Scaling of structural failure*

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This article attempts to review the progress achieved in the understanding of scaling and size effect in the failure of structures. Particular emphasis is placed on quasibrittle materials for which the size effect is important and complicated. After reflections on the long history of size effect studies, attention is focused on three main types of size effects, namely the statistical size effect due to randomness of strength, the energy release size effect, and the possible size effect due to fractality of fracture or microcracks. Definitive conclusions on the applicability of these theories are drawn. Subsequently, the article discusses the application of the known size effect law for the measurement of material fracture properties, and the modeling of the size effect by the cohesive crack model, nonlocal finite element models and discrete element models. Extensions to compression failure and to the rate-dependent material behavior are also outlined. The damage constitutive law needed for describing a microcracked material in the fracture process zone is discussed. Various applications to quasibrittle materials, including concrete, sea ice, fiber composites, rocks and ceramics are presented. There are 377 references included in this article.

1 INTRODUCTION

Scaling is the most important aspect of every physical theory. If scaling is not understood, the theory itself is not understood. Thus it is not surprising that the question of scaling has occupied a central position in many problems of physics and engineering. The problem of scaling acquired a prominent role in fluid mechanics more than a hundred years ago and provided the impetus for the development of the boundary layer theory, initiated by Prandtl (1904).

In solid mechanics, the scaling problem of main interest is the effect of structure size on its nominal strength. This is a very old problem, older than the mechanics of materials and structures. The question of size effect was discussed already by Leonardo da Vinci (1500s), who stated that "Among cords of equal thickness the longest is the least strong" (Fig 1a). He also wrote that a cord "is so much stronger...as it is shorter." This rule implies inverse proportionality of the nominal strength to the length of a cord, which is of course a strong exaggeration of the actual size effect.

More than a century later, the exaggerated rule of Leonardo was rejected by Galileo (1638) in his famous book (Fig 2) in which he founded mechanics of materials. He argued that cutting a long cord at various points (F, D, and E in Fig 1b) should not make the remaining part stronger. He pointed out, however, that a size effect is manifested in the dissimilar shapes of animal bones when small and large animals are compared (Fig 1c).

Half a century later, a major advance was made by Mariotte (1686). He experimented with ropes, paper, and tin and made the observation, from today's viewpoint revolutionary, that "a long rope and a short one always support the same

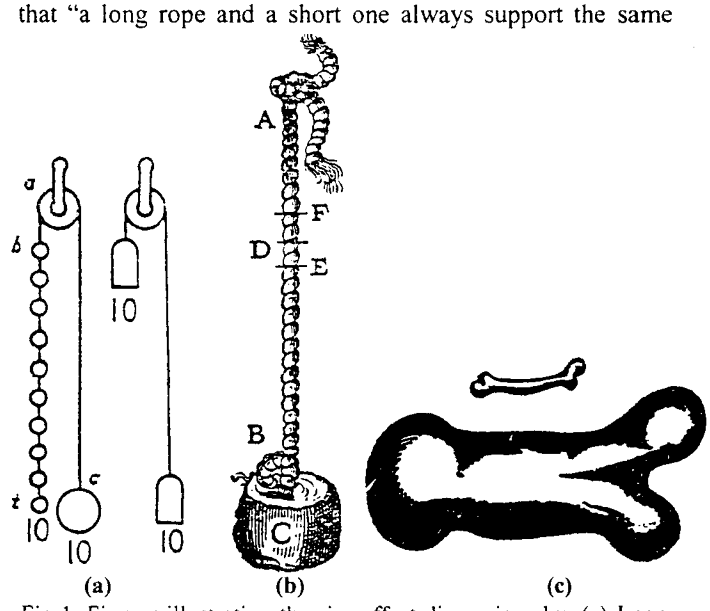

Fig 1. Figures illustrating the size effect discussions by: (a) Leonardo da Vinci in the early 1500s, and (b, c) Gallileo Galilei in 1638

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weight unless that in a long rope there may happen to be some faulty place in which it will break sooner than in a shorter." He proposed that this results from the principle of "the Inequality of the Matter whose absolute Resistance is less in one Place than another." In qualitative terms, he thus initiated the statistical theory of size effect, two and half centuries before Weibull. The probability theory, however, was at its birth at that time and not yet ready to handle the problem.

Marriote's conclusions were later rejected by Thomas Young (1807). He took a strictly deterministic viewpoint and stated that "a wire 2 inches in diameter is exactly 4 times as strong as a wire 1 inch in diameter," and that "the length has no effect either in increasing or diminishing the cohesive strength." This was a setback, but he obviously did not have in mind the random scatter of material strength. Later more extensive experiments clearly demonstrated the presence of size effect for many materials.

The next major advance was the famous paper of Griffith (1921). In that paper, he not only founded fracture mechanics but also introduced fracture mechanics into the study of size effect. He concluded that "the weakness of isotropic solids...is due to the presence of discontinuities or flaws...The effective strength of technical materials could be increased 10 or 20 times at least if these flaws could be eliminated." He demonstrated this conclusion in his experiments showing that the nominal strength of glass fibers was raised from 42,300 psi for the diameter of 0.0042 inch to 491,000 for the diameter of 0.00013 inch. In Griffith's view, however, the flaws or cracks deciding failure were only microscopic, which is not true for quasibrittle materials. Their random distribution determined the local macroscopic strength of the material. Thus, Griffith's work represented a physical basis of Mariotte's statistical concept, rather than a discovery of a new type of size effect.

DISCORSI
E
DIMOSTRAZIONI
MATEMATICHE,
intorno à due nuoue ſcienze
Attenenti alla
MECANICA & i MOVIMENTI LOCALI;
del Signor
GALILEO GALILEI LINCEO,
Filoſofo e Matematico primario del Sereniſſimo
Grand Duca di Toſcana.
Con vna Appendice del centro di grauità d'alcuni Solidi.

IN LEIDA,
Appreſſo gli Elſevirii. M. D. C. XXXVIII.

Fig 2. Title page of the famous book of Galileo (1638) which founded mechanics of materials

With the exception of Griffith, theoreticians in mechanics of materials paid hardly any attention to the question of scaling and size effect–an attitude that persisted into the 1980s. The reason doubtless was that all the theories that existed prior to the mechanics of distributed damage and quasibrittle (nonlinear) fracture use a failure criterion expressed in terms of stresses and strains (including the elasticity with allowable stress, plasticity, fracture mechanics with only microscopic cracks or flaws) exhibit no size effect (Bažant 1984). Therefore, it was universally assumed (until about 1980) that the size effect, if observed, was inevitably statistical. Its study was supposed to belong to the statisticians and experimentalists, not mechanicians. For example, the subject was not even mentioned by Timoshenko in 1953 in his comprehensive treatise *History of strength of materials*.

Progress was nevertheless achieved in probabilistic and experimental investigations. Peirce (1926) formulated the weakest-link model for a chain and introduced the extreme value statistics originated by Tippett (1925), which was later refined by Fréchet (1927), Fischer and Tippett (1928), von Mises (1936), and others (see also Freudenthal, 1968). This progress culminated with the work of Weibull (1939) in Sweden (see also Weibull, 1949, 1956).

Weibull (1939) reached a crucial conclusion: The tail distribution of extremely small strength values with extremely small probabilities cannot be adequately described by any of the known distributions. He proposed for the extreme value distribution of strength a power law with a threshold. Others (see, *eg*, Freudenthal, 1968; Selected Papers 1981) then justified this distribution theoretically, by probabilistic modeling of the distribution of microscopic flaws in the material. This distribution came to be known in statistics as the Weibull distribution.

With Weibull's work, the basic framework of the statistical theory of size effect became complete. Most subsequent studies until the 1980s dealt basically with refinements, justifications and applications of Weibull's theory (*eg*, Zaitsev and Wittmann, 1974; Mihashi and Zaitsev, 1981; Zech and Wittmann, 1977; Mihashi, 1983; Mihashi and Izumi, 1977; see also Carpinteri, 1986, 1989; Kittl and Diaz, 1988, 1989, 1990). It was generally assumed that, if a size effect was observed, it had to be of Weibull type. Today, we know this is not the case.

Weibull's statistical theory of size effect applies to structures that: 1) fail (or must be assumed to fail) right at the initiation of the macroscopic fracture (in detail, see Section 7), and 2) have at failure only a small fracture process zone causing negligible stress redistribution. This is the case, especially for metal structures embrittled by fatigue.

But this is not the case for *quasibrittle* materials. These materials are characterized by the existence of a large fracture process zone with distributed cracking damage. They include various types of concrete and mortar made with various cements and admixtures, polymers or asphalt (especially high strength concretes), various rocks, ice (especially sea ice), many composites (fiber or particulate), fiber-reinforced concretes, toughened ceramics, bone, biological shells, stiff clays, cemented sands, grouted soils, coal, paper, wood, wood particle board, various refractories, some special tough metal alloys, filled elastomers, etc. The size effect in these materials is due to stable growth of a large fracture or a large fracture process zone with microcracking before the maximum load is attained, and in particular to the stress redistribution and the release of stored energy engendered by such a large fracture and microcracking zone.

The most widely used quasibrittle material is concrete. Thus, the study of its fracture mechanics, initiated by Kaplan (1961), prepared the ground for the discovery of a different type of size effect. Kesler, Naus, and Lott (1971) concluded that the classical linear elastic fracture mechanics of sharp cracks does not apply to concrete. This conclusion was strengthened by Walsh (1972, 1976), who tested geometrically-similar notched beams of different sizes and plotted the results in a double logarithmic diagram of nominal strength *versus* size (Fig 3). Without attempting a mathematical description, Walsh made the point that this diagram deviates from a straight line of slope −1/2, and that this deviation signifies a departure from linear elastic fracture mechanics (LEFM).

A major step was made by Hillerborg *et al* (1976). Inspired by the softening and plastic fracture process zone models of Barenblatt (1959, 1962) and Dugdale (1960) (extended by Knauss, 1973, 1974; Wnuk, 1974; and Kfouri and Rice, 1977), they formulated the fictitious (or cohesive) crack model. Further they showed by finite element analysis that the failure of unnotched plane concrete beams in bending exhibits a size effect, and that the size effect is not of the Weibull type.

At the same time, Bažant (1976) analytically demonstrated that localization of strain-softening damage into bands engenders a size effect on post-peak deflections and energy dissipation of structures. In the early 1980s, Bažant (1983, 1984) derived, on the basis of approximate energy release analysis, a simple formula for the size effect law which describes the size effect on nominal strength of quasibrittle structures failing after large stable crack growth. He also formulated the crack band model, which permits capture of this size effect by finite elements in a very simple way (Bažant, 1982; Bažant and Oh, 1983) and is nowadays almost the only concrete fracture model used in industry and commercial codes (*eg* code DIANA, Rots, 1888; or code SBETA, Červenka and Pukl, 1994). A more general nonlocal approach to strain-softening damage capable of describing the size effect followed soon (Bažant, Belytschko, and Chang, 1984; Bažant, 1984; Pijaudier-Cabot and Bažant, 1987; Bažant and Pijaudier-Cabot, 1988; Bažant and Lin, 1988a, b; etc).

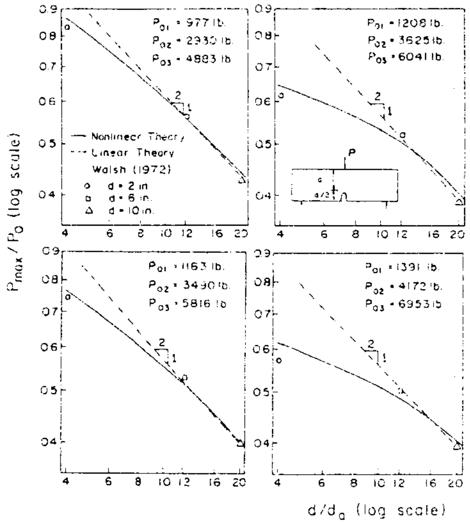

Fig 3. Data points obtained by Walsh (1972) in four of his six series of tests of geometrically similar notched three-point bend beams, and the fitting curves obtained by Bažant and Oh (1983) by finite element analysis with the crack band model

Beginning with the mid-1980s, the interest in the quasibrittle size effect surged enormously and many researchers made noteworthy contributions; to name but a few: Planas and Elices (1988, 1989, 1993), Petersson (1981), and Carpinteri (1986). The size effect has become a major theme at conferences on concrete fracture (Bažant (ed), 1992; Mihashi *et al* (eds), 1993; Wittmann (ed), 1995).

It was also recognized that measurements of the size effect on the maximum load allow a simple way to determine the fracture characteristics of quasibrittle materials. This line of investigation culminated with the Cardiff workshop (Barr, 1995) at which representatives of American and European societies endorsed a unified recommendation for a test standard based on the measurement of maximum loads alone.

An intriguing idea was injected into the study of size effect by Carpinteri *et al* (1993, 1995a, b, c), Carpinteri (1994a, b), and Carpinteri and Chiaia (1995). Inspired by numerous recent studies of the fractal characteristics of cracks in various materials have been conducted (Mandelbrot, 1984; Brown, 1987; Mecholsky and Mackin, 1988; Cahn, 1989; Chen and Runt, 1989; Hornbogen, 1989; Peng and Tian, 1990; Saouma *et al*, 1990; Bouchaud *et al*, 1990; Chelidze and Gueguen, 1990; Issa *et al*, 1992; Long *et al*, 1991; Måløy *et al*, 1992; Mosolov and Borodich, 1992; Borodich, 1992; Lange *et al*, 1993; Xie, 1987, 1989, 1993; Xie *et al* 1994, 1996; Saouma and Barton, 1994; Feng *et al*, 1995; etc), Carpinteri and Chiaia (1994) proposed that the difference in fractal characteristics of cracks or microcracks at different scales of observation is the principal source of size effect in concrete. However, recent mechanical analysis by Bažant (1997b) casts doubt on this proposition.

At present, there are three basic theories of scaling in solid mechanics:

1. *Weibull statistical theory of random strength* (Weibull 1939; see Section 7)
2. *Theory of stress redistribution and fracture energy release caused by large cracks* (Bažant, 1983, 1984; Section 3-5).
3. *Theory of crack fractality*, in which two types may be distinguished.
   a. *Invasive* fractality of the crack surface (*ie*, a fractal nature of surface roughness) (Carpinteri *et al*, 1993, 1995a, b, c; Carpinteri 1994a, b; Section 6), and
   b. *Lacunar* fractality (representing a fractal distribution of microcracks) (Carpinteri and Chiaia, 1995; Section 8).

Aside from these basic theories, there are four indirect size effects:

1. The boundary layer effect, which is due to material heterogeneity (*ie*, the fact that the surface layer of heterogeneous material such as concrete has a different composition because the aggregates cannot protrude through the surface), and to Poisson effect (*ie*, the fact that a plane strain state on planes parallel to the surface can exist in the core of the test specimen but not at its surface).
2. The existence of a three-dimensional stress singularity at the intersection of crack edge with a surface, which is also caused by the Poisson effect (Bažant and Estenssoro, 1979). This causes the portion of the fracture process zone near the surface to behave differently from that in the interior.
3. Time-dependent size effect caused by diffusion phenomena such as the transport of heat or the transport of moisture and chemical agents in porous solids (this is manifested *eg*, in the effect of size on shrinkage and drying creep, due to size dependence of the drying half time (Bažant and Kim, 1991) and its effect on shrinkage cracking (Planas and Elices, 1993).
4. Time-dependence of the material constitutive law, particularly the viscosity characteristics of strain softening, which impose a time-dependent length scale on the material (Tvergaard and Hutchinson 1982, 1987; Tvergaard and Needleman 1992, Sluys 1992).

Today, the study of scaling in quasibrittle materials is a lively, rapidly moving field. Despite considerable success in recent research, major questions remain open. The review that follows will focus on the three main theories of size effect and the indirect ones will be left out of consideration.

2 POWER SCALING AND TRANSITIONAL SIZE EFFECT

The basic and simplest type of scaling is obtained in any physical theory in which there is no characteristic length. We consider geometrically similar systems, for example the beams shown in Fig 4a, and are interested in the response Y (representing for example the maximum stress of the maximum deflection) as a function of the characteristic size (dimension) D of the structure; $Y = Y_0 f(D)$. We consider three structure sizes 1, D, and D' (Fig 4a). If size 1 is taken as the reference size, the responses for sizes D and D' are $Y = f(D)$ and $Y' = f(D')$. However, since there is no characteristic lengths, size D can also be taken as the reference size. This means that

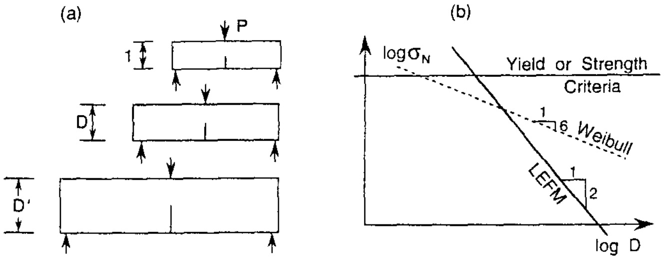

Fig 4. (a) Geometrically similar structures of different sizes D and (b) power scaling laws

$$\frac{Y'}{Y} = \frac{f(D')}{f(D)} = f\left(\frac{D'}{D}\right) \tag{1}$$

This is a functional equation for the unknown scaling law $f(D)$. It has one and only one solution, namely the power law. This may be shown by differentiating (Eq 1) with respect to D and then substituting $D' = D$, which yields the differential equation

$$\mathrm{d}f(D)/f(D) = m\mathrm{d}D/D \tag{2}$$

in which $m = \mathrm{d}f(D)/\mathrm{d}D$ for $D = 1$. This differential equation can be easily solved by separation of variables. The initial condition is $f(1) = 1$, and the solution is a power law with unknown constant exponent m:

$$f(D) = D^m \tag{3}$$

The foregoing derivation is true for every physical theory in which there is no characteristic length. In solid mechanics, such theories include elasticity and plasticity, as well as LEFM (the cracks must, of course, be geometrically similar; this excludes metallic structures with small flaws, which are a material property and do not change with the structure size).

The exponent m can be determined only if the failure criterion of the material is taken into account. For elasticity with allowable stress, or elastoplasticity with any failure criterion (*eg*, yield surface) expressed in terms of the stress or strain components, the exponent is $m = 0$ when response Y represents the stress, for example the maximum stress, or the stress at a particular point, or the nominal stress at failure (Bažant, 1994). This means that, according to all these classical theories, geometrically similar structures of different sizes fail at the same nominal stress (or at the same maximum stress). This is the basic, reference case, in which we say that there is no size effect (on the nominal strength).

Because $m = 0$ in plasticity, the size effect in structures is measured by the nominal strength. The nominal strength is a parameter of the maximum load P, defined as $\sigma_N = c_n P/bD$, in which b is the structure thickness in the third dimension, for the case of two-dimensional similarity, or $\sigma_N = c_n P/D^2$, in which c_n is a constant depending on structure shape but not size, which may be used to make σ_N coincide, for example, with the maximum stress or the average stress, or the stress at any particular point.

In LEFM, the situation is different, namely the exponent of the power law for the nominal strength is $m = -1/2$, provided the geometrically similar structures have geometrically similar cracks or notches. This may be derived by applying Rice's J-integral (Bažant, 1994).

In the plot of the logarithm of nominal strength *versus* the logarithm of size, the power law is a straight line (Fig 4b). For plasticity or elasticity with an allowable stress, the slope of this line is 0. For LEFM, the slope of this line is $-1/2$.

We may digress at this point to mention that, for Weibull-type statistical theories (in which the threshold value may usually be taken as 0), the scaling law is also a power law. According to the tests of Zech and Wittmann, the exponents for concrete are typically $-1/6$ or $-1/4$ for two- or three-dimensional similarity, respectively (see Fig 4b).

