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# Creep stability and buckling strength of concrete columns

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*SUMMARY*

*A solution is given for the long-term deflexions of a reinforced concrete column with an initial curvature according to a general theory of linear creep. This is used to produce solutions: (1) according to the Arutyunyan-Maslov theory (using an incomplete gamma function); (2) according to the approximate Dischinger-Whitney law; (3) by the use of an effective deformation modulus. It is shown that, whilst critical long-term loads may be calculated, these are very sensitive to inaccuracies in the creep theory and that it is more convenient to consider the long-term buckling strength of a column. A practical method of analysis is presented in which the long-term buckling strength is assessed by calculating the short-term buckling strength but using a reduced elastic modulus. Graphs are presented to aid analysis.*

NOTATION

| | |
|---|---|
| $\varepsilon$ | normal strain |
| $\sigma$ | normal stress |
| $E$ | Young's modulus of concrete |
| $t$ | time |
| $t_0$ | instant of load application |
| $x$ | length co-ordinate of bar |
| $y$ | ordinate of the deflexion line |
| $y_0$ | initial curvature |
| $y(t_0)$ | ordinate of the initial elastic deflexion |
| $l$ | buckling length of bar |
| $\eta = y/y(t_0)$ | relative deflexion |
| $\eta', \eta''$ | relative deflexions reduced with respect to strength variation and load variation (equations 39a) |
| $I$ | moment of inertia of the concrete part of section |
| $I_a$ | moment of inertia of the reinforcement |
| $E_a$ | Young's modulus of steel |
| $\rho_a$ | parameter of reinforcement of the section |
| $P$ | load |
| $P_E$ | elastic Euler critical load |
| $\bar{P}_E$ | reduced Euler critical load with respect to creep (equation 39) |
| $\bar{E}$ | corresponding reduced modulus |
| $\bar{P}$ | reduced load with respect to creep (equation 41) |
| $P_{cr}$ | long-time critical load |
| $P_p$ | additional short-time load |
| $C(t,\tau)$ | creep function (creep strain from time $\tau$ to time $t$ under unit stress) |
| $\mathbf{E}$ | creep operator in time |
| $\varphi_\infty(t)$ | creep factor (relative creep strain) from time $t$ to infinity |
| $\varphi_\infty(\infty)$ | creep factor for an infinitely old concrete |
| $\varphi_{\infty a}(t)$ | creep factor reduced with respect to reinforcement of section (equation 17) |
| $q, z, u$ | parameters for the calculation of $\eta$ (equations 17, 18, 20) |
| $E_{eff}$ | effective modulus (with respect to creep) |

## Introduction

In a concrete column, creep produces an increase of the lateral deflexion, so that the column may lose its stability in the course of time. Simultaneously with the deflexion, bending moments from the longitudinal force increase and, therefore, both buckling strength and bearing capacity become lower. If the column is later exposed to an additional short-term load, it behaves as if its initial curvature were greater. Consequently, buckling strength and bearing capacity are lower under the combined effects of a long- and a short-term load.

In the various existing regulations, these effects are considered only more or less approximately—usually only by introducing a reduced modulus of elasticity

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for long-term loads. For a correct approach, it is necessary to consider the actual stress-strain law of concrete.

Work which has so far been done on the problem of creep stability has been concerned mainly with metals at high temperatures[1], the creep of which has the character of non-linear viscous flow. Visco-elastic materials, invariant with time, have also been treated to a lesser extent[2,3]. As, according to the accepted creep law, concrete is not invariant with time, these solutions cannot be applied to concrete. For concrete columns, if the ageing of the concrete and the reinforcement are to be taken into account, probably the best solution—based upon the linear law of creep by Arutyunyan-Maslov—which has so far been presented is that by Distefano[4]. A solution for the creep buckling according to the approximate linear law by Dischinger-Whitney has already been published by the author.[5]

For bearing capacity, a non-linear concrete creep law has to be assumed. This problem, which is not the subject of the present paper, was investigated by Naerlovič-Veljkovič[6], who used a creep law of the Dischinger-Whitney type with a non-linear elastic component whilst, for simplification, the viscous flow component was considered as linear and the ageing of concrete was neglected. Recently Mauch and Holey[7] presented a numerical solution for a completely non-linear creep law of the Dischinger-Whitney type.

