^2 \left[ 1 - \left( \frac{2y}{s} \right)^2 \right]^2, \quad w_0 = \frac{225}{64} \quad (9)$$

for  $x \leq s/2, y \leq s/2$ , and  $w(x, y) = 0$  otherwise; constant  $w_0$  is selected so the integral of  $w(x, y)$  over the square  $s \times s$  be equal to  $s^2$ . It may be added that there is also a practical reason for introducing this weight function. If the weights were uniform over the square, function  $\Lambda$  would not have a smooth shape, which would be inconvenient and probably also unrealistic for a continuum model.

The stress field in an infinite medium caused by unit pressure applied on the faces of one single crack is described for the two-dimensional case by the well-known Westergaard's solution (see, e.g., [15], or [29]) which is given in [5, 6]. The component  $\sigma^{(1)}$  to be integrated in (8) depends on the orientation of the target crack, which is related to the macroscopic stress field. As a basic situation, we assume the directions of the maximum principal stress at the source and target points to be the same, which means that  $\sigma^{(1)}$  is the stress in the  $y$ -direction. The integral in (8) is difficult to evaluate analytically, and it is better to use numerical integration to obtain  $\Lambda$ . However, the asymptotic properties of function  $\Lambda$  for large  $r$  can be determined easily [13, 6] by considering the lines of influence from various possible source cracks to the given target crack as shown in Fig. 3a. If the target crack is very far from the square in which the source crack is centered, all the possible rays of influence are nearly equally long and come from nearly the same direction. Therefore, the integral in (8) should exactly preserve the long range asymptotic field.

As shown in [13, 6], the long-range ( $r \rightarrow \infty$ ) asymptotic crack influence function is

$$\Lambda_{\infty}(\mathbf{x}, \xi) = -k(r) \frac{1}{2\ell^2} [\cos 2\vartheta + \cos 2\psi + \cos 2(\vartheta + \psi)] \quad (10)$$

where  $\vartheta$  and  $\psi$  are the angles of ray  $r$  with the normals of the source crack and the target crack (see Fig. 3c),  $k(r) = a^2/r^2$ , and  $\ell$  is a certain constant representing what may be called the characteristic length for crack interactions.

It is convenient to replace the function  $k(r) = a^2/r^2$  by a function of the same asymptotic properties for  $r \rightarrow \infty$  which does not have a singularity at  $r = 0$ ;

$$k(r) = \left( \frac{ar}{r^2 + \ell^2} \right)^2 \quad (11)$$

It is now possible to represent the complete crack influence function given by (8) in the form:

$$\Lambda(\mathbf{0}, \xi) = \Lambda_{\infty}(\xi, \eta) + \Lambda_1(\xi, \eta) \quad (12)$$

where  $\Lambda_1$  represents a difference which is decaying to infinity faster (i.e., as a higher power of  $r$ ) than  $\Lambda_{\infty}$  and can therefore be neglected for sufficient distances