By the inverse of the preceding derivation, it follows that Weibull statistical theories imply the material to have no characteristic lengths. This immediately invites a question with regard to the applicability of these theories to quasibrittle materials such as concrete or composites, which obviously possess a characteristic length corresponding to the dimension in the inhomogeneities in the microstructure of the material. This is one reason why the Weibull-type statistical theory of size effect is not applicable to quasibrittle materials, except on scales so large that the size of their inhomogeneities becomes negligible and the large-scale material behavior changes from quasibrittle to brittle (see Section 7).

In quasibrittle materials, the problem of scaling is more complicated because the material possesses a characteristic length and this length is important. It is nevertheless clear that, for a sufficiently large size, the scale of the material inhomogeneities, and thus the material length, should become unimportant. So the power scaling law should apply asymptotically for sufficiently large sizes. If there is a large crack at failure, the exponent of this asymptotic power law must be −1/2, which is represented by the dashed asymptote in Fig 5. The material length must also become unimportant for very small structure sizes, for example when the size of the concrete specimen is only several times the aggregate size. This means that for very small sizes, the size effect should again asymptotically approach a power law. Because, for such small sizes, a discrete crack cannot be discerned as the entire specimen is occupied by the fracture process zone, the exponent of the power law should be 0, corresponding to the strength criterion (see the horizontal dashed asymptote in Fig 5). The difficulty is that most applications of quasibrittle materials fall into the transitional range between these two asymptotes, for which the scaling law may be expected to follow some transitional curve (see the solid curve in Fig 5).

Let us now give a simple explanation of the deterministic size effect due to energy release. Consider the rectangular panel in Fig 6, which is initially under a uniform stress equal to the nominal stress σ_N. Introduction of a crack of length a with a fracture process zone of a certain length and width h may be approximately imagined to relieve the stress and thus release the strain energy from the areas of the shaded triangles and the crack band shown in Fig 6. The slope of the effective boundary of the stress relief zone, k, is a constant when the size is varied. We may assume that, for the range of interest, the length of the crack at maximum load is approximately proportional to the structure size D while the size h of the fracture process zone is essentially a constant, related to the inhomogeneity size in the material (this assumption is usually, but not always, verified by experiment or nonlocal finite element analysis.)

For a very large structure size, the width h becomes negligible, and then the energy release is coming only from the shaded triangular zones (Fig 6) whose area is proportional to D^2. This means that the energy release is proportional to $D^2\sigma_N^2/E$ (E = Young's modulus). At the same time, the energy consumed is proportional to the area of the band of constant width h, which is proportional to D. So the energy consumed and dissipated by fracture is proportional to $G_f D$ where G_f is the fracture energy, a material property representing the energy dissipated per unit length and unit width (unit area) of the fracture surface. Thus, $\sigma_N^2 D^2/E \propto G_f D$, from which it immediately follows that the size effect law for very large structures is $\sigma_N \propto D^{-1/2}$.

On the other hand, when the structure is very small, the triangular stress relief zones have a negligible area compared to the area of the crack band, which means that the energy release is proportional to $D\sigma_N^2/E$. Therefore, energy balance requires that $D\sigma_N^2/E \propto G_f D$, from which it follows σ_N = constant. So, asymptotically for very small structures, there is no size effect.

The foregoing analysis (given in more detail in Bažant, 1983, 1984) is predicated on the assumptions that the crack lengths in small and large structures are similar. According to experimental observations and finite element simulations, this is often true for the practically interesting range of sizes. However, there are some cases where this similarity of cracks does not occur, and then, of course, the scaling becomes different.

The curves of nominal strength *versus* the relative structure deflection (normalized so that the initial slope in Fig 7 be independent of size) have, for small and large structures, the shapes indicated in Fig 7. Aside from the effect of size on the maximum load, there is a size ef-

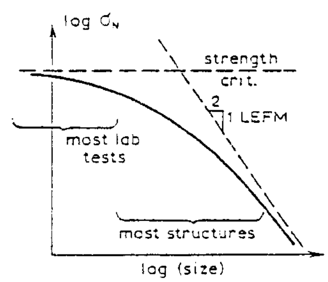

Fig 5. Transitional scaling of the nominal strength of quasibrittle structures failing only after large fracture growth

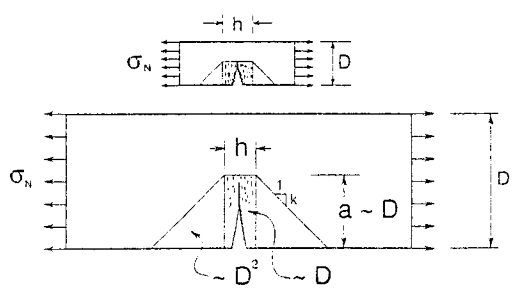

Fig 6. Approximate zones of stress relief caused by fracture in small and large specimens

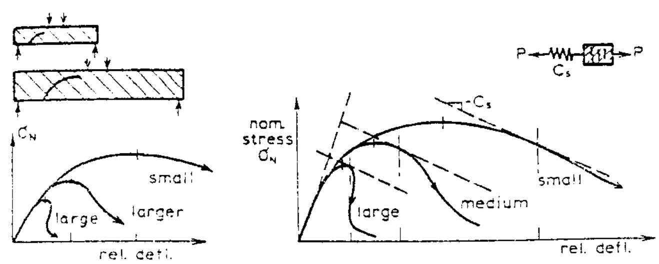

Fig 7. Left: Load-deflection curves of quasibrittle structures of different sizes; Right: stability is lost at the tangent points of lines of slope $-C_s$, with C_s = stiffness of loading device

fect on the shape of the post-peak descending load-deflection curves. For small structures, the post-peak curves descend slowly; for larger structures, steeper; and for sufficiently large structures, they may exhibit a snapback, that is, a change of slope from negative to positive. If such a structure is loaded by an elastic device with a spring constant C_s, it loses stability at the point where the load-deflection diagram first attains the slope$-C_s$ (if ever), as seen in Fig 7. These tangent points indicate failure. The ratio of the deflection at these points to the elastic deflection characterizes the ductility of the structure. Obviously, small quasibrittle structures have a large ductility while large quasibrittle structures have small ductility. The areas under the load-deflection curves characterize the energy absorption. The energy absorption capability of a quasibrittle structure decreases, in relative terms, as the structure size increases. This is important for blast loads and impact.

The progressive steepening of the post-peak curves in Fig 7 with increasing size and the development of a snapback can be most simply explained by the series coupling model, which assumes that the response of a structure may be at least partly modeled by the series coupling of the cohesive crack or damage zone with the elastic behavior of the structure (Bažant and Cedolin, 1991, Section 13.2).

3 SIZE EFFECT FOR THE CASE OF LARGE CRACKS AT FAILURE: ASYMPTOTIC ANALYSIS

In general, the scaling properties for the nominal strength of a structure reaching the maximum load after a large stable crack growth can be most generally deduced by an asymptotic analysis of the energy release, as recently shown by Bažant (1996). We will now briefly review this analysis, restricting attention to two-dimensional similarity, although the case of three-dimensional similarity could be analyzed similarly. We define the nominal stress as $\sigma_N = P/bD$ where P is the applied load or load parameter, b is the structure thickness in the third dimension, and D is the characteristic size (dimension) of the structure, for example taken as the depth of the notched three-point bend beam shown in Fig 8.

The fracture may be characterized by the dimensionless variables $\alpha_0 = a_0/D$, $\alpha = a/D$, $\theta = c_f/D$, in which a = the total crack length which gives (according to LEFM) the same specimen compliance as the actual crack with its fracture process zone, a_0 = length of the traction-free crack or the notch, and $c_f = a - a_0$ = effective size of the fracture process zone (or the effective length of the R-curve).

However, the interpretation in the sense of the cohesive crack or R-curve model is not essential for our analysis. We can equally well assume that c_f is, in general, any kind of material length, for example $c_f = G_f/W_d$ where G_f = fracture energy of the material (dimension J/m^2), and W_d = energy dissipated by distributed cracking in the fracture process zone per unit volume (dimension J/m^3) which is represented by the area under the total stress-strain curve with strain softening in the sense of continuum damage mechanics. Or, we can assume that $c_f = EG_f/f_t'^2$, where f_t' is the tensile strength of the material. The last expression is the characteristic size of the fracture process zone of the material according to Irwin (1958).

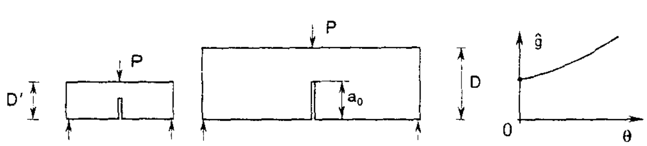

Fig 8. Similar structures with large cracks and function $\hat{g}$

The energy release from the structure can be analyzed either on the basis of the change of the potential energy of the structure Π at constant load-point displacement, or the change of the complementary energy of the structure, Π^*, at constant load. We choose the latter, and express Π^* in the following dimensionally correct form

$$\Pi^* = \frac{\sigma_N^2}{E} bD^2 f(\alpha_0, \alpha, \theta) \tag{4}$$

in which E = Young's elastic modulus of the material and f is a dimensionless function characterizing the geometry of the structure. Further, we must introduce two conditions for the maximum load.

First, the fracture at maximum load is propagating, which means that the energy release rate $\mathcal{G}$ must be equal to the energy consumption rate R, which we may interpret in the sense of the R-curve (resistance curve) giving the dependence of the critical energy release rate required for fracture growth on the crack length a. Most generally, the resistance to fracture can be characterized as $R = G_f r(\alpha_0, \alpha, \theta)$ in which r is a dimensionless function of the relative crack length α, the relative notch length α_0, and the relative size of the fracture process zone θ, having the property that $r \to 1$ when $\theta \to 0$ and $\alpha \to \alpha_0$. Obtaining the energy release rate $\mathcal{G} = (\partial \Pi^*/\partial\alpha)/b$ from Eq (4) by differentiation at constant nominal stress, we thus obtain the following first condition for the maximum load

$$b^{-1}\left[\partial \Pi^*/\partial\alpha\right]_{\sigma_N} = G_f r(\alpha_0, \alpha, \theta) \tag{5}$$

The second condition is that, under load control conditions, the maximum load represents the limit of stability. If the rate of growth of the energy release rate is smaller than the rate of growth of the R-curve, the fracture propagation is stable because the energy release change does not suffice to compensate for the rate of the energy consumed and dissipated by fracture. In the limit, both are equal, and so the second condition of the maximum load, corresponding to the stability limit, reads:

$$\left[\frac{\partial \mathcal{G}}{\partial \alpha}\right]_{\sigma_N} = \frac{\partial R}{\partial \alpha} \tag{6}$$

Geometrically, this represents the condition that the curve of the energy release rate must be tangent to the R-curve.

Substituting the expression for the complementary energy in Eq (4), one can show from the foregoing two conditions of maximum load that the nominal strength of the structure is given in the form:

$$\sigma_N = \sqrt{\frac{EG_f}{D\hat{g}(\alpha_0, \theta)}} \tag{7}$$

in which $\hat{g}$ is a dimensionless function expressed in terms of functions f and r and their derivatives (Bažant 1996). For fracture situations of positive geometry (increasing g), which is the usual case, the plot of function $\hat{g}$ at constant relative notch length α_0 looks roughly as shown in Fig 8. This function has the meaning of the dimensionless energy release rate modified according to the R-curve.

Obviously, function $\hat{g}$ must be smooth, and so it can be expanded into Taylor series with respect to the relative material length θ about the point (α_0, 0). In this way, the following series expansion of the nominal strength of the structure is obtained:

$$\sigma_N = \sqrt{\frac{EG_f}{D}} \left[\hat{g}(\alpha_0,0) + \hat{g}_1(\alpha_0,0)\frac{c_f}{D} + \frac{1}{2!}\hat{g}_2(\alpha_0,0)\left(\frac{c_f}{D}\right)^2 + \cdots\right]^{-1/2} = \frac{Bf_t'}{\sqrt{D}}\left(D_0^{-1} + D^{-1} + \kappa_2 D^{-2} + \kappa_3 D^{-3} + \cdots\right)^{-1/2} \tag{8}$$

Here, $\hat{g}_1$ and $\hat{g}_2$ are the first, second, etc, derivatives of function $\hat{g}$ with respect to θ, and D_0, κ_2, $\kappa_3 \cdots$ represent certain constants expressed in terms of function $\hat{g}$ and its derivatives at (α_0, 0). The series expansion is obviously an asymptotic expansion because the powers of size D are negative. So the expansion may be expected to be very accurate for very large sizes, but must be expected to diverge for $D \to 0$.

Further, it is interesting to obtain a small-size asymptotic expansion. To this end, one needs to use instead of θ the parameter $\eta = \theta^{-1} = D/c_f$. By a similar procedure as before, one can show that the nominal strength of the structure may be written in the form:

$$\sigma_N = \sqrt{\frac{EG_f}{c_f}}\left[\tilde{g}(\alpha_0, \eta)\right]^{-1/2} \tag{9}$$

This function again has the meaning of the dimensionless energy release rate (modified by the R-curve) but as a function of the inverse relative size of the process zone, η. Function $\tilde{g}$ must also be sufficiently smooth to permit expansion in Taylor series with respect to parameter θ about the point (α_0,0). This yields an asymptotic expansion of the following form:

$$\sigma_N = \sigma_P\left[1 + \left(\frac{D}{D_0}\right) + b_2\left(\frac{D}{D_0}\right)^2 + b_3\left(\frac{D}{D_0}\right)^3 + \cdots\right]^{-1/2} \tag{10}$$

in which σ_P, D_0, b_2, b_3,$\cdots$ are certain constants depending on the shape of the structure.

It must be emphasized that the large size and small size asymptotics are mere theoretical extrapolations. Obviously, at some very large size, the mechanism of failure will change, and a size smaller than the inhomogeneities of the material is meaningless. The purpose of the asymptotic expansions is not to describe the behavior at infinite and zero sizes, but to anchor the size effect curve for the size range of practical interest.

The results we obtained may be illustrated by Fig 9 showing the logarithmic size effect plot (for the case of geometrically similar structures with similar and large cracks). The large-size and small-size expansions in Eqs (8) and (10) are shown by the dashed curves. The large-size expansion asymptotically approaches the straight line of slope $-1/2$, corresponding to the scaling according to LEFM for the case of large and similar cracks. The small-size expansion approaches a horizontal line on the left, which corresponds to scaling according to the theory of plasticity or any strength theory.

The problem now is how to interpolate between these two expansions in order to obtain an approximate size effect law of general validity. This is the subject of the well-known theory of matched asymptotic. We have a situation in which the asymptotic behaviors, in our case for the large and small sizes are relatively easy to obtain but the intermediate behavior (in our case for the intermediate sizes) is very difficult to determine. This is a typical situation in which the technique of asymptotic matching is effective (Bender and Orszag, 1978; Barenblatt, 1979; Hinch, 1991). This technique was introduced at the beginning of the century in fluid mechanics by Prandtl in his famous development of the boundary layer theory.

In our case, the asymptotic matching is very simple because, as it turns out, the first two terms of both asymptotic series expansions leads to a formula of the same general form, namely

$$\sigma_N = \frac{Bf_t'}{\sqrt{1+\beta}}, \qquad \beta = \frac{D}{D_0} \tag{11}$$

where B is a dimensionless constant, and the tensile strength f_t' is introduced for reasons of dimensionality. (It should however be pointed that this is asymptotic matching in a simplified sense because the coefficients of both asymptotic expansion are not fixed numbers known *a priori* but are adjusted so as to match the same formula.)

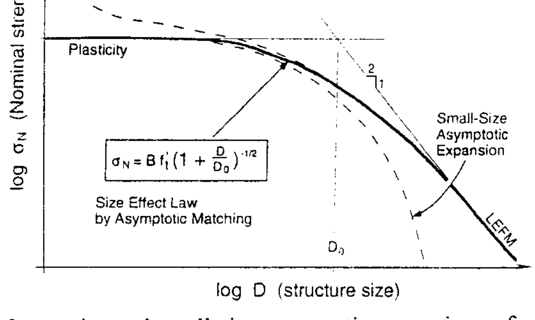

Fig 9. Large-size and small-size asymptotic expansions of size effect (dashed curves) and the size effect law as their asymptotic matching (solid curve)

The last formula is the size effect law derived initially by Bažant (1983, 1984) on the basis of simplified energy release arguments. The ratio β in this equation is called the *brittleness number* (Bažant, 1987; Bažant and Pfeiffer, 1987) because the case $\beta \to \infty$ represents a perfectly brittle behavior, and the case $\beta \to 0$ represents a perfectly nonbrittle (plastic, ductile) behavior. Because the constant D_0, representing the point of the intersection of the two asymptotes in Fig 9, depends on structure geometry, this definition of brittleness number is not only size independent but also shape independent. The brittleness is understood as the proximity to LEFM scaling.

The asymptotic analysis can be made more general by considering function $\hat{g}$ or $\tilde{g}$ to be a smooth function of θ^r or η^r, rather than θ or η, where r is some constant. Furthermore, it is also possible that, for very large sizes, there is a transition to a ductile failure mechanism which endows the structure with an additional residual nominal strength, σ_r (this may, for example, happen in the Brazilian split-cylinder test due to friction on sliding wedges under the platens). These modifications can be shown to lead to the following generalized formula:

$$\sigma_N = \sqrt{\sigma_P^2\left(1+\beta^r\right)^{-1/r}} + \sigma_r \tag{12}$$

in which σ_P = constant = small-size nominal strength. Exponent r is often more effective in approximating broad-range experimental results than adding higher-order terms of the series expansion. Equation (12) allows close approximation of numerical results obtained by nonlocal finite element analysis of the cohesive crack model for a very broad size range, at least 1:1000. The optimum values of exponent r depend on geometry (*eg*, $r = 0.44$ for standard three-point bend beams and 1.5 for a large center-cracked panel loaded on the crack).

4 APPLICATIONS OF THE SIZE EFFECT LAW BASED ON ENERGY RELEASE

The size effect law can also be expressed in terms of LEFM functions and material parameters, in the sense of an equivalent LEFM approximation. To this end, one may introduce the approximation $\hat{g}(\alpha_0, \theta) = g(\alpha_0 + \theta)$. With this approximation, which is asymptotically exact for large D, the size effect law corresponding to the asymptotic matching formula in Eq (11) acquires the form:

$$\sigma_N = \sqrt{\frac{EG_f}{g'(\alpha_0)c_f + g(\alpha_0)D}} \tag{13}$$

in which the parameters are given as:

$$D_0 = c_f\,\frac{g'(\alpha_0)}{g(\alpha_0)}, \qquad Bf_t' = \sqrt{\frac{EG_f}{c_f\,g'(\alpha_0)}} \tag{14}$$

Note that the transitional size D_0, delineating the brittle behavior from nonbrittle behavior, is proportional to the effective size of the fracture process zone and also to the ratio g'/g which depends on the geometry of the structure. Thus, the size effect law in Eq (13) expresses not only the effect of size but also the effect of structure geometry (shape). This law can be applied to structures or specimens that are not geometrically similar.