In the present paper we shall use an improved form[8,9] of the Arutyunyan-Maslov law of linear creep[4] which (unlike reference 4) also yields, as a special case, approximate solutions according to the Dischinger-Whitney law or according to an effective creep modulus. Meanwhile a considerably shorter method of derivation of basic equations will be shown which makes use of creep operators and an analogy of creep with elasticity. It will further be shown that, in view of the high sensitivity of instability to inaccuracies arising from the introduction of the linear creep law, it is necessary to consider also the problem of long-term buckling strength. A simple method will be suggested by which this problem can be converted, by a comparison of the internal forces, into a short-term (elastic) buckling strength, this being possible even in the more general case of a combination of long-term load with short-term load. The results of this solution have already been presented in preliminary publication by the author in 1965.[10]

## Lateral deflexions of a strut with an initial imperfection

For stresses in concrete of less than one-half of its ultimate compressive strength, and for increasing, constant or only slightly decreasing loads, *linear creep* may be assumed. Let us denote the measure of creep by $C(t,\tau)$, i.e. the creep strain at a time $t$ produced by

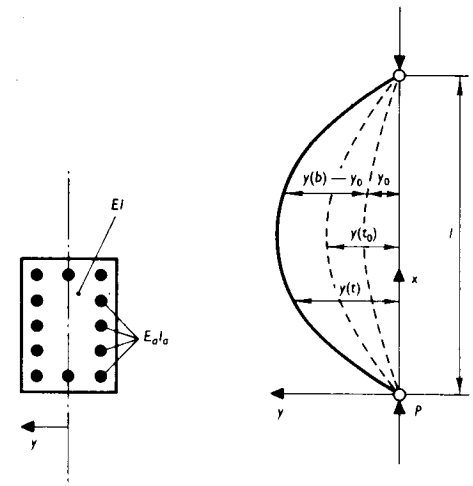



*Figure 1: Assumed initial curvature of column and deflexions.*

a constant unit stress applied at a time $\tau$. If we use the principle of superposition for the increments,

$$d\sigma(t) = \frac{\partial\sigma(t)}{\partial t}\,dt$$

and if we integrate by parts, the creep law between the stress $\sigma(t)$ and the strain $\varepsilon(t)$ is obtained in the form:[5,8,11]

$$\varepsilon(t) = \mathbf{E}^{-1}\sigma(t) = \frac{\sigma(t)}{E(t)} - \int_{t_0}^{t}\sigma(\tau)\frac{\partial}{\partial\tau}\left[\frac{1}{E(\tau)} + C(t,\tau)\right]d\tau \quad \ldots\ldots\ldots(1)$$

in which $\mathbf{E}^{-1}$ is the integral operator of Volterra[12], and $E(t)$ is the Young's modulus of concrete at a time $t$.

Let us assume a reinforced concrete column of symmetrical cross-section (for plain concrete, no symmetry is required). The centroidal moment of inertia of the (homogeneous) concrete part of the section with regard to the axis of symmetry will be designated by $I$, and that of the steel reinforcement $I_a$, the modulus of elasticity of the steel being $E_a$. Let $E$, $I$, $E_a$ and $I_a$ be constant along the axial co-ordinate $x$. The buckling length according to the theory of elasticity is $l$; the deflexion is $y = y(x)$. The initially curved column is assumed to have the form $y_0 = y_0(x)$ (Figure 1). In the theory of elasticity, according to the hypothesis that cross-sections remain plane and perpendicular, the following equation applies[13]:

$$(EI + E_aI_a)\frac{\partial^2}{\partial x^2}(y - y_0) = -Py \ldots\ldots\ldots(2)$$

The equation of any creep problem can most readily be obtained by the analogy of creep with elasticity[5,8], the elastic constants being replaced by the corresponding creep operators, e.g. $1/E$ by $\mathbf{E}^{-1}$ (these operators referring to forces or moments in the same way as to stresses and strains), provided that the parts of the section corresponding to the introduced internal forces

and deformations are homogeneous. Thus, from equation 1, we obtain:

$$(EI + E_aI_aE\mathbf{E}^{-1})\frac{\partial^2}{\partial x^2}(y - y_0) + E\mathbf{E}^{-1}(Py) = 0 \quad (3)$$

which, together with the end conditions, determines the solution. In this equation it is possible to separate the integration in the co-ordinate $x$ and in the time $t$ through a representation of $y_0$ in a series of characteristic functions of the corresponding homogeneous equation with homogeneous end conditions. Therefore we introduce

$$y_0 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}x \quad (4)$$

$$y(t) = \sum_{n=1}^{\infty} \eta_n(t)\frac{a_n}{1 - P(t_0)/[n^2P_E(t_0)]}\sin\frac{n\pi}{l}x \quad (5)$$

This satisfies, for $t = t_0$ with $\eta_n(t_0) = 1$, the initial condition, i.e. the well-known[13] expressions for the elastic deflexion. As a special case, there is evidently involved also the case of an eccentrically loaded straight column.

If equation 5 is substituted into equation 3, the following equation is obtained, by a comparison of coefficients (at the individual harmonic components):

$$(1 - \rho_a)n^2P_E\left(\eta_n - 1 + \frac{P(t_0)}{n^2P_E(t_0)}\right) = E\mathbf{E}^{-1}\left[P\eta_n - \rho_a n^2P_E\left(\eta_n - 1 + \frac{P(t_0)}{n^2P_E(t_0)}\right)\right] \quad (6)$$

($n = 1, 2, 3 \ldots$) where

$$\rho_a = \frac{E_aI_a}{EI + E_aI_a} \quad (6a)$$

expresses the influence of the degree of reinforcement (for plain concrete, $\rho_a = 0$; for a steel, $\rho_a = 1$); $P_E(t)$ is the Euler's critical load corresponding to $E(t)$ at time $t$,

$$P_E = \frac{(EI + E_aI_a)\pi^2}{l^2}$$

These equations also apply to elastically restrained columns and to struts in a frame, provided that creep occurs homogeneously in all cross-sections. Equation 6 represents Volterra integral equations for $\eta_n(t)$. For a general form of $C(t,\tau)$, the solution may be obtained by some numerical method.