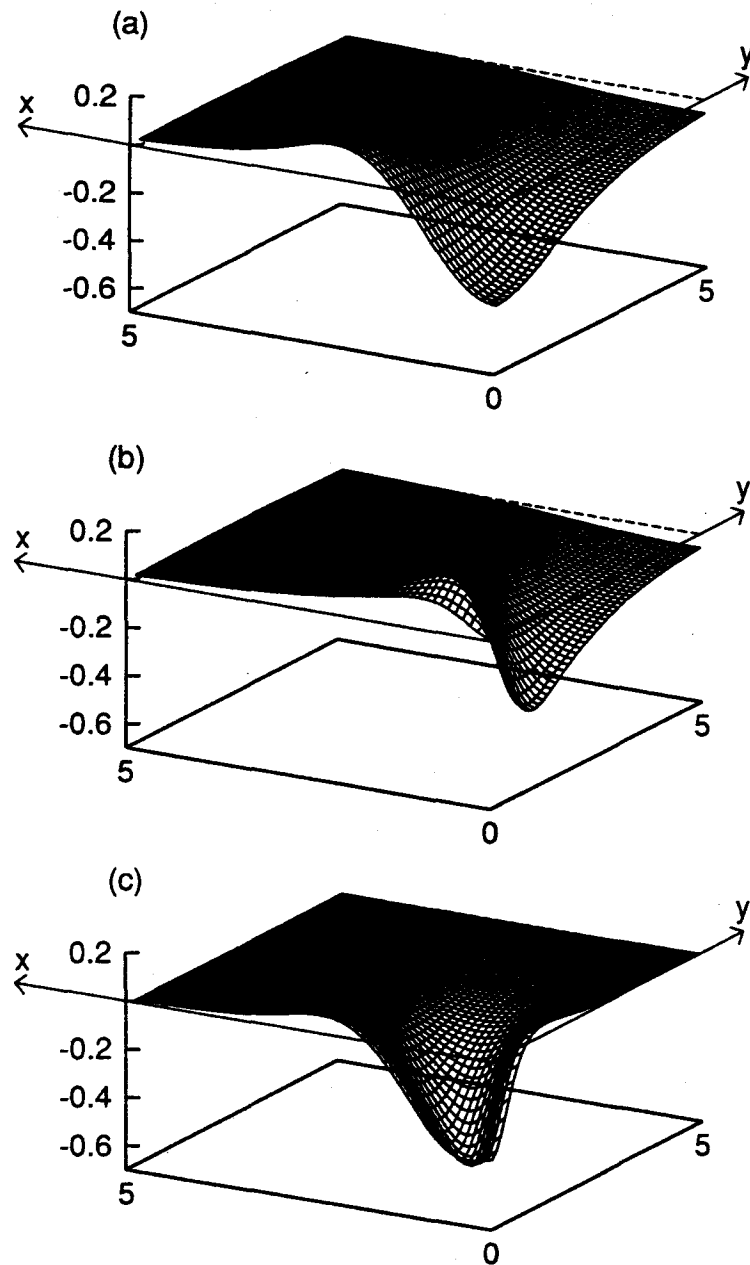


Figure 4: (a) Total crack influence function for the case of parallel source and target cracks, (b) analytical expression having the correct long-range asymptotic field, and (c) difference of the crack influence functions in (a) and (b)

Table 1: Values of the difference between the calculated crack influence function and an approximate analytical expression having the correct long-range asymptotic field

$x$	$y$	0.05	0.55	1.05	1.55	2.05	2.55	3.05	3.55	4.05
0.00		-.4133	-.1263	.0345	-.0388	.0230	.0126	.0070	.0040	.0024
0.50		-.4260	-.4157	-.1550	-.0396	-.0075	.0003	.0017	.0016	.0012
1.00		-.2835	-.3379	-.2435	-.1184	-.0504	-.0208	-.0087	-.0037	-.0016
1.50		-.0806	-.1205	-.1315	-.0937	-.0543	-.0288	-.0150	-.0078	-.0041
2.00		.0440	.0160	-.0214	-.0356	-.0317	-.0225	-.0144	-.0088	-.0053
2.50		.0680	.0510	.0210	-.0015	-.0112	-.0123	-.0101	-.0074	-.0051
3.00		.0408	.0333	.0190	.0063	-.0016	-.0051	-.0058	-.0051	-.0040
3.50		.0221	.0191	.0128	.0064	.0016	-.0013	-.0027	-.0031	-.0028
4.00		.0129	.0115	.0086	.0053	.0024	.0003	-.0010	-.0016	-.0018
4.50		.0080	.0073	.0059	.0041	.0024	.0010	-.0001	-.0007	-.0010
5.00		.0052	.0049	.0041	.0031	.0021	.0011	.0004	-.0002	-.0005

$r$  from the center of the source crack. This difference is minimized with the parameter  $\ell$  equal to the half crack length  $a$ .

Calculations have been made for the case that the target crack is parallel to the source crack and  $a/s = 0.25$ . The values of  $\Lambda$  were evaluated by numerical integration of (8) using a dense square mesh; see Fig. 4a. The analytical expression for the asymptotic crack influence function from (10) is plotted in Fig. 4b, and after its subtraction from the  $\Lambda_\infty$  values, the plot of the difference of the crack influence function shown in Fig. 4c was obtained. For the numerical values of  $\Lambda_1$ , see Table 1.

Function  $\Lambda_1(x, y)$  obviously depends on the relative crack size  $a/s$ . However, it has been found that it depends on  $a/s$  only very little when  $a/s \geq 0.25$ . For smaller  $a/s$ , the crack interactions are probably mostly unimportant. So perhaps a single crack influence function expression could be used for all the cases.

Another case to study is that of nonparallel target cracks. This is left for a subsequent journal article.

Analytical closed-form expressions for the stress field of a pressurized penny-shaped crack in three dimensions are also available. They have been used to determine the asymptotic crack influence function in three dimensions [5, 6]. A statistical definition in three dimensions that is analogous to (8) can obviously be also written. It will be studied in a subsequent journal article, along with other open questions.

## 4 Conclusions

1. The recently proposed nonlocal continuum damage model based on micro-

mechanical analysis of crack interactions, which has been reviewed in the present lecture, appears to represent a more rational and more realistic model than the previous nonlocal formulations. The analysis of crack interactions proves that the macroscopic continuum model ought to be nonlocal, except for deformations that are nearly homogeneous over regions that are much larger than the microcrack spacing and dimensions.

2. The continuum crack influence function, which describes the effect of a source crack pressurized by a unit pressure on the average stress across a closed target crack, may be defined by averaging over all the possible random realizations of the source crack within a square domain whose size represents the typical spacing of the dominant cracks. The numerical table of this function given in this paper can be used in practice.

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