One useful application of the size effect in Eq (13) has proven to be the determination of the nonlinear fracture parameters of the material. To this end, one must test a set of specimens with a sufficiently large range of the brittleness number β. The range depends on the degree of statistical

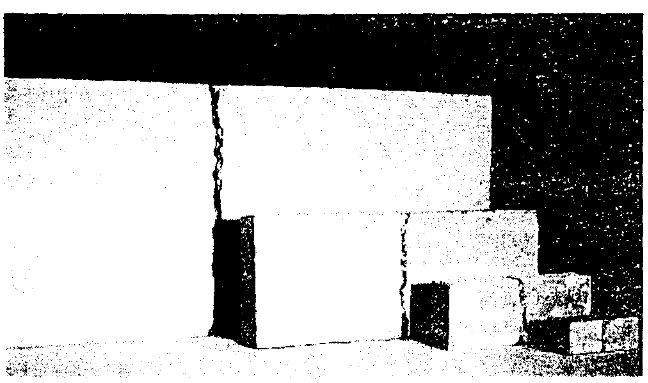

Fig 10. Similar three-point bend specimens tested by Bažant and Pfeiffer (1987)

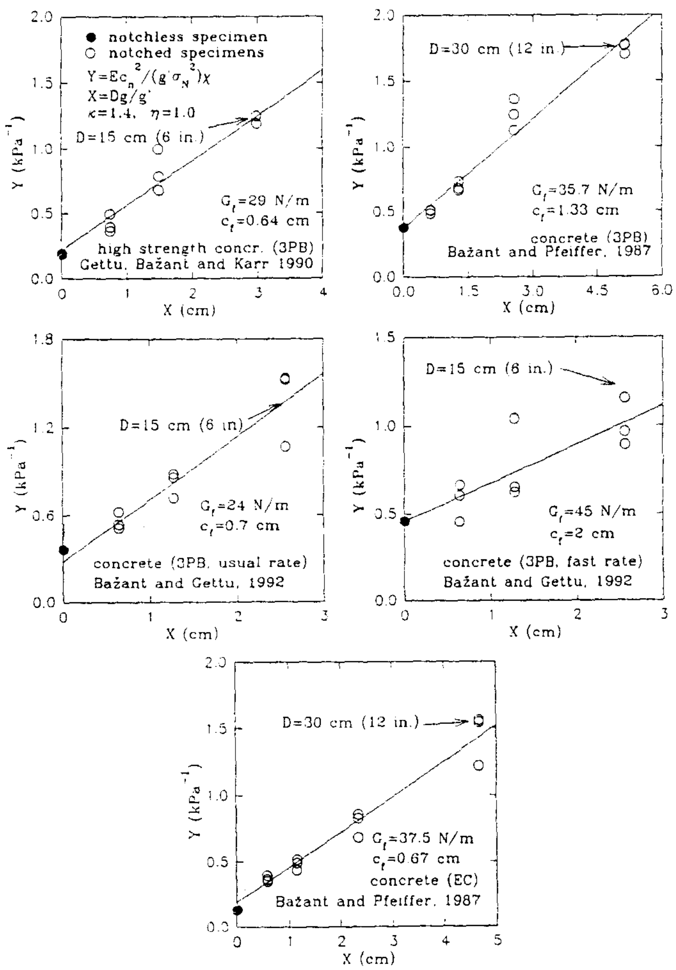

Fig 11. Linear regressions (according to the size effect law) of the nominal strength values of notched concrete specimens measured by Bažant and Pfeiffer (1987), Bažant and Gettu (1992) and Gettu *et al* (1990)

scatter of the results. If the scatter is very small, a small range of β is sufficient, and if the scatter is very large, a large range of β is needed. For the typical scatter observed in concrete and many other materials, the minimum range of the brittleness number is 1:4, and preferably, for more accurate results, 1:8. The broader the range, the more accurate the results. To achieve a sufficient range of brittleness numbers, one may test geometrically similar notched fracture specimens of sufficiently different sizes, as illustrated in Fig 10. However, geometric similarity is not necessary, although the results for geometrically similar specimens are somewhat more accurate because the effect of the changes of geometry is described by Eq (13) only approximately.

To determine the material fracture characteristics from the measured maximum loads of specimens of different brittleness numbers, one may rearrange Eq (13) into a linear regression plot (Fig 11): $Y = AX + C$ in which $Y = 1/g'\sigma_N^2$, $X = Dg/g'$, evaluated at α_0. The fracture characteristics are then obtained as $G_f = 1/AE$, $c_f = C/A$. From G_f and c_f, one can also obtain the critical crack-tip opening displacement

$$\delta_{CTOD} = (1/\pi)\sqrt{8G_f c_f / E} \tag{15}$$

(Bažant and Gettu, 1990; Bažant, 1996; Vol III) which was introduced in the early 1960s in the models of Wells (1961) and Cottrell (1963) for metals, and in the mid-1980s in a similar model for concrete by Jenq and Shah (1985). The size effect method has been adopted as a standard recommendation for concrete fracture testing by RILEM (1990).

Figure 12 shows the comparison of the size effect law with the data points obtained in the testing of Indiana limestone, carbon-epoxy fiber composites, silicone oxide ceramic and sea ice. The data for sea ice, obtained by Dempsey *et al* (1995) cover an unprecedented, large size range (also Mulmule 1995). In Dempsey's tests, floating notched square specimens of sea ice of sizes from 0.5m to 80m and thickness 1.8m were tested in situ in the Arctic Ocean. The results revealed a very strong size effect, rather close to the LEFM asymptote, revealing a high brittleness of sea ice at large scales.

Figure 13 illustrates the comparison with the size effect law for data obtained on specimens without notches (tests of diagonal shear failure of geometrically similar reinforced concrete beams Bažant and Kazemi, 1991, with size range 1:16). Figure 14 shows a comparison of the size effect law with data obtained on unnotched and unreinforced specimens (cylinders in double-punch loading, size 1:16; Marti, 1989).

The size effect law also closely agrees with the results of finite element analysis using the nonlocal damage concept (*eg*, Fig 15, Ožbolt and Bažant 1996), the crack band model (see the curve in Fig 3, Bažant and Oh, 1983), or the cohesive crack model (Bažant and Li, 1996). Furthermore, the size effect law was shown to approximately agree with the mean trend of maximum load values calculated by the discrete element method (random particle simulation, Bažant, Tabbara *et al*, 1990) or sea ice (Bažant and Jirásek, 1995; Fig 16).

There are nevertheless some instances in which the simple size effect law in Eqs (11) or (13) is insufficient because the logarithmic size effect plot of the data exhibits a positive

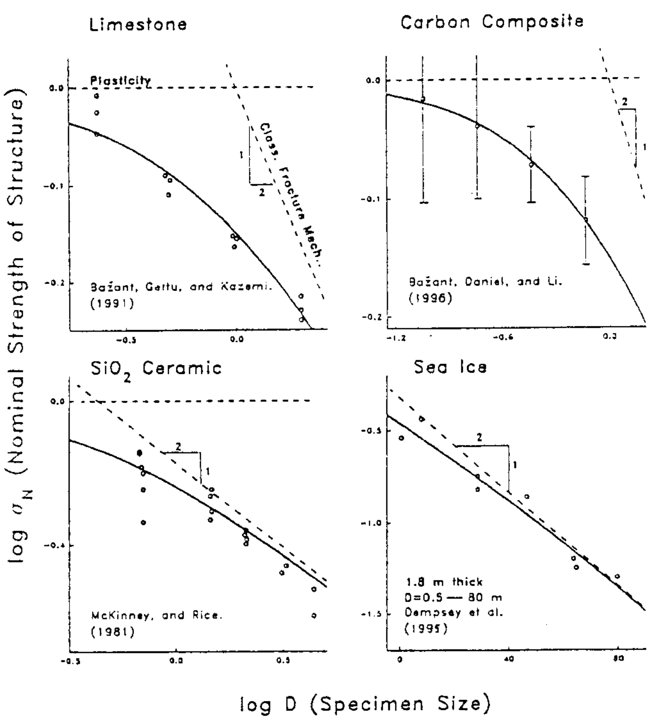

Fig 12. Nominal strength data from the tests of Indiana limestone (Bažant, Gettu and Kazemi 1991), carbon fiber epoxy laminates (Bažant, Daniel, and Li 1996), SiO_2 ceramics (McKinney and Rice 1981), and sea ice (Dempsey *et al*, 1995, Mulmule *et al*, 1995), and their fits by the size effect law

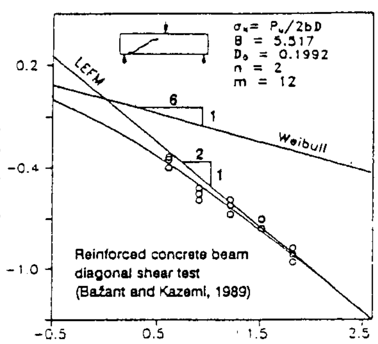

Fig 13. Nominal strength data from Bažant and Kazemi's (1991, and 1989 report) tests of diagonal shear failure of reduced-scale concrete beams with longitudinal reinforcement (of size range 1:16), their fit by the size effect law, and comparison with prediction of statistical Weibull-type theory for concrete

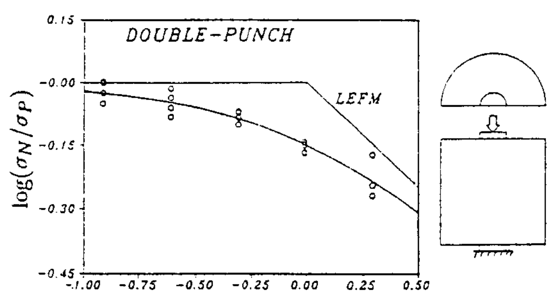

Fig 14. Nominal strength data from Marti's (1989) tests of double punch failure of concrete cylinders (of size range 1:16), and their fit by size effect law

respect to the zero initial crack length, the argument of the energy release function g is $\alpha = 0$. This means that the energy release rate $g(\alpha) = g(0) = 0$, and so the first term of the large-size expansion in Eq (8) vanishes. If we truncated the series after the second term, as before, no size effect would be obtained.

Therefore, we must in this case also include the third term of the large-size asymptotic expansion. This leads to the following approximation for the nominal strength for failure at crack initiation from a smooth surface:

$$\sigma_N = \sqrt{\frac{EG_f}{g'(0)c_f + \frac{1}{2}g''(0)c_f^2 D^{-1}}} \approx f_r^{\infty}\left(1 + \frac{D_b}{D}\right) \qquad (17)$$

The last expression is an approximation which preserves the asymptotic properties, and f_r^{∞} and D_b are constants, the former representing the nominal strength for a very large size and the latter having the meaning of the effective thickness of the boundary layer of cracking. The plot of the foregoing formula (17) for the size effect at crack initiation is shown in Fig 18. Furthermore, Fig 18 shows the plot of this formula in a linear form, with the coordinate D_b/D, and makes a comparison to the data points obtained in eight data series taken from the literature (after Bažant and Li, 1995).

The analysis we have outlined so far yields: the large size expansion of the size effect for long cracks, the small size expansion for long cracks, and the large size expansion for short cracks, while the small size expansion for short cracks can also be obtained. The question now is whether these expansions could be interpolated, or matched, so as to yield one formula approximating the intermediate situations and matching all the asymptotic cases. This formula has been obtained (Bažant, 1996):

$$\sigma_N = \sigma_0\left(1 + \frac{D}{D_0}\right)^{-1/2}\left\{1 + \left[\left(\overline{\eta} + \frac{D}{D_0}\right)\left(1 + \frac{D}{D_0}\right)\right]^{-1}\right\} \qquad (18)$$

in which $\overline{\eta}$ and σ_0 are empirical constants. The plot of this formula, which could be called the universal size effect law, is shown in Fig 19. Note that the discontinuity of slope on top left of the surface is due to expressing D_b, for the sake of simplicity, in terms of the positive part of the derivative of function g(this slope discontinuity could be avoided but at the expense of a more complicated formula).

The foregoing universal size effect law can be exploited for the testing of material fracture parameters. It al-

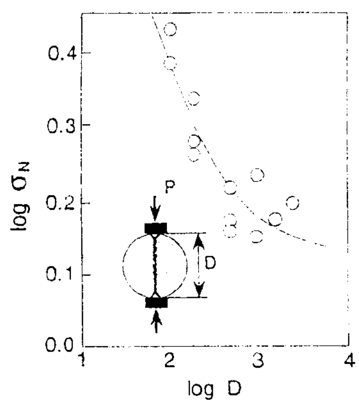

Fig 17. Nominal strength data from Brazilian split-cylinder tests of Hasegawa, Shioya and Okada (1985) and their fit by the size effect law with residual strength in Eq (12)

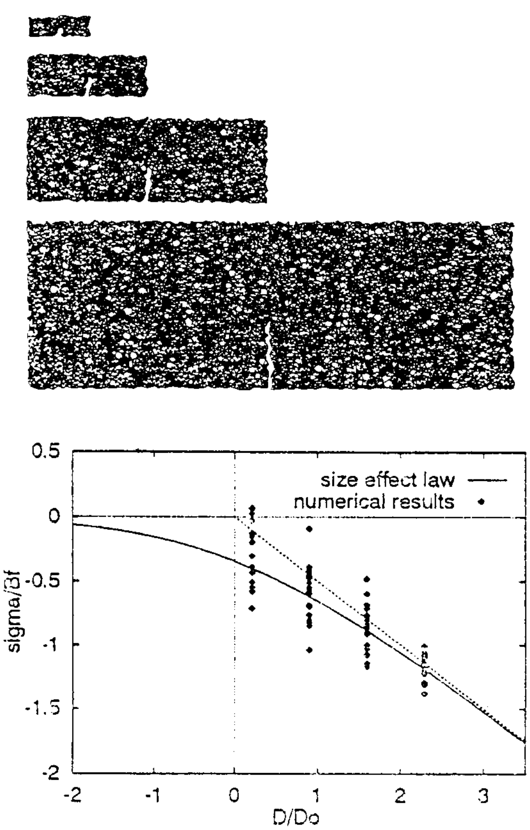

Fig 16. Nominal strength values obtained by discrete element method (random particle simulation of the specimens shown) and their comparison ti size effect law, exploited for determining the fracture characteristics of the random particle system (Jirásek and Bažant 1995a, b)

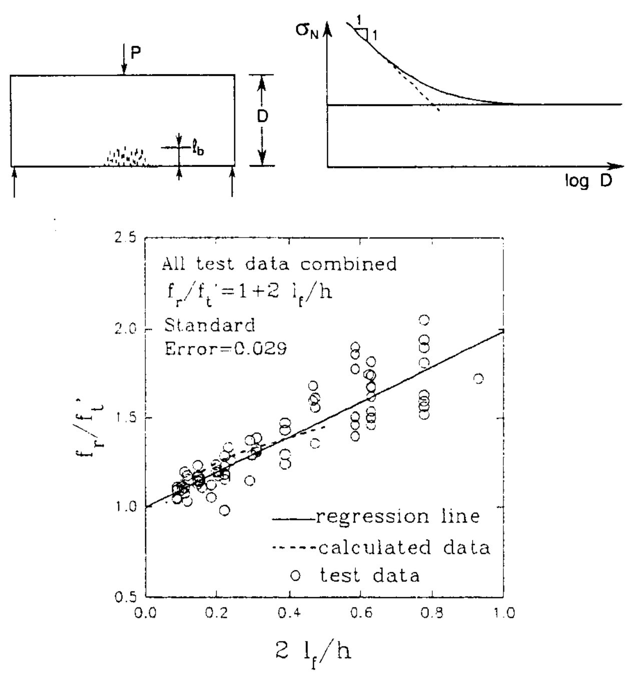

Fig 18. Cracking zone at maximum load P in a notchless quasibrittle specimen (top left); law of the size effect for quasibrittle failures at crack initiation (Bažant and Li, 1994, 1996) (top right); and use of this law in linear regression of test data for concrete obtained in eight different laboratories (Bažant and Li, 1994) (bottom)

the case of failure after a large stable crack growth, the matching of the large-size and small-size asymptotic expansions for the fractal fracture yields, instead of Eq (11), the result:

$$\sigma_N = \sigma_N^0 D^{(d_f-1)/2}\left(1+\frac{D}{D_0}\right)^{-1/2} \tag{21}$$

For failure at crack initiation, the asymptotic analysis yields instead of Eq (17) the result:

$$\sigma_N = \sigma_N^{\infty} D^{(d_f-1)/2}\left(1+\frac{D_b}{D}\right) \tag{22}$$

These expressions reduce to the nonfractal case when $d_f = 1$. The plots of these equations are shown in Fig 21 in comparison with the size effect formulas for the nonfractal case.

The hypothesis that the fracture propagation is fractal has been made and the consequences have been deduced (Bažant, 1997). Now, by judging the consequences, we may decide whether the hypothesis was correct. Looking at the plots in Fig 21, it is immediately apparent that the fractal case disagrees with the available experimental evidence. For failures after large crack growth, the rising portion of the plot has never been seen, and there are many data showing that the asymptotic slope is very close to $-1/2$, rather the much smaller value predicted from the fractal hypothesis. This is clear by looking at Figs 12-14. For failures at crack initiation, the kind of plots seen in Fig 21 (bottom), with a rising size effect curve for large sizes, is also never observed. Thus, it is inevitable to conclude that the hypothesis of a fractal source of size effect is contradicted by test data and thus untenable. (The existence of fractal characteristics of fracture surfaces in various materials is of course not questioned, and neither is the possibility that these fractal characteristics may influence the value of the fracture energy of the material and may have to be considered in micromechanical models which predict the fracture energy value.)

What is the physical reason that the fractal hypothesis fails? No doubt it is the fact that the front of the crack is surrounded by a large fracture process zone consisting of microcracks and frictional slips, as shown in Fig 22. Because the fracture energy G_f of quasibrittle materials is usually several orders of magnitude larger than the surface energy, the fracture process zone of microcracking dissipates far more energy than the crack curve. Therefore, from the energy viewpoint, the crack curve, which might be fractal, cannot matter.

There is another fractal concept, namely the lacunar fractality of microcrack distribution, which has recently been invoked by Carpinteri and Chiaia (1995, 1996). This is the concept, after a discussion of the Weibull theory, to which we turn attention next.

7 DOES WEIBULL STATISTICAL THEORY APPLY TO QUASIBRITTLE FRACTURE?

The statistical theory of size effect based on the concept of random strength was, in principle, completed by Weibull (1939) (also 1949, 1951, 1956). The Weibull theory has been enormously successful in applications to metal structures embrittled by fatigue. However, it took until the 1980s to realize that this theory does not really explain the size effect in quasibrittle structures failing after a large stable crack growth. The Weibull theory rests on two basic hypotheses:

1. The structure fails as soon as one small element of the material attains the strength limit.
2. The strength limit is random and the probability P_1 that the small element of material does not fail at a stress less than σ is given by the following Weibull cumulative distribution:

$$\varphi(\sigma) = \left\langle\frac{\sigma-\sigma_u}{\sigma_0}\right\rangle^m \left(\sigma \geq \sigma_u \approx 0\right) \tag{23}$$

It should be emphasized that this distribution is only the tail distribution of the extreme values. (Of course, far above this threshold there is a transition to some distribution such as normal, log-normal, or gamma but on the scale of the drawing in Fig 23 (top left) this occurs miles away.)

Weibull applied this distribution to the classical problem of a long chain (Fig 23 top right) or cable, for which the hypothesis obviously applies well. It also applies to any statically determinate structure consisting of many elements (for example bars), which fails if one element fails. But this is not the case for statically indeterminate structures and multidimensional bodies.