Next it will be shown that, for some special forms of $C(t,\tau)$, an analytical solution may be given. For the sake of simplicity, only the first term of the series (4) and (5), $\eta = \eta_1$ only, will be taken into account, that is an initial curvature of the column according to the first sine curve. This term has a dominant influence. The solution $\eta_n(t)$ for higher components could then be obtained, if necessary, by a mere substitution of $\eta_n$ for $\eta$ and of $n^2P_E$ for $P_E$.

For long-term stability, as will be seen, it is necessary to consider a relatively more complex creep law by Arutyunyan-Maslov:

$$C(t,\tau) = \frac{\varphi_\infty(\tau)}{E(\tau)}(1 - e^{-\gamma(t-\tau)}) \quad (7)$$

$\varphi_\infty(\tau)$ is a final factor of creep, i.e. from the time $\tau$ to $\infty$; usually $\varphi_\infty(\infty) = 0{\cdot}7$ to $1{\cdot}3$, $\varphi_\infty(1\text{ month}) = 1{\cdot}5$ to $2{\cdot}5$, $1/\gamma = 30$ to $60$ days.

It has been shown by the author[5,8,9,10,14] that this law can be represented by the well-known Boltzmann's rheological model (which consists of a spring and a dashpot in parallel, coupled in series with a spring) if its parameters are considered as time-dependent, the viscosity of the dashpot being $\lambda = E(t)/\gamma\varphi_\infty(t)$ and the spring constant of the spring parallel with the dashpot being $E(t)/\varphi_\infty(t) - \lambda(t)$.

If equation 7 were substituted into equation 1 and the first and second derivative of this equation with respect to $t$ written, it would be found that in both equations thus obtained the same integral appears. Eliminating it from both equations 2 and 3, a differential equation $\varepsilon = \mathbf{E}^{-1}\sigma$ would be obtained, where $\mathbf{E}^{-1}$ is a linear differential operator with variable coefficients:

$$\mathbf{E}^{-1} = \frac{1}{E(t)}\left[\frac{\partial^2}{\partial t^2} + \gamma\left(1 + \varphi_\infty(t) - \frac{\dot{E}(t)}{\gamma E(t)}\right)\frac{\partial}{\partial t}\right] \times \left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t}\right)^{-1} \quad (8)$$

$\dot{E} = \mathrm{d}E/\mathrm{d}t$. The initial elastic condition is $\sigma(t_0) = E(t_0)\varepsilon(t_0)$. The initial condition for the strain rate results from the first derivative of equation 1 for $t = t_0$[5,8]:

$$\mathbf{E}^{-1}(t_0) = \left[\left(\frac{\partial}{\partial t} + \gamma\varphi_\infty\right)\left(E\frac{\partial}{\partial t}\right)^{-1}\right]_{t=t_0} \quad (9)$$

Let us now introduce the operator given by equation 8 into equation 6 (for $n = 1$, $\eta_n = \eta$). This yields the following differential equation of the second order in time $t$ with variable coefficients:

$$-P_E\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} - P_E\gamma\left[1 + \rho_a\left(\varphi_\infty - \frac{\dot{E}}{\gamma E}\right)\right]\frac{\mathrm{d}\eta}{\mathrm{d}t} + \frac{\mathrm{d}^2(P\eta)}{\mathrm{d}t^2} + \gamma\left(1 + \varphi_\infty - \frac{\dot{E}}{\gamma E}\right)\frac{\mathrm{d}(P\eta)}{\mathrm{d}t} = 0 \quad (10)$$

In addition to an initial condition $\eta(t_0) = 1$, it is necessary to know the initial strain rate for which, by substituting the operator given by equation 9 into equation 6, we obtain:

$$P_E(t_0)\frac{\mathrm{d}\eta(t_0)}{\mathrm{d}t} = \left[\frac{\partial(P\eta)}{\partial t}\right]_{t=t_0} + [1 - \rho_a(t_0)]\gamma\varphi_\infty(t_0)P(t_0) \quad (11)$$

With a load $P$ variable with time, equation 10 also contains the term $\eta$ and cannot be integrated analytically. Consequently, let us assume that $P =$

constant for $t \geqslant t_0$. Equation 10 with 11 is then simplified as follows:

$$\frac{d^2\eta}{dt^2} = F(t)\frac{d\eta}{dt} \quad \ldots\ldots\ldots\ldots\ldots (12)$$

with an initial condition:

$$\frac{d\eta(t_0)}{dt} = \dot{\eta}(t_0) = \frac{1 - \rho_a(t_0)}{P_E(t_0)/P - 1}\gamma\varphi_\infty(t_0) \ldots (13)$$

where

$$F(t) = -\gamma\left\{1 - \frac{1 - \rho_a(t)P_E(t)/P}{P_E(t)/P - 1}\left[\varphi_\infty(t) - \frac{E(t)}{\gamma E(t)}\right]\right\} \quad \ldots\ldots\ldots (14)$$