Weibull's theory has been applied to such problems by many researchers, which is correct only if the multidimensional structure (Fig 23 bottom) fails as soon as one small element of the material fails. Such sudden failure occurs in fatigue-embrittled metal structures, in which the critical flaw at the moment the sudden failure is triggered is still of microscopic dimensions compared to the cross-section size. But this is not the case for concrete structures and other quasibrittle structures which are designed to fail only after a large stable crack growth. For example, in the diagonal shear failure of reinforced concrete beams the critical crack grows over 80% to 90% of the cross-section size before the beam becomes unstable and fails. During such large stable crack

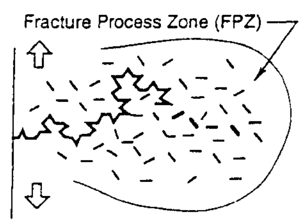

Fig 22. Fractal crack curve and its fracture process zone with distributed cracking

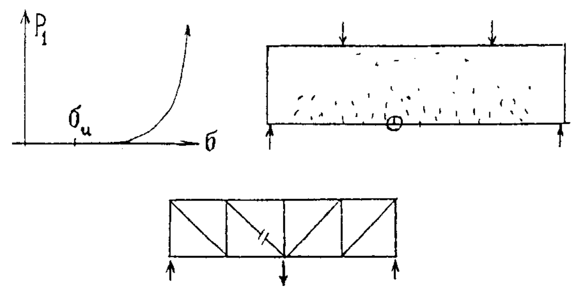

Fig 23. Weibull (cumulative) distribution of local material strength (top left), a critical flaw (encircled) in a field of many flaws (top right, and example of a multidimensional statically determinate structure that behaves as a chain and follow Weibull theory (bottom)

horizontal asymptote, and the size effect law in Eq (27) which begins by an asymptote of slope $-m/n$, both fit the test data about equally well, relative to the scatter of measurements.

It is interesting that the effect of material randomness completely disappears for large sizes, as revealed by the fact that the large size asymptote has the LEFM slope of $-1/2$. How can it be physically explained?

The reason is that, when the structures are sufficiently large, a further increase of the structure size is not accompanied by any increase in the size of the fracture process zone (Fig 26). The Weibull-type probability integral in Eq (26) is taken over the entire structure, however, the only significant contribution to the integral comes from the fracture process zone. Since the fracture process zone does not increase with an increase of the structure size, it is obvious that the failure probability should not be affected by a further increase of the structure size if it is already large.

8 CAN LACUNAR FRACTALITY OF MICROCRACKS CAUSE A SIZE EFFECT?

After discussing Weibull theory, we are ready to tackle another type of fractality–the lacunar fractality of microcracks, which is illustrated in Fig 27 (top). From a distance, we see one crack, but looking closer, we see it consists of several shorter cracks with gaps between them, and looking still closer we see that each of these cracks consists of several still shorter cracks with shorter gaps between them, and so forth. Refinement to infinity generates a Cantor set or a fractal set whose fractal dimension d_f is less than the Euclidean dimension of the space (which is 1 for a one-dimensional array of cracks on a line; Fig 27 top left). It seems that the microcrack systems in concrete do exhibit this type of fractality, but only to a limited extent. Quasibrittle materials are materials with large heterogeneities and a large characteristic length. So obviously the refinement to smaller and smaller cracks must have a cutoff.

The argument that lacunar (or rarefying) fractality is the cause of size effect in quasibrittle structures (Carpinteri and Chiaia, 1995) went as follows. The fractal dimensions of the arrays of microcracks are different at small and large scales of observation. For a small scale, the fractal dimension D_f is distinctly less than 1, and for a large scale it is nearly 1. For the failure of a small structure the small scale matters, and for the failure of a large structure the large scale matters. Therefore, as it was argued, there should be a transition from a power scaling law corresponding to small scale fractality to the power scaling law corresponding to the large scale fractality, the latter having exponent 0 for the strength, *ie*, no size effect. Thus, as it was claimed, the size effect should be given by a transitional curve between the two asymptotes of slope $-1/2$ and 0 shown in Fig 27 (bottom left). The slope of the initial asymptote was assumed to be $-1/2$. This size effect was described by a law called the *multifractal* scaling law (MFSL) (Carpinteri *et al*, 1993, 1995a, b, c)

$$\sigma_N = \sqrt{A_1 + \frac{A_2}{D}} \tag{28}$$

in which A_1 and A_2 are constants. It was shown that some test data for concrete can be reasonably well described by this formula (although they can be equally well described by several other formulas based on nonfractal mechanisms, particularly Eq (12)).

There are, however, test data that clearly disagree with the MFSL, Eq (28). Many test data exhibit, in the logarithmic size effect plot, an initial slope much less than $-1/2$, particularly for specimen sizes that are as small as possible for the given size of aggregate. Many data approach an asymptote of slope $-1/2$ at very large sizes. Also, there are many data that exhibit a negative rather than positive curvature in the plot of $\log \sigma_N$ and $\log D$. These features disagree with the MFSL.

At closer scrutiny, there are also mathematical and physical reasons why the lacunar fractality cannot be the source of the observed size effect. If the failure is assumed to be controled by lacunar fractality, that is by microcracks, it obviously implies that the failure occurs at crack initiation, in which case the mathematical formulation must be akin to Weibull theory. Labeling the aforementioned small and large scales of observations by superscripts A and B, the Weibull distributions of the strength of a small material element in the fracture process zone with lacunar microcracks may be written as

$$\varphi\left[\sigma(\boldsymbol{x}); d_f^A\right] = \left\langle \frac{\sigma_N S(\xi) c_f^{1-d_f^A} - \hat{\sigma}_u^A}{\hat{\sigma}_0^A} \right\rangle^m \tag{29}$$

$$\varphi\left[\sigma(\boldsymbol{x}); d_f^B\right] = \left\langle \frac{\sigma_N S(\xi) c_f^{1-d_f^B} - \hat{\sigma}_u^B}{\hat{\sigma}_0^B} \right\rangle^m \tag{30}$$

Here the stress in the small material element of random strength has been written as $\sigma = \sigma_N S(\xi)$, in which S is the same function for all sizes of geometrically similar structures, and $\xi = \boldsymbol{x}/D$, for the nonfractal (non-lacunar) case. For the fractal (lacunar) case, this is generalized as $\sigma = \sigma_N S(\xi) c_f^{1-d_f}$ because the stress of the material element, in the case of lacunar microcracks, must be considered to have a non-standard, fractal dimension. Obviously, the Weibull con-

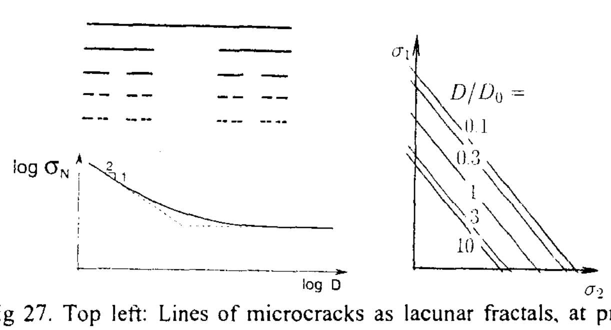

Fig 27. Top left: Lines of microcracks as lacunar fractals, at progressive refinements; bottom left: *MFSL* proposed by Carpinteri *et al* (1995); right: interaction diagrams for different size structures with two loads and constant a_0

curvature, as illustrated in Fig 17. This is, for example, observed for the Brazilian split cylinder test. The cause is that, for a very large structure, the load to produce the diagonal cracks in a cylinder becomes negligible but failure cannot occur because wedge regions under the load must slide frictionally, which imposes a certain residual strength σ_r. Another reason may be that the crack length at failure ceases to increase in proportion to the specimen size. Such data can be well described by the generalized size effect law in Eq (12) in which D_0 is very small, smaller than the smallest D in the data set (see Fig 17).

Applications to the fiber composite laminates are more intricate. One reason is that the orthotropy of the material must be taken into account. This has been done, obtaining the expression for the energy release rate in the form

$$\mathcal{G}(\alpha) = \sigma_N^2 D g(\alpha) Q(\rho)/\bar{E} \tag{16}$$

in which $\bar{E} = \left[2\sqrt{E_x E_y^3}/(1+\rho)\right]^{1/2}$, $\rho = \left(\sqrt{E_x E_y}/2G_{xy}\right) - \sqrt{\upsilon_{xy}\upsilon_{yx}}$ and $Q(\rho)$ is a function capturing the effect of orthotropy, the material, and specimen shape, as recently shown by Bao *et al* (1992). A further difficulty is that the size of the process zone, c_f, depends on the direction of fracture propagation with respect to the fibers. The results obtained with this analysis by Bažant, Daniel, and Li (1996) are shown in Fig 12.

Complex questions, however, remain with regard to the role of pullout and breakage of fibers in the scaling of failure of fiber composites.

When the values for material fracture parameters are determined by a method that is not based on the size effect, one faces the question of spurious size dependence of these values. For example, the fracture energy can be conveniently determined from the area under the measured load-deflection diagram, which is called the *work-of-fracture method* (Nakayama, 1965; Tattersall and Tappin, 1966) and has been pioneered for concrete by Hillerborg *et al* (1976) (see also Hillerborg, 1985a, b). But the values of the fracture energy thus obtained depend on the size of the specimen (Bažant, 1996; Bažant and Kazemi, 1991). Methods to eliminate this dependence were discussed by Planas and Elices (1994).

5 SIZE EFFECT FOR CRACK INITIATION AND UNIVERSAL SIZE EFFECT LAW

The foregoing analysis applies only to structures that fail after a large stable crack growth. This is typical for quasibrittle materials and is also the objective of a good design because the large stable crack growth endows the structure with a large energy dissipation capability and a certain measure of ductility. For example, the objective of reinforcing concrete structures, of toughening ceramics, of putting fibers in composites, etc, may be recognized as the attainment of a large stable crack growth prior to failure.

In some situations, however, quasibrittle fractures fail at crack initiation. For example, this happens for a plain concrete beam. This nevertheless does not mean that the fracture process zone size would be negligible. Because of heterogeneity of the material, the process zone size is still quite large, as illustrated in Fig 18 (top left). The maximum load is obtained typically when this large cracking zone coalesces into a continuous crack capable of growing further. Because a large cracking zone forms prior to the maximum load, one cannot expect the Weibull theory to be applicable, as will be explained in Section 7.

As described in detail in Bažant (1996), the failure at crack initiation from a smooth surface can also be analyzed on the basis of the expansions in Eqs (8) or (10), however, with one modification. Since the expansions are made with

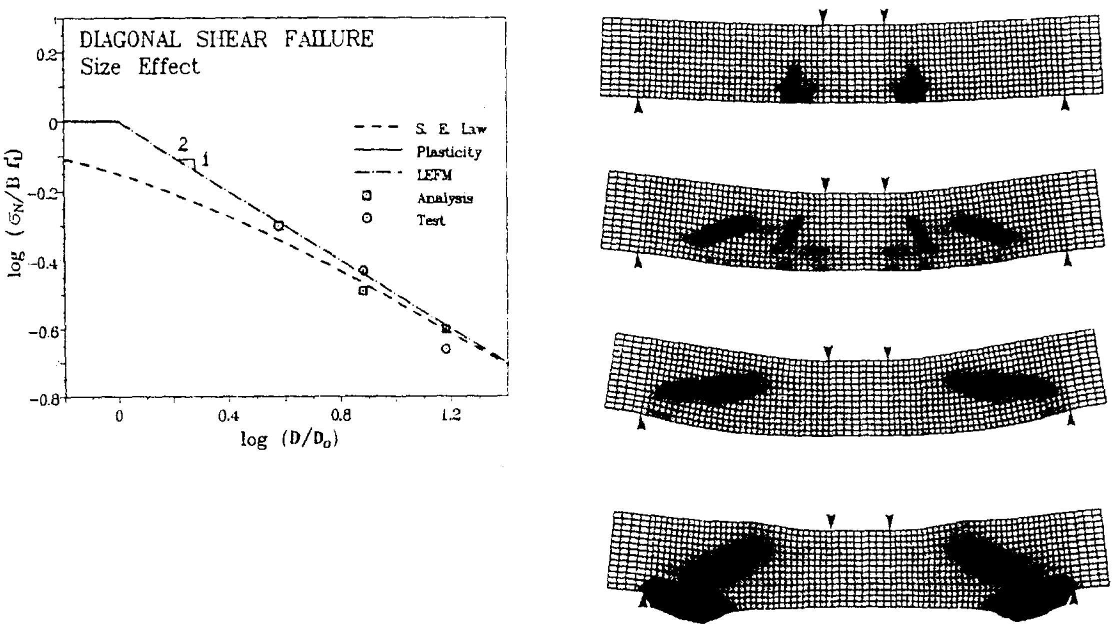

Fig 15. Left: Nominal strength values obtained by finite element analysis using the nonlocal model with crack interactions (Ožbolt and Bažant 1996) compared to test data of Bažant and Kazemi (1991) for diagonal shear failure of longitudinally reinforced concrete beams and to the size effect law (dashed curve); Right: cracking damage zone in subsequent stages of loading.

lows using specimens of one size, notched and notchless. For such specimens, it is possible to obtain a sufficient range of brittleness number β (more than 1:4) without varying the specimen size. On the other hand, if unnotched specimens are not included, a sufficient range cannot be obtained just by varying the notch length.

For the purpose of data fitting, Eq (18) may be reduced to a series of nonlinear regressions (Bažant and Li, 1996). The linear regression plots for some previously reported test data are shown in Fig 11, for which we have already discussed the empty data points which correspond to notched specimens of different sizes. The solid data points correspond to unnotched specimens. The fact that the solid points are approximately aligned with the trend of the empty data points confirms the approximate applicability of the universal size effect law in Fig 19. Obviously, it is possible to delete the empty data points for specimens of all sizes except the largest and obtain about the same results using only the data points for the notched specimen of the largest size and the unnotched specimen of the same size. This approach may simplify the determination of material fracture parameters from test data.

6 IS FRACTURE FRACTALITY THE CAUSEOF OBSERVED SIZE EFFECT?

This intriguing question was recently raised by Carpinteri, (1994a, b) (see also Carpinteri *et al* 1993, 1995a, b, c; Carpinteri and Ferro, 1994; and Carpinteri and Chiaia, 1995). The arguments Carpinteri offered, however, were not based on mechanical analysis and energy considerations. Rather they were strictly geometrical and partly intuitive. Recently, Bažant (1996) attempted a mechanical analysis of the problem, which will now be briefly outlined. The answer has been negative. However, the fact that the surface roughness of cracks in many materials can be described, at least over a certain limited range, by fractal concepts, is not in doubt (*eg*, Mandelbrot *et al*, 1984; Brown, 1987; Mecholsky and Mackin, 1988; Cahn, 1989; Chen and Runt, 1989; Hornbogen, 1989; Peng and Tian, 1990; Saouma *et al*, 1990; Bouchaud *et al*, 1990; Chelidze and Gueguen, 1990; Issa *et al*, 1992; Long *et al*, 1991; Måløy *et al*, 1992; Mosolov and Borodich, 1992; Borodich, 1992; Lange *et al*, 1993; Xie, 1987, 1989, 1993; Xie *et al*, 1994, 1996; Saouma and Barton, 1994; Feng *et al*, 1995.)

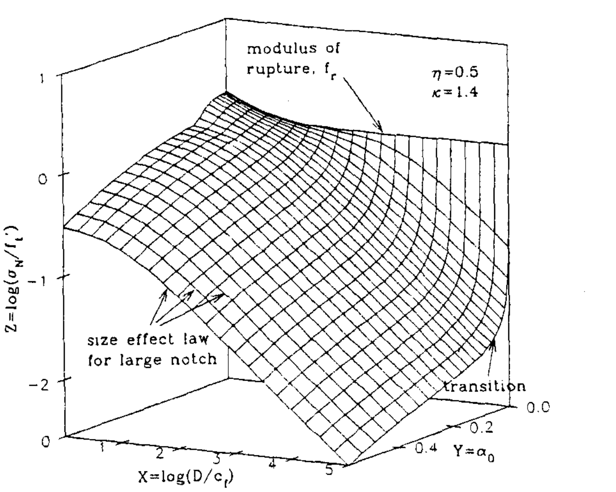

Fig 19. Universal size effect law for failure both at crack initiation and after large crack growth (Bažant 1996)

In two dimensions, a fractal curve, which can be imagined to represent a crack, can be illustrated, for example, by the von Koch curves shown in Fig 20. Progressive refinements are obtained by adding self-similar bumps into each straight segment. If the length of this curve is measured by a ruler of a certain resolution δ_0, imagined as the ruler length, the length measured will obviously depend on the length of the ruler and if the length of the ruler approaches zero, the measured length will approach infinity. This is described by the equation

$$a_\delta = \delta_0 (a/\delta_0)^{d_f} \tag{19}$$

where a_δ is the measured length along the curve, a is the projected (smooth, Euclidean) crack length, and the exponent d_f is called the fractal dimension, which is greater than 1 if the curve is fractal, and equal to 1 if it is not.

Obviously, the total energy dissipation W_f for the crack length a_δ would be infinite if we would assume that a finite amount of energy G_f is dissipated per unit crack length. This is a conceptual difficulty for fracture mechanics of fractal cracks. In a sequel to the study of Mosolov and Borodich (1992), Borodich (1992) proposed to resolve this difficulty by setting

$$W_f / b = G_{fl}\, a^{d_f} \tag{20}$$

in which W_f = total energy dissipation; G_{fl} represents what may be called the fractal fracture energy whose dimension is not J/m^2 but J/m^{d_f+1}.

Based on this fractal concept of fracture energy, one may carry out a similar asymptotic analysis as we have outlined for non-fractal cracks (see Bažant, 1997). For

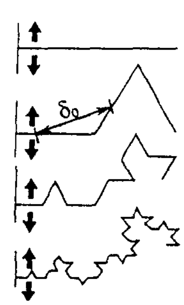

Fig 20. Von Koch fractal curve at progressive refinements and measurement of its length by a ruler of length δ_0

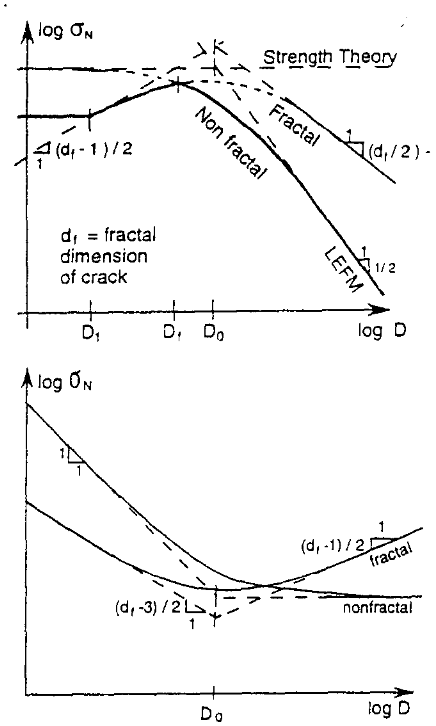

Fig 21. Size effect curves predicted by nonfractal and fractal energy-based analyses, for failures after large crack growth (right) or at crack initiation (bottom)

growth, enormous stress redistributions occur and cause a large release of stored energy which, as already discussed, produces a large deterministic size effect.

The size effect in Weibull theory comes from the fact that, in a larger structure, the probability of encountering a small material element of a certain small strength increases with the structure size. By considering the joint probability of survival of all the small material elements in the structure, one obtains for the structure strength a probability integral of a similar form as that for a long chain or a series coupling of many elements (Tippett, 1925; Peirce, 1926; Fréchet, 1927; Fischer and Tippett, 1928; von Mises, 1936):

$$-\ln\left(1-P_f\right)=\int_V \varphi\left[\sigma(\boldsymbol{x})\right]dV(\boldsymbol{x})/V_r \tag{24}$$

in which P_f = failure probability of the structure, V = volume of the structure, V_r = small representative volume of the material whose strength distribution is given by φ (σ), and $\boldsymbol{x}$ = spatial coordinate vector. By virtue of the fact that the Weibull distribution is a power law (and that σ_u may be neglected), the aforementioned probability integral always yields for the size effect a power law. It is of the form

$$\sigma_N = k_v V^{-1/m} = k_0 D^{-n/m} \tag{25}$$

where k_0 = constant characterizing the structure shape, and n = number of dimensions of the structure (1, 2 or 3). For two-dimensional similarity (n = 2) and typical properties of concrete, the exponent is approximately n/m = 1/6.

As already mentioned, the fact that the scaling law of Weibull theory is a power law implies that there is no characteristic size of the structure, and thus no material length (this is also obvious from the fact that no material length appears anywhere in the formulation). This observation makes the Weibull-type scaling suspect for the case of quasibrittle structures whose material is highly heterogeneous, with a heterogeneity characterized by a non-negligible material length.

To take into account stress redistributions, various phenomenological theories of load sharing and redistribution in a system of parallel elements have been proposed. Although they are useful if the redistributions and load-sharing are relatively mild, they are insufficient to describe the large stress redistributions caused by large stable crack growth. They lack the fracture mechanics aspects of the problem.

To take into account the stress redistribution due to large fracture, one might wish to substitute the near-tip stress field of LEFM into the probability integral in Eq (24). However, for normal values of the Weibull modulus m, the integral diverges. So this is not a remedy. However, Weibull theory can be extended to capture large stress redistributions approximately – by introducing a nonlocal generalization (Bažant and Xi, 1991), in which the probability integral Eq (24) is replaced by the following integral:

$$\ln\left(1-P_f\right)=k_1\int_V \varphi\left[E\bar{\epsilon}(\boldsymbol{x})\right]dV(\boldsymbol{x})/V_r \tag{26}$$

Here the stress at a given point in the structure is replaced by the average (over a certain neighborhood, Fig 24) of the strain field, $\bar{\epsilon}$ (times the elastic modulus E, to get a quantity of the stress dimension. In other words, the failure probability at a certain point x of the structure is assumed to depend not on the stress (stress according to the continuum theory) at that point but on the average strain in a certain neighborhood of the point, as in nonlocal theories for strain localization in strain-softening materials. With this nonlocal generalization, the analytical evaluation of the integral Eq (26) seems prohibitively difficult, however it is easy to obtain the asymptotic behavior for $D \to \infty$ and $D \to 0$. Also, for $m \to \infty$, the solution should approach the size effect law based on energy release, Eq (11). It was shown that a simple formula that interpolates between these three asymptotic cases, *ie*, achieves asymptotic matching, is as follows (Bažant and Xi, 1991):

$$\sigma_N = \frac{\sigma_P}{\sqrt{\beta^{2n/m}+\beta}} \qquad \beta = \frac{D}{D_0} \tag{27}$$

This formula is sketched in Fig 25 (top), which also shows the aforementioned asymptotic scaling laws. They turned out to be the same as the Weibull type scaling law for small sizes (line of slope $-m/n$), and the LEFM scaling law for large similar cracks and large sizes (line of slope −1/2). According to this result, the scaling law of the classical Weibull theory should be applicable for sufficiently small structures. However, comparisons with test data for concrete show that the deterministic size effect law which begins by a

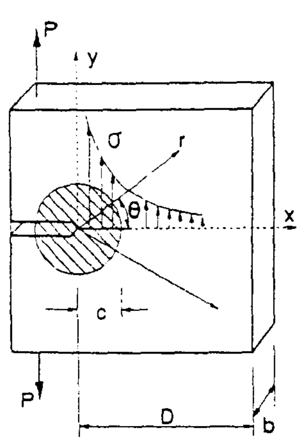

Fig 24. Neighborhood, simulating the fracture process zone, over which the strain field is averaged in the nonlocal generalization of Weibull theory (Bažant and Xi, 1991)

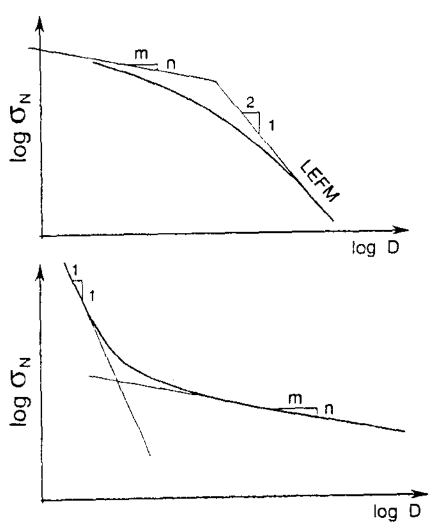

Fig 25. Scaling law according the nonlocal generalization of Weibull theory for failures after large crack growth (left) and at crack initiation (right)

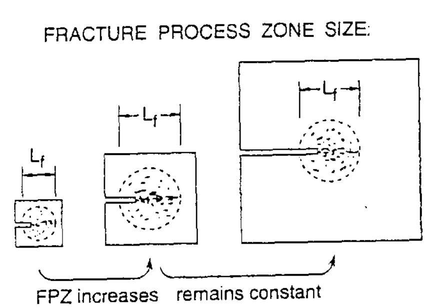

Fig 26. Changes of fracture process zone size with increasing structure size

stants $\hat{\sigma}_0$ and $\hat{\sigma}_u$ must now be considered to have fractal dimensions as well, but Weibull modulus m must not. An equation of the type of Eq (29) or (30) was written by Carpinteri *et al*, however, further analysis consisted of geometric and intuitive arguments. We will now sketch a recently published mechanical analysis (Bažant, 1997b).

In Weibull theory (failure at initiation of macroscopic fracture), every structure is equivalent to a long bar of variable cross section (Bažant, Xi, and Reid, 1991; Fig 28). Carpinteri *et al's*, argument means that a small structure is subdivided into small material elements (Fig 28a) and a large structure is subdivided into proportionately larger material elements (Fig 28c). However, this is not an objective view of the failure mechanism of two structures made of the same material.

The large elements of the larger structure shown in Fig 28c must be divisible into the small elements considered for the structure in Fig 28a, which are the representative volumes of the material for which the material properties are defined. If the large elements were not divisible into the small ones, it would imply that the material of the small structure is not the same.

Having in mind the subdivision of the large elements into the small elements, we may now calculate the failure probability of the large structure on the basis of the refined subdivision into the small elements, as shown in Fig 28b, or else it would imply that the small and large structures are not made of the same material. We note that the failure probability P_f of the large structure subdivided into large elements ΔV_{Bj} ($j = 1,2,\cdots N$), and the failure probability P^B_{fj} of the large element ΔV_{BJ} the large structure subdivided into small elements ΔV_{Aij}, must satisfy the following relations of Weibull theory:

$$-\ln\left(1-P_f\right)=\sum_j \varphi\left(\sigma_N S^B_j; d^B_f\right)\Delta V_{Bj}/V_r \tag{31}$$

$$-\ln\left(1-P^B_{fj}\right)=\sum_j \varphi\left(\sigma_N S^A_{ij}; d^A_f\right)\Delta V_{Aij}/V_r \tag{32}$$

Now, since we may subdivide each element B of the large structure into the small elements A if the material is the same, we have

$$\begin{aligned}-\ln\left(1-P_f\right)&=-\sum_j \ln\left(1-P^B_{fj}\right)\\&=\sum_j\sum_i \varphi\left(\sigma_N S^A_{ij}; d^A_f\right)\Delta V_{Aij}/V_r\end{aligned} \tag{33}$$

Equating this to Eq (8), we see that, in order to meet the requirement of the objective existence of the same material, the Weibull characteristics on scales A and B must be different and such that

$$\varphi\left(\sigma_N S^B_j; d^B_f\right)=\left(\Delta V^B_j\right)^{-1}\sum_i \varphi\left(\sigma_N S^A_{ij}; d^A_f\right)\Delta V^A_{ij} \tag{34}$$

Eqs (33) and (34) imply that consideration of different scales cannot yield different scaling laws. The same power law (in the case of zero Weibull threshold) must result from the hypothesis of lacunar fractality of microcrack distribution, regardless of the scale considered.

If the argument for the MFSL were accepted it would imply that a large material volume with no fractality may not be subdivided into smaller representative material volumes exhibiting lacunar fractality. As long as both the small and large volumes are sufficiently larger than the size of material inhomogeneities (the maximum aggregate size in concrete), this implication is selfcontradicting. The argument would be acceptable if the size of the smaller material volume were less than the size of these inhomogeneities, for instance on the scale of the matrix between the inhomogeneities (*eg*, the mortar between the large aggregates in concrete). The fractal dimensions of the systems of the tiny cracks in the matrix and of the cracks in concrete as a composite must of course be different. By definition, the failure of even the smallest concrete structure is not governed by the mortar alone. The aggregates are essential, and the failure is governed by the properties of the composite. So Carpinteri's argument is unacceptable as a basis of the scaling law for one and the same material.

To sum up, the scaling law of a structure failing at the initiation of fracture from a fractal field of lacunar microcracks must be identical to the scaling of the classical Weibull theory. The only difference is that the values of Weibull parameters depend on the lacunar fractality. This difference could be taken into account if the values of these parameters could be predicted by micromechanics. But as long as the Weibull parameters are determined by experiments, the concept lacunar fractality of microcracks contributes nothing. The lacunar fractality can have no effect on the scaling law.

9 ASYMPTOTIC SCALING AND INTERACTION DIAGRAM FOR THE CASE OF SEVERAL LOADS

The asymptotic analysis presented in Sections 3-5 may be easily extended to the case of several loads P_i, characterized by nominal stresses $\sigma_i = P_i/bD$. The energy release rates of the individual loads are not additive, but the stress intensity factors of the individual loads, K_{Ii}, are. Therefore, by superposition, $\sum_i \sigma_{Ni}\sqrt{Dg_i(\alpha)} = \sqrt{EG_f}$ where $g_i(\alpha)$ are the dimensionless energy release rates corresponding to loads P_i. The condition of stability limit (tangency of the total energy release rate curve to the R-curve) gives for the maximum

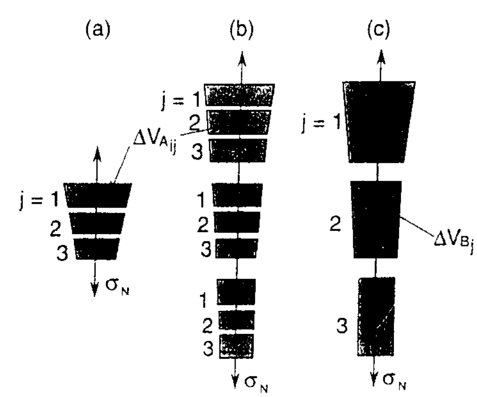

Fig 28. Subdivision of: (a) a small structure into small elements, (b) a large structure into small elements, and (c) a large structure into large elements

load the relative crack length $\alpha = \alpha_m(\alpha_0, \theta)$ (this argument is similar to that which led to Eq (7) but we now consider functions g_i right away as functions of one variable, rather than introducing such a simplification at the end). Inserting this value into the last relation and expanding functions $g_i(\alpha) = g_i(\alpha_0 + \theta)$ into a Taylor series in terms of θ about point $\theta = 0$, we obtain the relation $g_i(\alpha_0 + \theta) = g_i(\alpha_0) + g'(\alpha_i)\theta + \frac{1}{2}g_i''(\alpha_0)\theta^2 + \ldots$

For the case of a large crack, we may truncate this series after the second (linear) term. We also consider a positive geometry (*ie*, $g_i'(\alpha_0) > 0$ for all i). Furthermore, we may set $\sigma_{Ni} = \mu\sigma_{Di}$ where σ_{Di} are the given (fixed) design loads and μ = safety factor. After rearrangements:

$$\mu = \sqrt{EG_f}\left(\rho_1\sigma_{D1} + \rho_2\sigma_{D2} + \ldots + \rho_n\sigma_{Dn}\right)^{-1/2} \tag{35}$$

$$\rho_i = \sqrt{g_i(\alpha_0)D + g_i'(\alpha_0)c_f} \tag{36}$$

This equation gives the size effect, as well as the geometry effect, for the case of a large crack. At the same time, it may be regarded as the interaction diagram (failure envelope) for many loads. If α_0 is constant, these interaction diagrams are linear for any given size D (Fig 27 right). But in other than notched fracture specimens of positive geometry, α_0 is not fixed and it is of course possible for α_0 (the traction-free crack length at P_{max}) to depend on the ratios P_2/P_1, $P_3/P_1, \ldots$; then the interaction diagrams are not linear.

For the case of macroscopic crack initiation from a smooth surface, we have $g_i(0) = 0$. Therefore, similar to the case of one load, the series expansions cannot be truncated after the linear term. We may truncate them after the quadratic terms. A similar procedure as in Section 5 then yields for μ the same expression as Eq (35), but with

$$\rho_i = \sqrt{g_i'(0)c_f + \frac{1}{2}g_i''(0)\frac{c_f^2}{D}} \tag{37}$$

Equations (35) and (37) represent the large-size asymptotic approximations of size effect. Small-size asymptotic approximations for the case of many loads can be derived similarly, replacing the variable θ with $\eta = 1/\theta$.

Similar to the case of one load, it is further possible to find, for the case of many loads, a universal size effect law that has the correct asymptotic properties for large and small sizes and large cracks or crack initiation. It is analogous to Eq (18) and may again be written in the form of Eq (35) but with

$$\rho_i = r_i\left[1 + \left(\frac{D_{0i}}{D}\right)^r\right]^{1/2r} \left\{1 + s\left[\left(\bar{\eta} + \frac{D}{D_{bi}}\right)\left(1 + \frac{D}{D_{0i}}\right)\right]^{-1}\right\}^{-1/s} \tag{38}$$

$$r_i = \left[c_f g_i'(\alpha_0)\right]^{-1/2r}, \quad D_{0i} = c_f g_i'(\alpha_0)/g_i(\alpha_0),$$
$$D_{bi} = c_f\left\langle -g_i''(\alpha_0)\right\rangle/4g_i'(\alpha_0) \tag{39}$$

Here r, s and $\bar{\eta}$ are empirical constants whose values may probably be taken as 1 for many practical purposes.

10 SCALING OF FRACTURE OF SEA ICE

Different types of size effect are exhibited by sea ice failures. The scaling of failure of floating sea ice plates in the Arctic presents some intricate difficulties. One practical need is to understand and predict the formation of very long fractures (of the order of 1 km to 100 km) which cause the formation of open water leads or serve as precursors initiating the build-up of pressure ridges and rafting zones.

Large fractures can be produced in sea ice as a result of the thermal bending moment caused by cooling of the surface of the ice plate (Fig 29). Due to buoyancy, the floating plate behaves exactly as a plate on elastic Winkler foundation, with the foundation modulus equal to the unit weight of sea water. Under the assumption that the ice plate is infinite and elastic, of constant thickness h, that the temperature profiles for various thicknesses h are similar, and that the thermal fracture is semi-infinite and propagates statically (*ie*, with insignificant inertia forces), it was found (Bažant, 1992) that the critical temperature difference

$$\Delta T_{cr} \propto h^{-3/8} \tag{40}$$

This means that the critical nominal thermal stress $\sigma_N \propto h^{-3/8}$. The analysis was done according to LEFM. Despite the existence of a large fracture process zone, LEFM is justified because a steady-state propagation must develop. The fracture process zone does not change as it travels with the fracture front, and thus it dissipates energy at a constant rate, as in LEFM.

It has been shown that the scaling law in Eq (40) must apply to failures caused by bending cracks of any type, provided that they are full-through cracks propagating along the plate (created by any type of loading, *eg*, by vertical load; Slepyan, 1990; Bažant, 1993).

It may be surprising that the exponent of this large size asymptotic scaling law is not $-1/2$. However, this apparent contradiction may be explained if one realizes that the plate thickness is merely a parameter but not actually a dimension in the plane of the boundary value problem, that is, the horizontal plane (x, y). In that plane, the problem has only one characteristic length–namely the well-known flexural wavelength of a plate on elastic foundation, L_0. As it happens, L_0 is not proportional to h but to $h^{3/4}$. Thus it follows that the exponent of L_0 in the scaling law is $(-3/8)(4/3) = -1/2$. So the scaling of thermal bending fracture does in fact obey the previously mentioned LEFM scaling law:

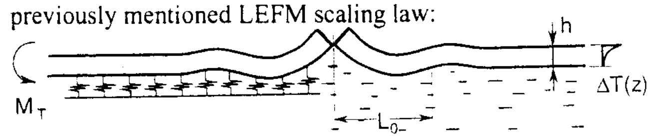

Fig 29. Bending fracture of floating sea ice plate caused by temperature difference

$$\Delta T_{cr} \propto L_0^{-1/2}, \tag{41}$$

Simplified calculations (Bažant, 1992c) have shown that, in order to propagate such a long thermal bending fracture through a plate 1m thick, the temperature difference across the plate must be about 25°C, while for a plate 6m thick the temperature difference needs to be only 12°C. This is a large size effect. It may explain why very long fractures in the Arctic Ocean are often seen to run through the thickest floes rather than through the thinly refrozen water leads between and around the floes (as observed by Assur in 1963).

An important practical problem, in which the scaling is different, is the failure caused by vertical (downward or upward) penetration through the floating ice plate (Fig 30). In that case, the fractures are known to form a star pattern of radial cracks (Fig 30, top left) which propagate outward from the loaded area. The failure occurs when the circumferential cracks begin to form, as indicated by the load-deflection diagram in Fig 30 (bottom).

This problem was initially analyzed under the assumption of full-through bending cracks, in which case the asymptotic scaling law for large cracks again appears to be of the type $h^{-3/8}$ (Slepyan, 1990; Bažant 1992). However, experiments as well as finite element analyses show that the radial cracks before failure do not reach through the full thickness of the ice plate, as shown in Fig 30 (top). This enormously complicates the analysis.

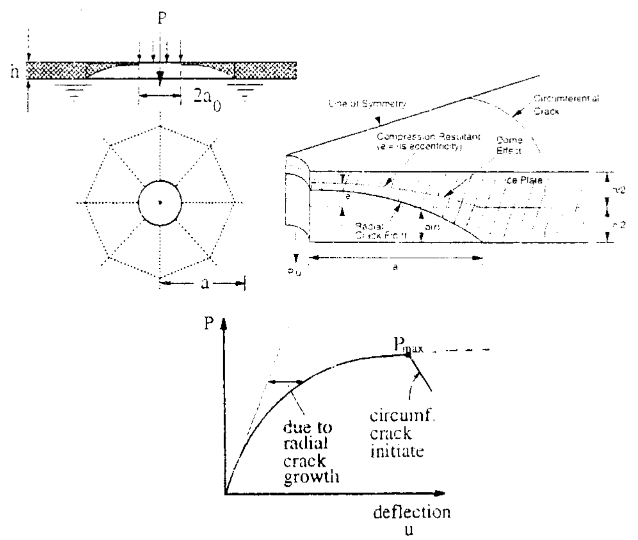

Fig 30. Top left: Radial and circumferential cracks caused by vertical penetration of an object through floating sea ice plate; Top right: Part-through radial crack and shift of compression resultant causing dome effect; Bottom: Typical load deflection diagram

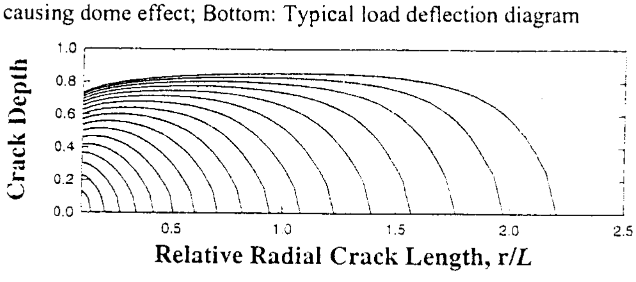

Fig 31. Calculated subsequent profiles of the radial part-through crack (the plate thickness is strongly exaggerated)

To solve this problem (Bažant and Kim, 1997), the elasticity of one half of the sector of the floating plate limited by two adjacent radial cracks is characterized by a compliance matrix obtained numerically. The radial cross section with the crack is subdivided into narrow vertical strips. In each strip, the crack is assumed to initiate through a plastic stage (representing an approximation of the cohesive zone). This is done according to a strength criterion (in the sense of Dugdale model), with constant in-plane normal stress assumed within the cross section part where the strain corresponding to the strength limit is exceeded. For the subsequent fracture stage, the relationship of the bending moment M and normal force N in each cracked strip to the additional rotation and in-plane displacement caused by the crack is assumed to follow the nonlinear line spring model of Rice and Levy (1972). The transition from the plastic stage to the fracture stage is assumed to occur as soon as the fracture values of M and N become less than their plastic values (to do this consistently, the plastic flow rule is assumed such that the ratio M/N would always be the same as for fracture).