Equation 12 represents an equation of the first order for $d\eta/dt$ that can be solved by a separation of the variables:

$$\eta(t) = 1 + \dot{\eta}(t_0)\int_{t_0}^{t} \exp\left(\int_{t_0}^{\theta} F(t)dt\right)d\theta \ldots (15)$$

The integral (15) can be expressed analytically only for certain specific functions $\varphi_\infty(t)$ and $E(t)$.

For concrete, it is usually assumed that $E(t)$ = constant and, according to Arutyunyan[11], $\varphi_\infty(\tau)/E(\tau) = C_0 + B/\tau$, which makes possible a solution using incomplete gamma functions. For the latter function a solution for deflexion has already been submitted by Distefano[4]. This function, however, is not particularly suitable, because in the corresponding Boltzmann model for fresh concrete, $t \to 0$, the spring constant of a spring coupled in parallel with a dashpot tends to a negative quantity of $-1/B\gamma$, this being an absurdity. The solution will therefore be carried out for a function[8]

$$\varphi_\infty(\tau)/E(\tau) = C_0 + Ae^{-\gamma\tau}$$

for which the above-mentioned paradox does not appear. This function is also more convenient because, unlike the first function[4], the Dischinger-Whitney law of creep is its special case for $\varphi_\infty(\infty) = C_0 = 0$, the creep factor being $\varphi(t) = 1 - e^{-\gamma t}$.

By substituting for $\varphi_\infty(t)$ into equation 15, we obtain:

$$\eta(t) = 1 + \dot{\eta}(t_0)\int_{t_0}^{t} \exp\left\{\gamma[q(\infty) - 1](\theta - t_0) + z(t_0)(1 - e^{-\gamma(\theta - t_0)})\right\} d\theta \ldots\ldots\ldots\ldots\ldots (16)$$

where

$$q(t) = \frac{\varphi_{\infty a}(t)}{P_E/P - 1} \quad \ldots\ldots\ldots\ldots (17)$$

$$\varphi_{\infty a}(t) = \left(1 - \rho_a\frac{P_E}{P}\right)\varphi_\infty(t) \ldots\ldots (17a)$$

$$z(t) = q(\infty)\left(\frac{\varphi_\infty(t)}{\varphi_\infty(\infty)} - 1\right) = z(t_0)e^{-\gamma(t - t_0)} \ldots (18)$$

If the function $z(t)$ is substituted as a new variable, that is,

$$\gamma(t - t_0) = \ln\frac{z(t_0)}{z(t)}$$

$$\gamma dt = -\frac{dz}{z}$$

we obtain, after the corresponding modifications:

$$\eta(t) = 1 + u(t_0)\int_{z(t)}^{z(t_0)} z^{-q(\infty)}e^{-z}dz \ldots\ldots\ldots (19)$$

where

$$u(t_0) = (1 - \rho_a)q(t_0)\exp\{z(t_0) + [q(\infty) - 1]\ln z(t_0)\} \quad \ldots\ldots\ldots (20)$$

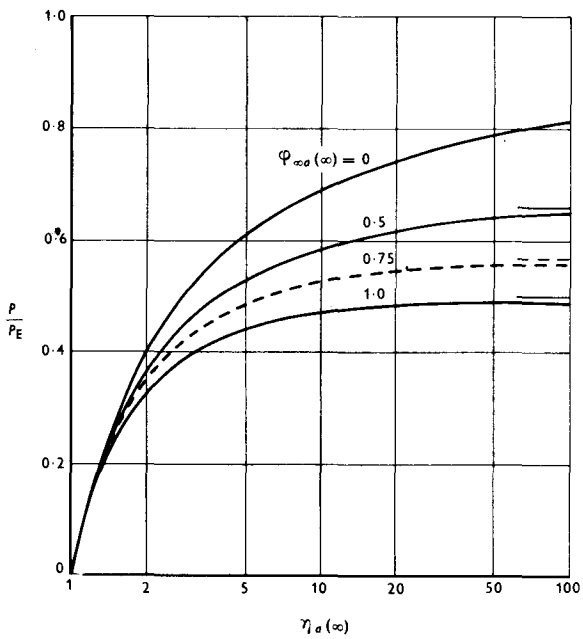
The integral in equation 17 represents the incomplete gamma function, Tables of which are available (reference 11, p. 54), for $0 \leqq q \leqq 0{\cdot}5$[15-18]. Let us note that the integral 15 might be similarly expressed in terms of an incomplete gamma function for a function $\varphi_\infty/E = C_0 + Ae^{-\gamma_1\tau}$ of a more general character where $\gamma_1 \neq \gamma$. An advantage in comparison with reference 4 is that $\eta(\infty)$, according to equation 19, does not depend upon $\gamma$.

For practical application a Table has been computed of $\eta(\infty)$ according to equation 17 for different values of $\varphi_\infty(t_0)$, $\varphi_\infty(\infty)$ and $P/P_E$. For $\rho_a = 0$, i.e. for plain concrete, the graphs of the quantities $\eta(\infty)$ are given in Figure 2. These graphs can, however, also be used for determining $\eta(\infty)$ with a non-zero percentage of reinforcement. If, for given $\varphi_{\infty a}(t_0)$, $\varphi_{\infty a}(\infty)$ and $P/P_E$, the corresponding value in Figure 2 is designated as $\eta_a$, the value of $\eta(\infty)$ is $1 + (1 - \rho_a)(\eta_a - 1)$. Among the individual graphs for $\varphi_\infty(t_0) = 1{\cdot}0, 1{\cdot}5, 2{\cdot}0$ etc., it is possible to interpolate linearly, but for greater accuracy it is more expedient to interpolate, for example, by using a curve plotted from three points (or to assume for interpolation the same intermediate course as according to the approximate formula below equation 25. The values $\eta(t)$ for $t < \infty$ may be determined by multiplying $\eta(\infty)$ by the ratio $\eta(t)/\eta(\infty)$ found from the approximate formula 25.

The approximate but widely used Dischinger-Whitney law of creep[5, 8, 19, 20] is, in principle, based upon the assumption that all creep deformations are irreversible, so that the curves $C(t,\tau)$ for different values of $\tau$ are identical and parallel, i.e.

$$C(t,\tau) = \frac{\varphi(t) - \varphi(\tau)}{E(t_0)} \ldots\ldots\ldots\ldots (21)$$

where $\varphi(t)$ is called the creep factor; $\varphi(t)$ should be introduced especially in order to make the value $[\varphi(\infty) - \varphi(t_0)]/E(t_0)$ conform to the actual measured value $C(\infty, t_0)$ for a given $t_0$. Because $\varphi(\infty)$ has to be of finite value, a zero creep corresponds to changes of stress in the very old concrete. This is the main drawback of this theory. (This is evident also from a




(*a*) *For* $\varphi_{\infty a}(t_0) = 0{\cdot}5$

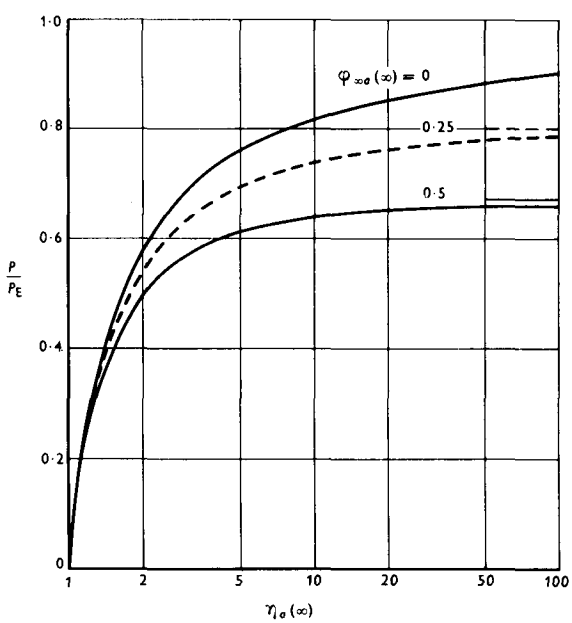
$$\eta(t) = \frac{y(t,x)}{y(t_0,x)}$$

$$y_0 = a \sin \frac{nx}{l}$$

$$P_E = (EI + E_a I_a)\frac{\pi^2}{l^2}$$

$$\varphi_{\infty a}(t) = \varphi_\infty(t)\left(1 - \rho_a \frac{P_E}{P}\right)$$