This analysis (Bažant and Kim, 1997) has provided the profiles of crack depth shown in Fig 31, where the last profile corresponds to the maximum load (the plate depth is greatly exaggerated in the figure). The figure also shows the radial distribution of the nominal stresses due to bending moment and to normal force. The normal forces transmitted across the radial cross section with the crack are found to be quite significant. They cause a dome effect which helps to carry the vertical load.

An important question in this problem is the number of radial cracks that form. The solution (Bažant and Li, 1995) shows that the number of cracks depends on the thickness of the plate and has a significant effect on the scaling law.

Numerical solution of the integral equation along the radial cracked section, expressing the compatibility of the rotations and displacements due to crack with the elastic defor mation of the plate sector between two cracks, yields the size effect plot shown in Fig 32. The numerical results shown by data points can be relatively well described by the generalized size effect law of Bažant, shown in the figure. The top of

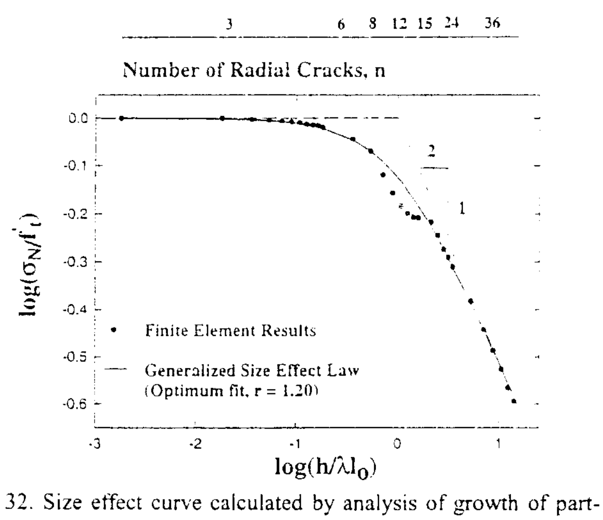

Fig 32. Size effect curve calculated by analysis of growth of part-through cracks, with varying number of radial cracks for different thickness ranges

the figure indicates the number of radial cracks for each range of crack thicknesses. The deviation of the numerical results from the smooth curve, seen in the middle of the range in the figure, is probably caused by insufficient density of nodal points near the fracture front. As confirmed by Fig 32, the asymptotic size effect does not have the slope −3/8 but the slope −1/2. Obviously, the reason is that, at the moment of failure, the cracks are not full-through bending cracks but grow vertically through the plate thickness.

11 SIZE EFFECT IN THE COHESIVE (FICTITIOUS) CRACK MODEL

According to the cohesive crack model, introduced for concrete under the name fictitious crack model by Hillerborg *et al* (1976), the crack opening in the fracture process zone (cohesive zone) is assumed to be a unique decreasing function of the crack-bridging stress (cohesive stress) σ; $w = g_i(\sigma)$. The basic equations of the cohesive crack model express that the crack opening calculated from the bridging stresses must be compatible with the elastic deformation of the surrounding structure, and the condition that the stress intensity factor K at the tip of the cohesive crack must be zero in order for the stress to be finite. They read:

$$g[\sigma(\xi)] = -\int_{\alpha_0}^{\alpha} D\, C^{\sigma\sigma}(\xi,\xi')\sigma(\xi')d\xi' + D\, C^{\sigma P}(\xi)P \quad (42)$$

$$K = -\int_{\alpha_0}^{\alpha} k_\sigma(\xi)\sigma(\xi)D d\xi + Pk_p = 0 \quad (43)$$

in which $\xi = x/D$, x = coordinate along the crack (Fig 33), $\alpha = a/D$, $\alpha_0 = a_0/D$, a, a_0 = total crack length and traction free crack length, $C^{\sigma\sigma}(\xi,\xi')$, $C^{\sigma P}(\xi)$ = compliances of the surrounding elastic structure for loads and displacements at the crack surface and at the loading point (Fig 33), and $k_\sigma(\xi)$, k_P = stress intensity factors at the tip of cohesive crack ($x = a$) for unit loads applied at the crack surface or at the loading point.

The usual way to solve the maximum load of a given structure according to the cohesive crack model was to integrate these equations numerically for step-by-step loading (Petersson, 1981). However, recently it was discovered that, under the assumption that there is no unloading in the cohesive cracks (which is normally the case), the size effect plot can be solved directly, without solving the history of loading before the attainment of the maximum load. As shown by Li and Bažant (1996), it is convenient to invert the problem such that one looks for the size D for which a given relative crack length $\alpha = a/D$ corresponds to the maximum load P_{max}. Then it is found that this size D represents the first eigenvalue of the following integral equation over the crack bridging zone:

$$D\int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi,\xi')\upsilon(\xi')d\xi' = -g'[\sigma(\xi)]\upsilon(\xi) \quad (44)$$

in which the eigenfunction $\upsilon(\xi)$ has the meaning of the derivative $\partial\sigma(\xi)/\partial\alpha$. The maximum load is then given by the following quotient

$$P_{max} = \frac{\int_{\alpha_0}^{\alpha} \upsilon(\xi)d\xi}{D\int_{\alpha_0}^{\alpha} C^{\sigma P}(\xi)\upsilon(\xi)d\xi} \quad (45)$$

These results have also been generalized to obtain directly the load and displacement corresponding, on the load deflection curve, to a point with any given tangential stiffness, including the displacement of the snap-back point which characterizes the ductility of the structure.

The cohesive crack model nicely illustrates the transition from failure at a relatively large fracture process zone for the case of small structures to the failure at a relatively small process zone for the case of large structures. See the plot of the profiles of the normal stress ahead of the tip of the traction-free crack length (notch length) shown in Fig 34. The points at the tip of the cohesive zone represent the maximum stress points in these stress profiles. Note how the maximum stress points move, in relative coordinates, closer to the tip of the notch if the structure size is increased. These results of the cohesive crack model confirm that, for large sizes, the size effect of LEFM should be approached.

12 INFLUENCE OF LOADING RATE AND FATIGUE ON SIZE EFFECT

Strictly speaking, fracture is always a time dependent phenomenon. In polymers, strong time dependence of fracture growth is caused primarily by viscoelasticity of the material (see the works of Williams, Knauss, Schapery, and others beginning with the 1960s). In rocks and ceramics, the time dependence of fracture is caused almost exclusively by the time dependence of the bond ruptures that cause fracture. In other materials such as concrete, both sources of time dependence are very important (Bažant and Gettu, 1992; Bažant and Wu, 1993; Bažant and Li, 1995). Both sources of time dependence have a significant but rather different influence on the scaling of fracture.

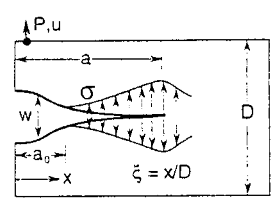

Fig 33. Cohesive crack and distribution of bridging stress

Consider first the rupture of an interatomic bond, which is a thermally activated process. The fre-

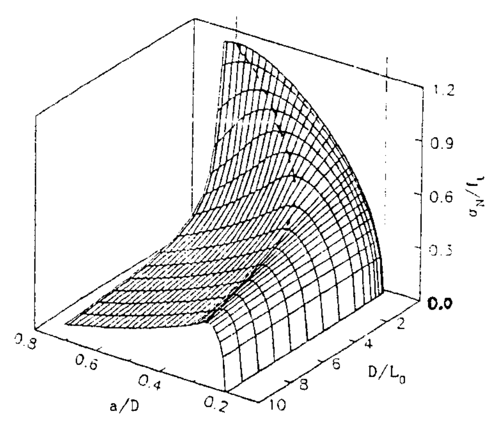

Fig 34. Stress profiles along the crack line for the maximum load and for various sizes of similar specimens (the peaks represent the tips of the cohesive crack)

quency of ruptures is given by the Maxwell-Boltzmann distribution, defining the frequency f of exceeding the strength of atomic bonds, $f \propto e^{-\mathcal{E}/RT}$, where T = absolute temperature, R = gas constant and $\mathcal{E}$ = energy of the vibrating atom. When

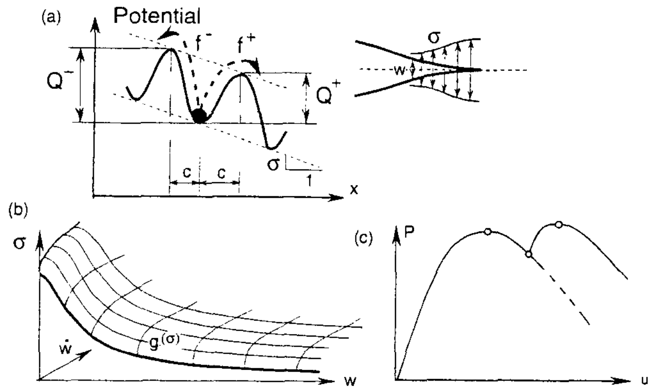

Fig 35. (a) Skewing of the potential surface of interactomic bond caused by applied stress, with corresponding reduction of activation energy Q^+; (b) Dependence of cohesive stress on crack opening and cohesive stress; (c) response change after a sudden increase of the loading rate

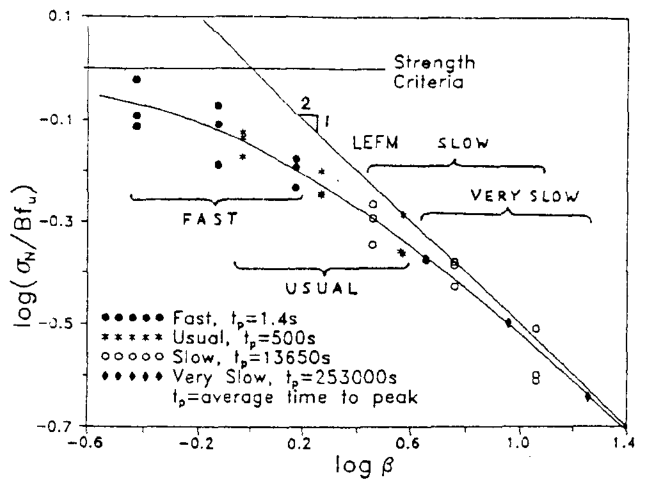

Fig 36. Nominal strengths of 4 groups of 3 specimens of different sizes tested at 4 different times to peak, t_p, plotted as as function of size relative size $\beta = D/D_0$ (after Bažant and Gettu, 1992)

a stress is applied, the diagram of the potential energy surface of the interatomic bonds is skewed as sketched in Fig 35a. This causes the activation barrier for bond breakages to be reduced from Q to a smaller value $Q - c\sigma$, and the activation barrier for bond restorations to be increased from Q to $Q + c\sigma$, where Q = activation energy = energy barrier at no stress, and c = constant. This causes that the frequency of bond ruptures, f^+, becomes greater than the frequency of bond restorations, f^-, with the net difference

$$\Delta f = f^+ - f^- \propto e^{-(Q-c\sigma)/RT} - e^{-(Q+c\sigma)/RT} \propto \sinh(c\sigma/RT)e^{-Q/RT} \tag{46}$$

The rate of the opening w of the cohesive crack may be assumed approximately proportional to Δf. From this, the following rate-dependent generalization of the crack-bridging (cohesive) law for the cohesive crack has been deduced (Bažant, 1993, 1995; Bažant and Li, 1995):

$$w = g\left[\sigma - \kappa e^{Q/RT} \operatorname{asinh}\left(\frac{\dot{w}}{c_0}\right)\right] \tag{47}$$

The dependence of the stress displacement curves for the cohesive crack on the crack opening rate $\dot{w}$ is shown in Fig 35b.

The effect of linear viscoelasticity in the bulk of the structure can be introduced into the aforementioned equations of the cohesive crack model on the basis of elastic-viscoelastic analogy (correspondence principle). Numerical solutions of fracture specimens show that viscoelasticity in the bulk (linear creep) causes the points in the size effect plot to shift to the right, toward increasing brittleness. This explains the observations of Bažant and Gettu (1992), which show the data points on the size effect plot for groups of similar small, medium and large notched specimens tested at various rates of crack mouth opening displacement (Fig 36). These rates are characterized by the time t_p to reach the peak. As revealed by Fig 36, the groups of data points move to the right with an increasing t_p.

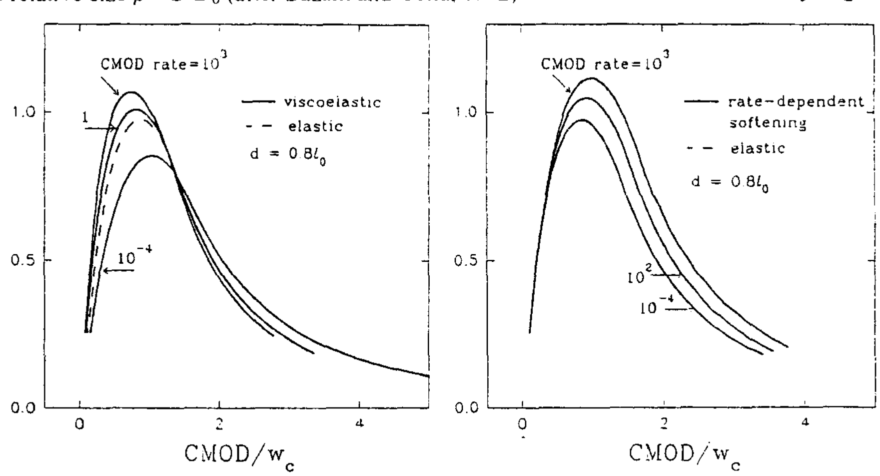

Fig 37. Curves of nominal stress *versus* relative crack mouth opening displacement (CMOD) for different CMOD rates, calculated by cohesive crack model under the assumption that the material exhibits only viscoelasticity in the bulk (left) or only rate-dependent crack opening (right) (Li and Bažant, 1995)

The fact that the brittleness of response is increasing with a decreasing rate of loading or increasing load duration may at first be surprising but can be explained (as revealed by calculations according to the time dependent cohesive crack model) by relaxation of the stresses surrounding the fracture process zone, which cause the process zone to become shorter. This behavior is also clarified by the plot of the nominal strength (normalized with respect to the material strength f_t') *versus* the crack mouth opening displacement (normalized with respect to the critical crack opening w_c). For a specimen in which the only source of time dependence is creep, the peaks of these stress displacement curves shift with an increasing rate of loading to the left and the softening curves cross (Fig 37 left). On the other hand, when the rate dependence is caused only by the bond breakages, the peaks shift to the right, as seen in Fig 37 (right), and in that case there is no shift of brittleness of the kind seen in Fig 36. It must be emphasized that these results are valid only in the range of static loading, that is, in absence of inertia forces and wave propagation effects. The behavior becomes more complicated in the dynamic range.

Related to the time dependence is the influence of fatigue on fracture (Paris and Erdogan, 1967). The rate of growth of a crack caused by fatigue loading is approximately given by the Paris law (or Paris-Erdogan law) which reads: $\Delta a/\Delta N = \kappa(\Delta K_I/K_{Ic})^n$, in which a = crack length, N = number of cycles, ΔK_I = amplitude of the applied stress intensity factor: κ, n = dimensionless empirical constants; and K_{Ic} = fracture toughness introduced only for the purpose of dimensionality. The interesting point is that the rate of growth does not depend on the maximum and minimum values of K_I, as a good approximation.

This law has found wide applicability for fatigue growth of cracks in metals. If similar structures with similar cracks are considered, this equation implies the size effect of LEFM, which is however too strong for not too large quasibrittle structures. It was shown (Bažant and Xu, 1991; and Bažant and Schell, 1993) that the Paris law needs to be combined with the size effect law for monotonic loading, yielding the following generalization of Paris law in which the effect of structure size D is taken into account:

$$\frac{\Delta a}{\Delta N} = \kappa\left(\frac{\Delta K_I}{K_{Ic}}\sqrt{1+\frac{D_0}{D}}\right)^n \tag{48}$$

in which D_0 is the same exponent as in Paris law, and K_{Ic} is a constant denoting the fracture toughness of an infinitely large structure.

The necessity of the size correction is demonstrated by the test results of Bažant and Xu (1991) for concrete in Fig 38. At constant size D, the logarithmic plot of the crack growth rate *versus* the amplitude of K_I should be approximately a straight line. This is clearly verified by Fig 38. However, for different specimen sizes, different lines are obtained. The spacing of these straight lines is well predicted by Eq (48), while for the classical Paris law these three lines would have to be identical.

13 SIZE EFFECT IN COMPRESSION FRACTURE

The fracture of quasibrittle materials due to compressive stress is one of the most difficult aspects of fracture mechanics. In compression fracture, one must distinguish two distinct phenomena: 1) micromechanics of initiation of compression fracture, and 2) mechanics of global compression fracture causing failure. The first problem has been investigated much more than the second, and various micromechanical mechanisms that initiate fracture under compressive stresses have been identified; *eg*, the growth of axial splitting cracks from voids (Cotterell 1972, Sammis and Ashby 1986; Kemeny and Cook, 1987, 1991; Steif, 1984; Wittmann and Zaitsev, 1981; Zaitsev, 1985; Fairhurst and Cornet, 1981; Ingraffea Heuzé, 1980; Nesetova and Lajtai, 1973; Carter, 1992; Yuan *et al*, 1993) or near inclusions, the creation of axial splitting cracks by groups of hard inclusions, and the formation of wing-tip cracks from sliding inclined surfaces (Hawkes and Mellor, 1970; Ingraffea, 1977; Ashby and Hallam, 1986; Horii and Nemat-Nasser, 1982, 1986; Sanderson, 1988; Schulson, 1990; Costin, 1991; Batto and Schulson, 1993; Schulson and Nickolayev, 1995; Lehner and Kachanov, 1996; and a critique by Nixon, 1996).

It must be realized, however, that these mechanisms do not explain the global failure of the structure. They can cause only a finite extension of the axial splitting cracks whose length is of the same order of magnitude as the size of the void, the inclusion, or the inclined microcrack. Each of these mechanisms can produce a zone of many splitting cracks approximately parallel to the uniaxial compressive stress or, under triaxial stress states, to the compressive principal stress of the largest magnitude. Biot (1965) proposed that the cause of compression failure may be three-dimensional internal buckling which can occur either in the bulk of specimen or within an inclined band. However, he considered only elastic behavior and did not conduct any energy analysis. Finite strain analysis of compression failure caused by internal buckling of an orthotropically damaged material or orthotropic laminate was analyzed by Bažant (1967). Kendall (1978) showed that, with the consideration of buckling phenomena under eccentric compressive loads, the energy balance condition of fracture mechanics yields realistic predictions of compression fracture of test cylinders loaded only on a part of the end surface.