$$\varphi(\infty) = 1 + (1 - \rho_a)\,[\eta_a(\infty) - 1]$$

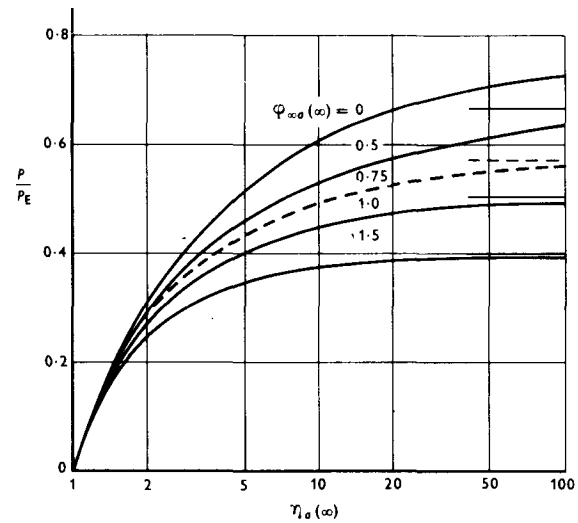



(*b*) *For* $\varphi_{\infty a}(t_0) = 1{\cdot}0$

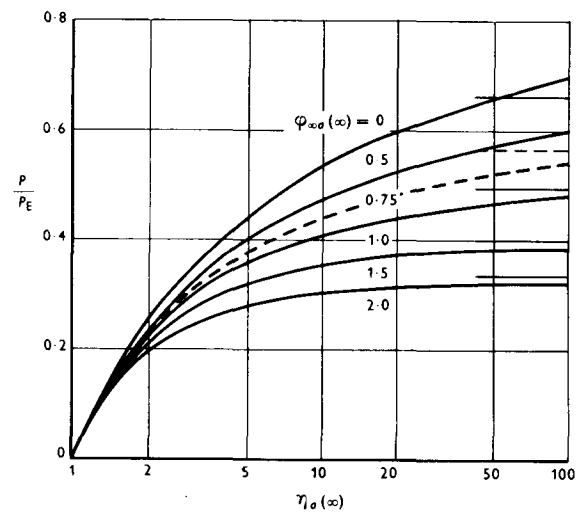
$$\eta(t) = 1 + u(t_0)\int_{z(t)}^{z(t_0)} z^{-q(\infty)} e^{-z}\,dz$$

$$u(t_0) = (1 - \rho_a)q(t_0)\exp\left\{2(t_0) + [q(\infty)-1]\ln z(t_0)\right\}$$

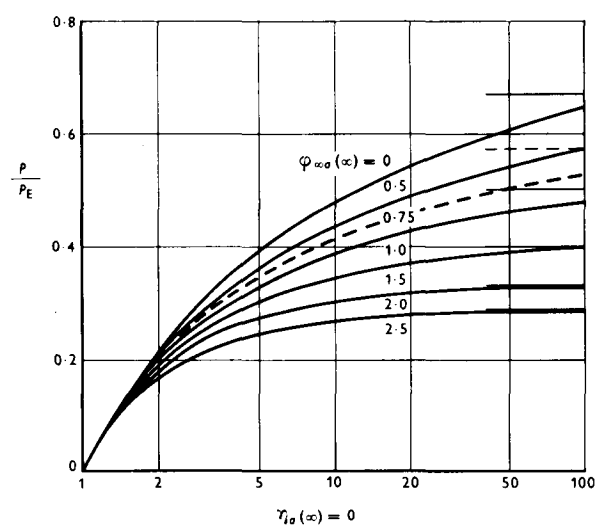
$$q(t) = \frac{\varphi_{\infty a}(t)}{P_E/P - 1}, \quad z(t) = q(\infty)\left(\frac{\varphi_\infty(t)}{\varphi_\infty(\infty)} - 1\right)$$




(*c*) *For* $\varphi_{\infty a}(t_0) = 1{\cdot}5$




(*d*) *For* $\varphi_{\infty a}(t_0) = 2{\cdot}0$




(*e*) *For* $\varphi_{\infty a}(t_0) = 2{\cdot}5$

*Figure 2: Graphs for determining the effect of the load P upon the increase of deflexion* $\eta_a(\infty)$ *for an infinite time for different values of the creep factors* $\varphi_{\infty a}(\infty)$ *and* $\varphi_{\infty a}(t_0)$.

rheological model, the Maxwell model with a dashpot viscosity $E/\varphi$ tending to infinity for $t \to \infty$.) The Dischinger-Whitney law with $\varphi(t) = 1 - e^{-\gamma t}$ is a special case of the Arutyunyan-Maslov law for $\varphi_\infty/E = e^{-\gamma\tau}$, i.e. $\varphi_\infty(\infty) = 0$.

By substituting equation 2 into equation 1 and differentiating with respect to $t$, we obtain an equation giving $\mathbf{E}^{-1}$:

$$\mathbf{E}^{-1} = \left(\frac{1}{E(t)}\frac{\partial}{\partial\varphi} + \frac{1}{E(t_0)}\right)\left(\frac{\partial}{\partial\varphi}\right)^{-1} \quad \ldots\ldots(22)$$

After this has been substituted into equation 6, there results a linear differential equation of the first order for $\eta(t)$:

$$(P_E - P)\frac{d\eta}{d\varphi} - \frac{E}{E(t_0)}\left(P - \rho_a P_E + \frac{E(t_0)}{E}\frac{dP}{d\varphi}\right)\eta$$
$$= \rho_a \frac{E}{E(t_0)} P_E \left(1 - \frac{P(t_0)}{P_E(t_0)}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(23)$$

Specifically for $P$ = constant and when $E$ is assumed to be approximately constant this equation has the form:

$$\frac{d\eta}{d\varphi} - c\eta = \rho_a$$
$$c = \frac{1 - \rho_a P_E/P}{P_E/P - 1} \quad \ldots\ldots\ldots\ldots(24)$$

and its solution is:

$$\eta(t) = -\frac{\rho_a}{c} + \left(1 + \frac{\rho_a}{c}\right) e^{c[\varphi(t) - \varphi(t_0)]} \quad \ldots(25)$$

For this law of creep, the solution can be expressed simply even with a stepwise variable load $P$. Let in time interval $(t_0, t_1)$ the load $P_0$ and for $t \geqq t_1$ the load $P_1$ be acting. In the time $t_1$ the ordinate $\eta$ is elastically changed, in the ratio

$$\alpha_{01} = \frac{1 - P_0/P_E}{1 - P_1/P_E}$$

The solution of equation 24 is:

$$\eta(t) = -\frac{\rho_a}{c_1} + \left[-\frac{\alpha_{01}\rho_a}{c_0}\right.$$
$$\left. + \alpha_{01}\left(1 + \frac{\rho_a}{c_0}\right) e^{c_0[\varphi(t_1) - \varphi(t_0)]} + \frac{\rho a}{c_1}\right] e^{c_1[\varphi(t) - \varphi(t_1)]}$$
$$\ldots\ldots(26)$$

where $c_0$ or $c_1$ is the value of $c$ according to equation 24 for $P_0$ or $P_1$. This expression is considerably simplified for plain concrete, i.e. for $\rho_a = 0$.

For a small percentage of reinforcement, it is also possible to assume $\rho_a \approx 0$ and to express the influence of the reinforcement approximately by a reduction of $\varphi(t)$ to a value $\varkappa\varphi(t)$, where[5] $\varkappa = EI/(EI + E_aI_a) = 1 - \rho_a$. For $P$ = constant and $E$ = constant we then obtain:

$$\eta \approx e^{c\varkappa[\varphi(t) - \varphi(t_0)]} \quad \ldots\ldots\ldots\ldots(27)$$

whilst for a variable $P(t)$:

$$\eta(t) \approx \alpha_{0t} \exp \int_{\varphi(t_0)}^{\varphi(t)} c(t)\varkappa d\varphi \ldots\ldots\ldots(28)$$

$$\alpha_{0t} = \frac{1 - P(t_0)/P_E}{1 - P(t)/P_E} \quad \ldots\ldots\ldots\ldots(28a)$$

and for a stepwise variable load:

$$\eta(t) \approx \alpha_{01} \exp\left\{c_0\varkappa[\varphi(t_1) - \varphi(t_0)] + c_1\varkappa[\varphi(t) - \varphi(t_1)]\right\}$$
$$\ldots\ldots\ldots(29)$$

The equations for this creep law also result from the equations for the Arutyunyan-Maslov law if we put $\varphi_\infty(\infty) = 0$.

Another approximate law for the creep of concrete is the law of 'effective creep modulus' (also called 'ideal modulus')[8]. In principle, this theory is based upon the assumption that, once a load has been applied, the concrete does not age any more, i.e. the creep law is time-invariant, with parameters merely depending upon $t_0$. All curves $C(t, \tau)$ for different values of $\tau$ are identical and mutually translated in the direction of axis $t$. Whenever the load tends to a constant value for $t \to \infty$, the state of stress tends to a steady state $\sigma(\infty) = E_{eff}\varepsilon(\infty)$ where

$$E_{eff} = \frac{1}{C(\infty, t_0)} = \frac{E}{1 + \varphi(\infty) - \varphi(t_0)} = \frac{E}{1 + \varphi_\infty(t_0)}$$

is the effective creep modulus. Hence

$$\eta(\infty) = \frac{1 - P/P_E}{1 - P/P_{E_{eff}}} \quad \ldots\ldots\ldots(30)$$

where

$$P_{eff} = (E_{eff}I + E_aI_a)\pi^2/l^2$$

is the Euler load corresponding to $E_{eff}(t_0)$. Within the range of the validity of the principle of superposition, all deformations according to this creep law are perfectly reversible. This law diverges from the real behaviour in a sense contrary to that of the Dischinger-Whitney law, because for increments of stress in a very old concrete it assumes a creep of the same magnitude as that for the first load.

If $C(t, \tau)$ is introduced according to equation 7 with $\varphi(\infty)$ = constant = $C(\infty, t_0)$, the stress-strain law (equation 1) may be reduced to a differential form with

$$\mathbf{E}^{-1} = \frac{1}{E(t_0)}\left(\frac{\partial}{\partial t} + \gamma[1 + \varphi_\infty(t_0)]\right)\left(\frac{\partial}{\partial t} + \gamma\right)^{-1} \ldots(31)$$

The solution of the corresponding equation 6 is a special case of equations 15 and 19.

A comparison of the values of $\eta(\infty)$, which can be obtained according to the different creep laws can be clearly seen in Figure 1.