The global compression fracture has been analyzed (Bažant 1993, Bažant and Xiang 1997) under the hypothesis that some of the aforementioned micromechanisms creates a band of axial splitting cracks as shown in Fig 39, which

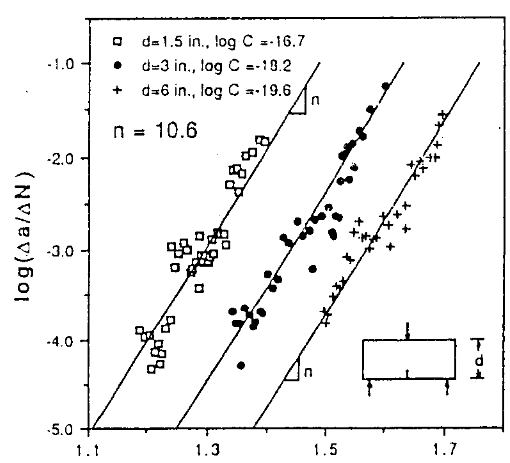

Fig 38. Crack growth per cycle *versus* amplitude or relative stress intensity factor for three different sizes of concrete specimens (after Bažant and Xu, 1991)

propagates laterally, in a direction either inclined or normal to the direction of the compressive stress of the largest magnitude (Bažant, 1993; Bažant and Xiang, 1997). In the post-peak regime, the axial splitting cracks interconnect to produce what looks as a shear failure although there is no shear slip before the post-peak softening (in fact, shear failure *per se* is probably impossible in concrete). The energy analysis of the propagating band of axial splitting cracks shows that, inevitably, there ought to be a size effect. Let us discuss it for the prismatic specimen shown in Fig 39.

Formation of the axial splitting cracks causes a narrowing of the band and, in an approximate sense, a buckling of the slabs of the material between the splitting cracks as shown in the figure (alternatively, this can be modeled as internal buckling of damaged continuum). This causes a reduction of stress, which may be considered to occur approximately in the shaded triangular areas. For the calculation of the energy change within the crack band one needs to take into account the fact that the slabs of material between the axial splitting cracks ought to undergo significant post-buckling deflections corresponding to the horizontal line 3-5. Thus, the energy change in the splitting crack band is given by the difference of the areas 0120 and 03560 (the fact that there is a residual stress σ_{cr} in compression fracture is an important difference from a similar analysis of tensile crack band propagation). The energy released must be consumed and dissipated by the axial splitting cracks in the band. This is one condition for the analysis.

The second condition is that the narrowing of the band due to microslab buckling must be compatible with the expansion of the adjacent triangular areas due to the stress relief. One needs to write the condition that the shortening of segment HI in Fig 39 on top left is compensated for by the extension of segments GH and IJ, which is a compatibility condition. The energy release from the crack band is given by the change of the areas under the stress-strain diagrams in the middle of Fig 39 (bottom), caused by the drop of stress from the initial compressive stress σ_0 to the final compressive stress σ_{cr} carried by the band of splitting cracks.

The resulting size effect on the nominal strength of large structures failing in compression has, according to this analysis, the form:

$$\sigma_N = C_1 D^{-2/5} + C_0 \qquad (49)$$

where C_1, C_0 = constants.

Mathematical formulation of the foregoing arguments (Bažant, 1993; Bažant and Xiang, 1997) provided a formula for the compression failure which exhibits a size effect. This size effect is plotted in Fig 39(f), with the logarithm of size D as a coordinate and either $\log \sigma_N$ or $\log (\sigma_N - \sigma_r)$ as the ordinate. In the latter plot (Fig 39 f), the size effect is shown to approach an asymptote of slope −2/5. This is another interesting feature, which results from the fact that the spacing of the axial splitting cracks is not constant but depends on the overall energy balance. The solution of the nominal strength of σ_N has been obtained under the assumption of arbitrary spacing s, and it was noted that σ_N exhibits a minimum for a certain spacing s, which depends on size D. It is this condition of minimum which causes the asymptotic slope to be −2/5 instead of −1/2.

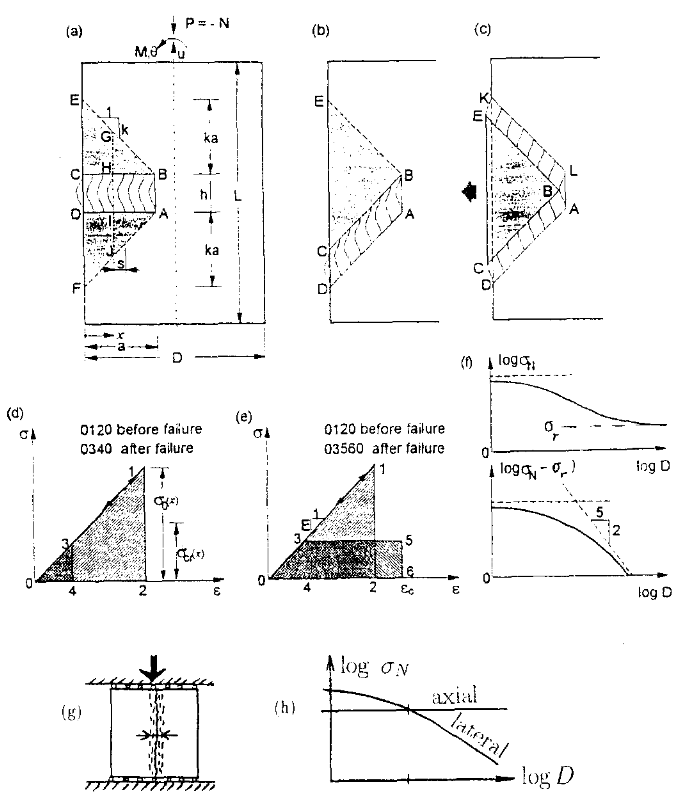

Fig 39. (a, b, c) Sideways propagations of a band of axial splitting cracks, with energy release zones, (d, e) reduction of strain energy density outside and inside the band, and (f) resulting approximate size effect curve

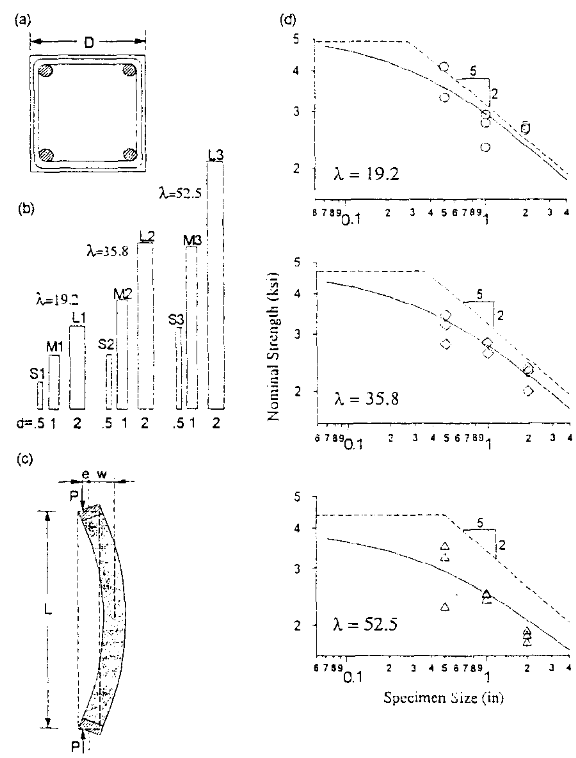

Fig 40. (a, b, c) Reduced-scale reinforced concrete columns of different sizes and slendernesses, tested by Bažant and Kwon (1993); (d) Measured nominal strength *versus* column size, and fits by formula (Bažant and Xiang, 1997)

The foregoing approximate theoretical results, given by simple formulas (Bažant, 1993; Bažant and Xiang, 1997) have been compared to the test results on size effect in reduced-scale tied reinforced concrete columns of three different sizes (in the ratio 1:2:4) and three different slendernesses, λ = 19.2, 35.8 and 52.5. The columns were made of concrete with reduced aggregate size. The test results indicated a size effect which is seen in Fig 40 (and is ignored by the current design codes). The formulas obtained by the foregoing approximate energy analysis of the propagation of a band of axial splitting cracks are shown by the solid curves in the figures, indicating a satisfactory agreement.

Why do small uniaxial compression specimens fail by an axial splitting crack and exhibit no size effect? In a uniform uniaxial stress field, a sharp planar axial crack does not change the stress and thus releases no energy. Therefore a damage band of finite width (Fig 39g) must precede the formation of an axial splitting crack. The energy is released only from this band but not from the adjacent undamaged solid. Therefore, the energy release is proportional to the length of the axial splitting crack, which implies that there is no size effect (Fig 39h). Thus, the lateral propagation of a band of splitting cracks, which involves a size effect, must prevail for a sufficiently large specimen size (Fig 39h, Bažant and Xiang, 1996). The reason that the axial splitting prevails for a small enough size is that the overall fracture energy consumed (and dissipated) by a unit axial extension of the splitting crack band is smaller than that consumed by a unit lateral extension, for which new cracks must nucleate.

A size effect is known to occur also in the breakout of boreholes in rock, as experimentally demonstrated by Nesetova and Lajtai (1992), Carter (1992), Carter *et al* (1992), Yuan *et al* (1992), and Haimson and Herrick (1989). It is known from the studies of Kemeny and Cook (1987, 1981) and others that the break out of boreholes occurs due to the formation of splitting cracks parallel to the direction of the compressive stress of the largest magnitude, $\sigma_{y\infty}$. An approximate energy analysis of the breakout was conducted under the simplifying assumption that the splitting cracks occupy a growing elliptical zone (although in reality this zone is narrower and closer to a triangle). The assumption of an elliptical boundary permitted the energy release from the surrounding infinite solid to be easily calculated according to Eshelby's theorem for eigenstrains in ellipsoidal inclusions (Bažant, Lin, and Lippmann, 1993). According to the theorem, the energy release from the infinite rock mass can be approximated as

$$\Delta\Pi = -\pi\Big[(a+2R)R\sigma_{x\infty}^2 + (2a+R)a\sigma_{y\infty}^2 - 2aR\sigma_{x\infty}\sigma_{y\infty} - 2a^2\sigma_{cr}^2\Big](1-\nu^2)/2E \tag{50}$$

in which R = borehole radius, a = principal axis of the ellipse (Fig 41), $\sigma_{x\infty}$ and $\sigma_{y\infty}$ = remote principal stresses, E = Young's modulus of the rock, and ν = Poisson ratio. A similar analysis as that for the propagating band of axial splitting cracks, already explained, has provided a formula for the breakout stress which has a plot similar to those in Fig 39f, and has the asymptotic behavior described by Eq (49).

14 ASYMPTOTIC SCALING FOR TENSILE OR COMPRESSIVE CRACKS WITH RESIDUAL COHESIVE STRESS

In the case of compression fracture due to lateral propagation of a band of axial splitting cracks (to be discussed in Section 13), a residual stress given by the critical stress for internal buckling in the band remains. Lumping the fracturing strains in the band into a line, one may approximately treat such a fracture as a line crack in which interpenetration of the opposite faces is allowed and the softening compressive stress-displacement law terminates with a plateau of residual constant stress σ_Y. Likewise, a constant residual stress σ_Y may be assumed for characterizing the tensile stress-displacement law for a crack in a fiber-reinforced composite (*eg*, fiber-reinforced concrete).

The asymptotic formulae Eqs (35)-(37) for the case of many loads can be applied to this case because the uniform pressure σ_Y along the crack can be regarded as one of two loads applied on the structure. We write the stress intensity factors due to the applied load P and the uniform crack pressure σ_Y as $K_I^2 = \sigma_N^2 D\, g(\alpha_0+\theta)$ (with $\theta = c_f/D$), and $K_I^2 = \sigma_y^2\, \gamma(\alpha_0+\theta)$, respectively, where g and γ are dimensionless functions taking the role of g_1 and g_2 in the preceding formulae. In this manner, Eqs (35) and (36) yield, after rearrangements, the following formula for the size effect (and shape effect) in the case of a large crack:

$$\sigma_N = \frac{\sqrt{EG_f} + \sigma_Y\sqrt{\gamma'(\alpha_0)c_f + \gamma(\alpha_0)D}}{\sqrt{g'(\alpha_0)c_f + g(\alpha_0)D}} \tag{51}$$

For geometrically-similar structures and size-independent α_0, this formula yields a size effect curve that terminates, in the log D scale, with a horizontal asymptote in the manner shown in Fig 18 (top right) and 27, but has also a horizontal asymptote on the left.

In the case of initiation of a crack with uniform residual stress σ_Y, equations Eqs (35 and (37) can be reduced to the following size (and shape) effect formula:

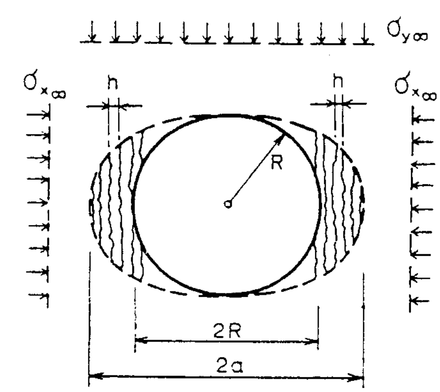

Fig 41. Borehole in rock and growth of an elliptical zone of axial splitting cracks (after Bažant, Lin, and Lippmann, 1993)

$$\sigma_N = \frac{\sqrt{EG_f} + \sigma_Y \sqrt{\gamma'(0)c_f + \frac{1}{2}\gamma''(0)\frac{c_f^2}{D}}}{\sqrt{g'(0)c_f + \frac{1}{2}g''(0)\frac{c_f^2}{D}}} \tag{52}$$

whose logarithmic plot also terminates with a horizontal asymptote as in Fig 18 (top right).

If the residual stress is compressive and is determined by internal buckling in a band of axial splitting cracks of arbitrary spacing, then σ_Y in the foregoing equations is not constant. As already explained, minimization of σ_N with respect to the crack spacing s shows that the crack spacing in the band should vary with D. Then, in Eqs (51) and (52):

$$\sqrt{EG_f} \quad \text{must be replaced by} \quad \sqrt{EG_f}\,D^{1/10} \tag{53}$$

Furthermore, the σ_Y value will also depend on the crack spacing, according to the formula for the critical buckling load. The overall trend will be well approximated by Eq (49), and in particular σ_N will approach the large-size asymptotic limit as $D^{-2/5}$.

The last two formulae ought to be also applicable to the compression kink bands in wood or in composites reinforced by parallel fibers. This problem has so far been treated by elasto-plasticity, and solutions of failure loads which give good agreement with the existing test data have been presented (Rosen, 1965; Argon, 1972; Budianski, 1983; Budianski *et al*, 1997; Budianski and Fleck, 1994; Kyriakides *et al*, 1995; Christensen and DeTeresa, 1997). Measurements of the size effect over a broad size range, however, appear to be unavailable at present, and there is good reason to suspect that a size effect exists. This is indicated by observing that: the shear slip and fracture along the fibers in the kink band probably exhibits softening, *ie*, a gradual reduction of the shear stress to some final asymptotic value, and the kink band does not form simultaneously along the entire kink band but has a front that propagates, in the manner of the band of parallel compression splitting cracks (analyzed in Section 13).

Arguing in favor of his MFSL, Carpinteri made the point that some measured size effect plots (of log σ_N *versus* log D) exhibit a positive curvature and approach a horizontal asymptote (as in Fig 18). However, as we have seen by now, this can have any of the following four (deterministic nonfractal) causes:

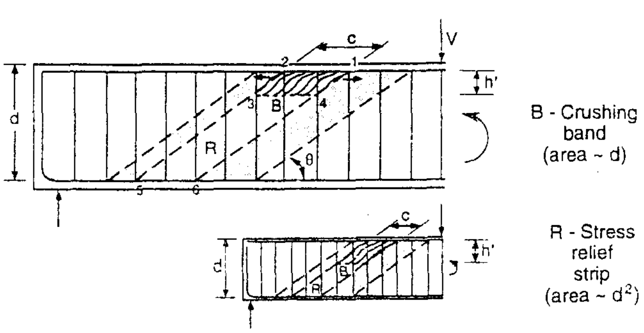

Fig 42. Fracture adaptation of truss model for diagonal shear failure of reinforced concrete beams: Compression crushing zone and energy release zone in beams of different sizes (after Bažant, 1996)

1. In unnotched structures, the relative length α_0 of traction-free crack might not be constant but may decrease with increasing size D.
2. There may be a residual cohesive (crack-bridging) stress σ_Y in the crack.
3. The failure may occur at the initiation of macroscopic crack growth.
4. Above a certain size D, there may be a transition to some plastic failure mechanism (as in the Brazilian test discussed in Section 4).

15 FRACTURING TRUSS MODEL FOR SHEAR FAILURE OF REINFORCED CONCRETE

It appears that compression failure is also the final failure mechanism in shear failures of reinforced concrete beams, such as diagonal shear of beams and torsion of beams, punching of plates, pullout of anchors, failure of corbells and frame connections, etc. The importance of the size effect in shear failure of beams has been experimentally documented by many investigators (Leonhardt and Walter, 1962; Kani, 1967; Kupfer, 1964; Leonhardt, 1977; Walraven, 1978, 1995; Iguro *et al*, 1985; Shioya *et al*, 1989; Shioya and Akiyama, 1994; Bažant and Kazemi, 1991; Walraven and Lehwalter, 1994; see also Bažant and Kim, 1994; Bažant and Cao, 1986, 1987; Bažant and Sun, 1987; Bažant, Şener and Prat, 1988; Mihashi *et al*, 1993). Let us briefly outline the mechanics (Bažant, 1996) of the size effect in the diagonal shear failure of reinforced concrete beams.

According to the truss model of Ritter (1899) and Mörsch, (1903), refined by Nielsen and Braestrup (1975), Thürlimann (1976), Collins (1978), Collins *et al* (1976, 1996), Marti (1980, 1985), Collins and Mitchell (1980), Hsu (1988, 1995), and Schlaich *et al* (1987), and others, and recently called the strut-and-tie model, a good approximation is to assume that a system of inclined parallel cracks forms in the high shear zone of a reinforced concrete beam before the attainment of the maximum load (Fig 42). The cracks are assumed to be continuous and oriented in the direction of the principal compressive stress (which is, of course, an approximation). This assumption implies that there is no shear stress on the crack planes and that the principal tensile stress has been reduced to 0. According to this simplified picture, the beam acts as a truss consisting of the longitudinal reinforcing bars, the vertical stirrups (which are in tension, and the inclined compression struts of concrete between the cracks. If the reinforcing bars and stirrups are designed sufficiently strong, there is only one way the truss can fail – by compression of the diagonal struts.

In the classical approach, the compression failure of the struts has been handled according to the strength concept which, however, cannot capture the localization of compression fracture and implies the compression fracture to occur simultaneously everywhere in the inclined strut. In reality, the compression fracture, called the crushing, develops within only a portion of the length of the strut (in a region with stress concentrations, as on the top of beam in Fig 42). Then it propagates across the strut. For the sake of simplicity, the band of axial splitting cracks forming the crushing zone may be assumed to propagate as shown in Fig 42 and reach, at maximum load, a certain length c. The depth of the crushing band may be expected to increase initially but later to stabilize at a certain constant value h governed by the size of aggregate.

It is now easy to explain how the size effect arises. Because of the existence of parallel inclined cracks at maximum load, the formation of the crushing band reduces stress in the entire inclined white strip of width c and depth d (beam depth shown in Fig 42). The area of the white strip is cd or $(c/d)d^2$ and its rate of growth is $(c/d)\,2d\dot{d}$, in which c/d is approximately a constant when similar beams of different sizes are compared. So, the energy release rate is proportional to $\sigma_N^2 d\dot{d}/E$, where the nominal strength is defined as $\sigma_N = V/bd$ = average shear stress, V = applied shear force, and b = beam width. The energy consumed is proportional to the area of the crushing band, ch or $(c/d)hd$, that is, to $G_f d/s$, and its rate of $G_f \dot{d}/s$ where G_f = fracture energy of the axial splitting cracks (s = crack spacing). This expression applies asymptotically for large beams because for beams of a small depth d the full width h of the crushing band cannot develop. Equating the derivatives of the energy release and energy dissipation expressions, *ie*, $\sigma_N^2 d\dot{d}/E \propto G_f \dot{d}/s$, we conclude that the asymptotic size effect ought to be of the form:

$$\sigma_N \propto s^{-1}\sqrt{EG_f d} \tag{54}$$

The complete size effect represents a transition from a horizontal asymptote to the inclined asymptote in the size effect plot given by this equation. Relatively simple design formulas are obtained in this manner (Bažant, 1996). The analysis can also be done in a similar way for the diagonal shear failure of beams with longitudinal reinforcement but without vertical stirrups, and further for torsion, etc.

16 NUMERICAL SIMULATION OF FRACTURE OR DAMAGE WITH SIZE EFFECT

A broad range of numerical methods which can simulate damage localization, fracture propagation and size effect is now available. They can be classified as follows:

1. Discrete fracture, with elastic analysis:
   a. R-curve model
   b. cohesive (fictitious) crack model
2. Distributed cracking damage - nonlinear analysis by:
   a. Finite elements:
      i Crack band model
      ii Nonlocal damage model:
         A averaging type (semi-empirical)
         B based on crack interactions (micromechanics)
      iii Gradient localization limiter:
         A 1st gradient
         B 2nd gradient
         C diffusion-type limiter
   b. Discrete elements - random particle model:
      i with axial forces only (random truss model)
      ii with transmission of both normal and shear forces between particles.

The simplest is the R-curve approach, which can often yield an analytical solution. The cohesive (or fictitious) crack model is efficient if the behavior of the elastic body surrounding the cohesive crack is characterized a priori by a compliance matrix or a stiffness matrix. A great complication arises in general applications in which the direction of fracture propagation is usually unknown. For such situations, Ingraffea (1977, with later updates) has had great success in developing an effective remeshing scheme (in his computer program FRANC); however, this approach has not yet spread into practice.

The engineering firms and commercial finite element programs (*eg* DIANA; SBETA, Červenka and Pukl, 1994), as it seems, use almost exclusively the crack band model. This model is the simplest form of finite element analysis that can properly capture the size effect. The basic idea in the crack band model (Bažant, 1982; Bažant and Oh, 1983) is to describe fracture or distributed cracking by a band of smeared cracking damage that has a single element width, and to treat the band width, *ie*, the element size in the fracture zone, as a material property (as proposed by Bažant, 1976). This is the simplest approach to avoid spurious mesh sensitivity and ensure that the propagating crack band dissipates the correct amount of energy (given by the fracture energy G_f).

A more general and more powerful but also more complex approach is the nonlocal damage approach, in which the stress at a given point of the continuum does not depend only on the strain and that point but also on the strains in the neighborhood of the point. While the crack band model can be regarded as a simplified version of the nonlocal concept, the truly nonlocal finite element analysis involves calculation of the stress from the stress values in the neighboring finite elements. The simplest and original form (Bažant, Belytschko, and Chang, 1984; Bažant, 1984) involves an empirical weighted averaging rule. There are many possible versions of nonlocal averaging. But the most realistic results (Jirásek, 1996) are apparently obtained with a nonlocal approach in which the secant stiffness matrix for the strain-softening stress-strain relation (which describes the evolution of damage or smeared cracking) is calculated from the spatially averaged strains and the stress is then obtained by multiplying with this matrix the local strain.

Physically, a more realistic nonlocal damage model is obtained by continuum smearing of the matrix relations that describe interactions among many cracks in an elastic solid. One type of such a matrix interaction relation, due to Kachanov (1985, 1987), has led to the following field equation (Bažant 1994):

$$\Delta\bar{S}^{(1)}(\boldsymbol{x}) - \int_V \Lambda(\boldsymbol{x},\xi)\Delta\bar{S}^{(1)}(\xi)dV(\xi) = \left\langle \Delta S^{(1)}(\boldsymbol{x})\right\rangle \tag{55}$$

This is a Fredholm integral equation in which V = volume of the structure; $\Lambda(\boldsymbol{x}, \xi)$ = crack influence function, characterizing in a statistically smeared manner the normal stress across a frozen crack at coordinate $\boldsymbol{x}$ caused by a unit pressure applied at the faces of a crack at ξ; $\langle..\rangle$ is a spatial averaging operator; $\Delta S^{(I)}$ or $\Delta \bar{S}^{(I)}$ = increment (in the current loading step) of the principal stress labeled by (I) before or after the effect of crack interactions. The integral in this equation is not an averaging integral because its kernel has spatial average 0. The kernel is positive in the amplification sector of crack interactions and negative in the shielding sector. So, in this nonlocal damage model, aside from an averaging integral there is an additional nonlocal integral over the inelastic stress increments in the neighborhood. These increments model the stress changes that relax or enhance the crack growth. They reflect the fact that a neighboring crack lying in the shielding zone of a given crack inhibits the crack growth, while another crack lying in the amplification zone enhances the crack growth (Bažant, 1994; Bažant and Jirásek, 1994a, 1994b).

This formulation shows that the nonlocality of damage is principally a consequence of the interactions among microcracks and provides a physically based micromechanical model. Application of this concept in conjunction with the microplane constitutive model for damage has provided excellent results for fracture and size effect in concrete (Ožbolt and Bažant, 1996). However, the analysis is more complex than with the classical empirical averaging approach to nonlocal damage. In practical terms, what has been gained from the crack interaction approach is that the failures dominated by tensile and shear fractures could be described by one and the same material model with the same characteristic length for the nonlocal averaging. This proved impossible with the previous models.

If the characteristic length involved in the averaging integral of a nonlocal damage model is at least three times larger than the element size, the directional bias for crack (or damage) propagation along the mesh lines gets essentially eliminated. However, in some cases this may require the finite elements to be too small (although it is possible to adopt an artificially large characteristic length, provided that this is compensated by modifying the post-peak slope of the strain-softening constitutive equation so as to ensure the correct damage energy dissipation). If the characteristic length is too small, or if the crack band model is used, then it is desirable either to know the crack propagation direction in advance and lay the mesh lines accordingly, or to use a remeshing algorithm of the same kind as developed by Ingraffea (1977) for the discrete crack model.

The earliest nonlocal damage model, in which not only the damage but also the elastic response was nonlocal, exhibited spurious zero-energy periodic modes of instability, which had to be suppressed by additional means, such as element imbrication (Bažant *et al*, 1984; Bažant 1984). This inconvenience was later eliminated by the formulation of Pijaudier-Cabot and Bažant (1987) (see also Bažant and Pijaudier-Cabot, 1988), in which the main idea was that only the damage, considered in the sense of continuum damage mechanics (and later also yield limit degradation, Bažant and Lin 1988), should be nonlocal and the elastic response should be local. The subsequent nonlocal continuum models with an averaging type integral were various variants on this idea.

From the viewpoint of finite element analysis, the principal purpose of introducing the nonlocal concept is to prevent arbitrary spurious localization of damage front into a band of vanishing width. Because, in the damage models with strain softening, the energy dissipation per unit volume of material (given by the area under the complete stress-strain curve) is a finite value, a vanishing width of the front of the damage band implies the fracture to propagate with zero-energy dissipation, which is obviously physically incorrect. This phenomenon also gives rise to spurious mesh sensitivity of the ordinary (local) finite element solutions according to continuum damage mechanics with strain softening.

From the physical viewpoint, the strain softening, characterized by a non-positive definite matrix of tangential moduli, appears at first sight to be a physically suspect phenomenon because it implies the wave speed to be complex (and thus wave propagation to be impossible), and because it implies the type of partial differential equation for static response to change from elliptic to hyperbolic (Hadamard, 1903; Hill, 1962; Mandel, 1964; Bažant and Cedolin, 1991; Chapter 13). These problems are in general avoided in two ways: 1) by introducing some type of a mathematical device, called the localization limiter, which endows the nonlocal continuum damage model with a characteristic length, and 2) by recognizing that the rate-dependence of softening damage is not negligible.

The conclusion that strain softening causes the wave speed to be complex rather than real, however, is an oversimplification, because of two phenomena. First, a strain softening material can always propagate unloading waves, because the tangent stiffness matrix for unloading always remains positive definite, as discovered experimentally in the 1960s (Rüsch and Hilsdorf, 1963; Evans and Marathe, 1968). Second, as revealed by recent tests at Northwestern University (Bažant and Gettu, 1992; Bažant, Guo, Faber, 1995; Tandon), a real strain-softening material can always propagate loading waves with a sufficiently steep front. The latter phenomenon is a consequence of the rate effect on crack propagation (bond breakage), which causes that a sudden increase of the strain rate always reverses strain softening to strain hardening (followed by a second peak); see Fig 35c. This phenomenon, which is mathematically introduced by Eq (47), is particularly important for the finite element analysis of impact.

Another type of localization limiter are the gradient limiters, in which the stress at a given point of the continuum is considered to depend not only of the strain at that point but also of the first or second gradients of strains at that point. This concept also implies the existence of a certain characteristic length of the material. It appears to give qualitatively reasonable results for various practical problems of damage propagation, as well as the size effect. However, it should be kept in mind that the gradient localization limiters have not been directly justified physically. They can be derived in the sense of an approximation to the nonlocal damage model with an integral of averaging type. Indeed, expansion of the kernel of the integral and of the strain field into Taylor series and truncation of these series yields the formulation with a gradient localization limiter (Bažant, 1984), and thus also justifies it physically (provided the integral formulation is based on the smearing of crack interactions).

The discrete element models for damage and fracture are a fracturing adaptation of the model for granular solids proposed by Cundall (1971) and Cundall and Strack (1979). They are very demanding for computer power. It is becoming more and more feasible as the power of computers increases. In these models, the material is represented by a system of particles whose links break at a certain stress. The typical spacing of the particles acts as a localization limiter, similar to the crack band model, and controls the rate of energy dissipation per unit length of fracture extension (Bažant, Tabbara *et al*, 1990). The particles can simulate the actual aggregate configurations in a material such as concrete, or may simply serve as a convenient means to impose a certain characteristic length on the model, as in the case of the simulation of sea ice floes (Jirásek and Bažant, 1995a, b).

In the case of isotropic materials, it is important that the configuration of particles be random. With a regular particle arrangement there is always a bias for fracture propagation along the mesh lines, even when all the properties of the particle links are randomized (Jirásek and Bažant, 1995b).

In the simplest discrete element model, the interactions between particles are assumed to be only axial. But that causes the Poisson ratio of the homogenizing continuum to be 1/4 for the three-dimensional case, or 1/3 for the two-dimensional case, and so materials with other Poisson ratios cannot be modeled (unless some artifices are used). Another disadvantage is that the damage band appears to be too narrow. An arbitrary Poisson ratio and a wider damage band can be achieved by a particle model in which the links between particles transmit not only axial forces but also shear forces. This is the case for the model of Zubelewicz (1983) and Zubelewicz and Bažant (1987), as well as the model of Schlangen and van Mier (1992) and van Mier and Schlangen

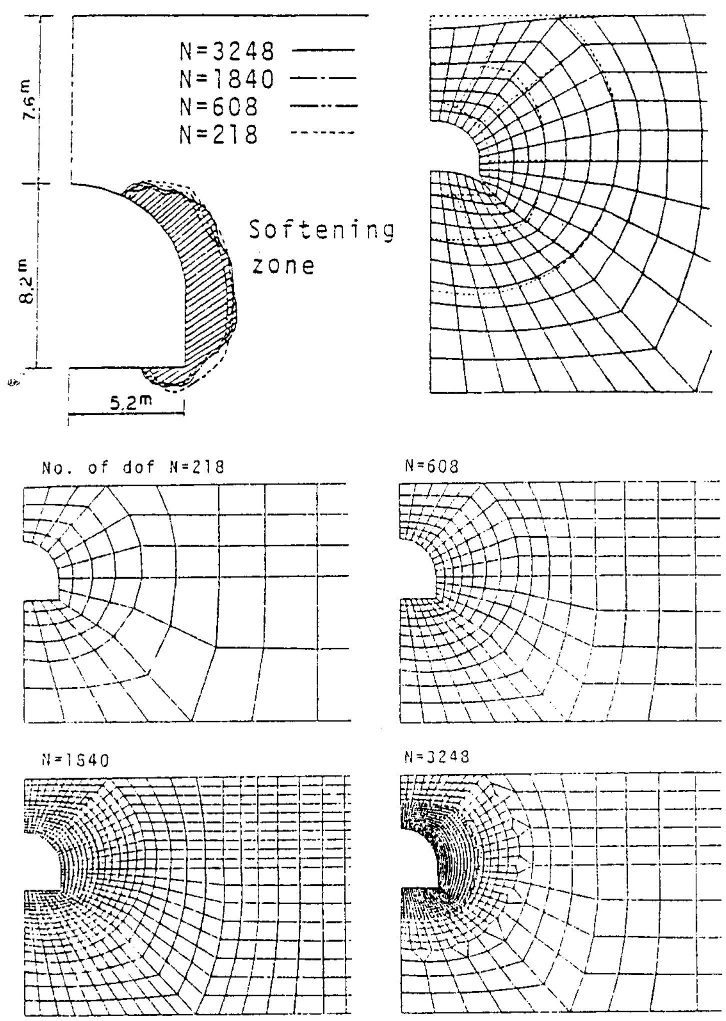

Fig 43. Analysis of tunnel excavation using nonlocal yield limit degradation, with deformed mesh (top right), and meshes of different refinements used (bottom) (after Bažant and Lin, 1988)

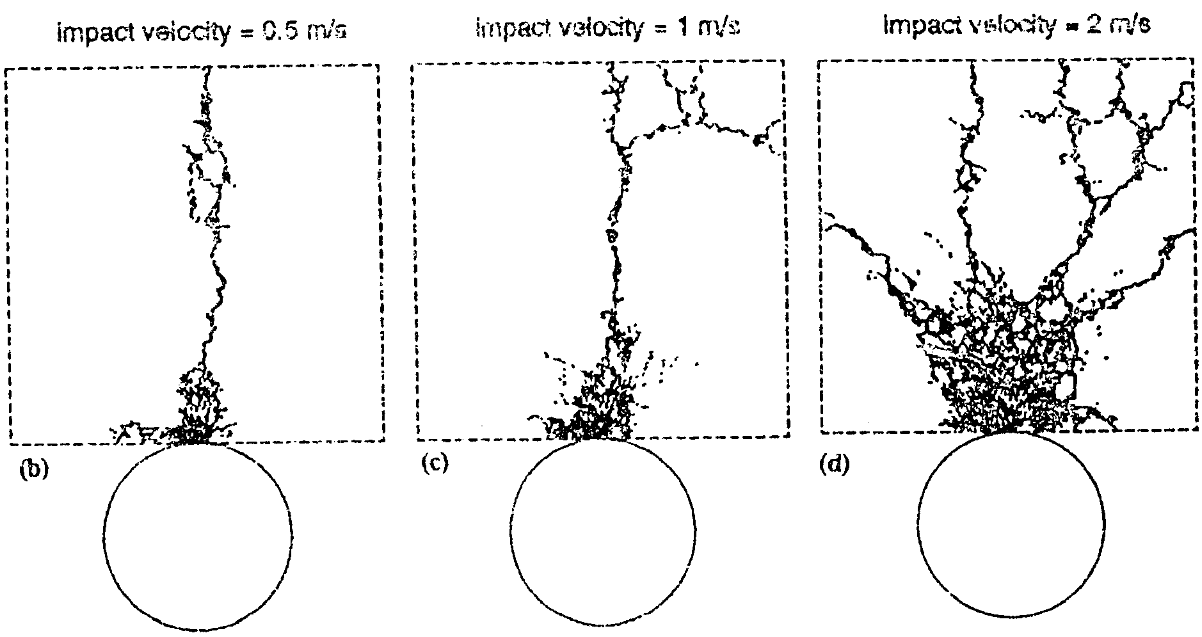

Fig 44. Random particle simulation of the breakup of an ice floe traveling at different velocities, after it impacts a rigid obstacle (Jirásek and Bažant, 1995b)

(1993). In the latter, the particle system is modeled as a frame with bars that undergo bending (the bending of the bars is of course fictitious and unrealistic, but it does serve the purpose of achieving a shear force transmission through the links between particles). Van Mier and co-workers have had considerable success in modeling concrete fracture in this manner.

An example of numerical solutions with nonlocal models and random particle models have already been given in Figs 16 and 17. Further two examples are presented in Figs 43 and 44, which show applications of a nonlocal finite element damage model to the analysis of failure of a tunnel excavated without lining, and to the simulation of the break-up of a traveling sea ice floe after it impacts a rigid obstacle.

17 CLOSING COMMENTS AND VIEW TO THE FUTURE

To close on a philosophical note, consider the gradual expansion of human knowledge (Fig 45). What is unknown may be imagined to form a circle. What is unknown lies outside. What can be discovered at any given stage of history is only what is in contact with the circle. Questions about what lies farther into the future cannot even be raised. In our field, the problem of strength of elastic frames was not even posed before Hooke. It started to be tackled in the middle of the 19th century and has been for the most part solved around 1960.

One of the most formidable problems in physics and mathematics has been that of turbulence, which has occupied the best minds for over a century and, as experts say, complete understanding is not yet in sight. The problem of scaling in quasibrittle materials is a part of damage mechanics, in which serious research started around 1960. Although much has been learned, it appears that damage mechanics is a formidable problem whose difficulty may be of the same dimension as turbulence. It will take a long time to resolve completely.

For the immediate future–and only such a view is possible now, the following is a sample of research directions that may be identified as necessary and potentially profitable:

1. Micromechanical basis of softening damage.
2. Physically-justified nonlocal model (based on the interactions of cracks and inclusions).
3. Scaling of brittle compression fracture and shear fracture.
4. Scaling of fracture at interfaces (bond rupture).
5. Rate and load duration effects on scaling, and size effects in long-time fracture or fatigue.
6. Softening damage and scaling for large strains.
7. Size effect on ductility of softening structures, and on their energy absorption capability.
8. Acquisition of size effect test data for all kinds of quasibrittle materials, many of them high-tech materials (see the Introduction), and data for real structures of various types.
9. Statistical characteristics of the size effect due to energy release and stress redistribution during fracture.
10. Scaling problems in geophysics, *eg*, earthquake prediction or ocean ice dynamics.
11. Downsize extrapolation of size effect into a range of reduced brittleness, which is of interest for miniature electronic components and micromechanical devices.
12. Incorporation of size effect into design procedures and code recommendations for concrete structures, geotechnical structures, fiber composites (*eg* for aircrafts and ships), nuclear power plants, ocean oil platforms, mining and drilling technology (especially rockburst and borehole breakout), etc.

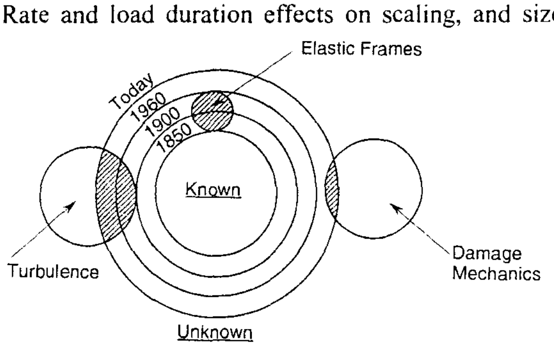

Fig 45. Damage mechanics in the perspective of the expansion of human knowledge

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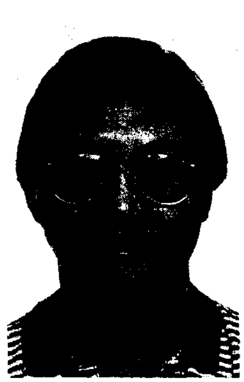

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