## Distribution of the internal forces in a cross-section

The bending moment $M_a(t)$ carried by an elastic reinforcement is obviously $-E_a I_a \partial^2[y(t) - y_0]/\partial x^2$. If equations 4 and 5 are substituted here and $l^2$ is expressed by means of $P_E(t_0)$, the following expression is obtained:

$$M_a(t) = -\rho_a(t_0)P_E(t_0)\left(\frac{\eta(t)}{1 - P(t_0)/P_E(t_0)} - 1\right)a_0 \sin\frac{\pi}{l}x \quad \ldots\ldots\ldots(32)$$

The relative increase of the moment carried by the reinforcement, $m_a(t) = M_a(t)/M_a(t_0)$, is then given by

$$m_a(t) = 1 + [\eta(t) - 1]\frac{P_E(t_0)}{P(t_0)} > \eta(t) \ldots\ldots(33)$$

The bending moment $M_b(t)$ transmitted by concrete according to the condition of equilibrium is then $-P(t)y(t) - M_a(t)$. The initial value of $M_b(t_0)$ according to the theory of elasticity is

$$M_b(t_0) = -P(t_0)y(t_0)\,[1 - \rho_a(t_0)]$$

Further,

$$M_a(t_0)/M_b(t_0) = \rho_a(t_0)\,[1 - \rho_a(t_0)]$$

By using these relationships it is possible to determine the relative increase of the moment transmitted by the concrete, $m_b(t) = M_b(t)/M_b(t_0)$, in the form:

$$m_b(t) = \frac{\eta(t)P(t)/P(t_0) - m_a(t)\rho_a(t_0)}{1 - \rho_a(t_0)} < \eta(t) \ldots(34)$$

It is interesting that, for relatively low loads, $P < P_E(t_0)\rho_a(t)$, the moment $M_b(t)$ decreases even when the deflexion increases. [The value $m_b(t)$ with $P$ = constant is lower than for $m_a < 1 + (\eta - 1)/\rho_a(t_0)$.]

The distribution of the normal force $P$ into a force $N_b(t)$ in concrete and a force $N_a(t)$ in steel is independent of the deflexion. The elastic compatibility condition is $N_b/EF_b = (P - N_b)/E_0F_a$, where $F_a$ and $F_b$ are sectional areas of steel and concrete. Interchanging $1/E$ with $\mathbf{E}^{-1}$, the basic equation for $N_b(t)$ is:

$$EF(P - N_b) = E_aF_aE\mathbf{E}^{-1}N_b \ldots\ldots\ldots(35)$$

## Long-term stability

In steel struts at high temperatures, because of the progressively non-linear creep, a loss of stability takes place within a finite time called the critical time. As regards concrete columns, the concept of a critical time has no sense under linear creep; for a long-term load, a loss of stability can occur only within a limit for $t \to \infty$ (a condition which, of course, is applicable only for a theory of small deflexions). As a condition of stability, it will be required here—just as is required by Distefano[4]—that for $t \to \infty$ the deflexion should tend to a finite value, i.e.

$$\lim_{t\to\infty} \eta(t) = \eta(\infty) < \infty \ldots\ldots\ldots\ldots(36)$$

For the Arutyunyan-Maslov law, according to equation 15, the integral $\int_{t_0}^{\infty} F(t)dt$ has to be negative, for which it is necessary that $\lim_{t\to\infty} F(t) < 0$. According to equation 14, we thus obtain the following condition of stability ($\lim \dot{E} = 0$):

$$P(\infty) < P_{cr} = \frac{1 + \rho_a\varphi_\infty(\infty)}{1 + \varphi_\infty(\infty)} < P_E \ldots(37a)$$

The critical load $P_{cr}$ is equivalent to the Euler critical load for a column, the concrete of which has a modulus $E/[1 + \varphi_\infty(\infty)]$ instead of $E$, that is an effective creep modulus for infinitely old concrete.

The same result would also be obtained from equation 19, in which $\lim_{t\to\infty} z(t) = 0$ and the integrand in equation 19 becomes $\infty$. For a convergence, it is necessary and sufficient that $q(\infty) < 1$, from which equation 37a again results.

For the Dischinger-Whitney law, equation 25, there results $c < \infty$, from which, according to equation 24, $P_{cr} = P_{E\,eff}$ will be obtained. This does not conform to the actual conditions and is due to the fact that the relative creep for stress increments produced here by an increase of the deflexion after a very long time tends to zero (and in the corresponding Maxwell model the viscosity of the dashpot tends to infinity) although, in reality, it tends to a non-zero value, $\varphi_\infty(\infty) > 0$. For the law of effective deformation modulus (equation 30), there is $\varphi_\infty(\infty) = \varphi_\infty(t_0)$ in (37a), $P_{cr} = P_{E\,eff}$. This is too small a value, which is again due to the approximate character of the theory of an effective deformation modulus, according to which the relative creep for increments of stress produced by an increment of deflexion is, after some time, the same as the initial one, i.e. $\varphi_\infty(t_0)$, although it has to be smaller and to tend to $\varphi_\infty(\infty)$. The values of critical loads are represented in the graphs (Figure 2) by horizontal asymptotes to the curves.

It is thus evident that any estimation of the stability is very sensitive to the inaccuracies of the creep law, particularly to the value $\varphi_\infty(\infty)$ of the creep factor to infinity for an infinitely old concrete. It is, consequently, necessary to use at least the Arutyunyan-Maslov law. All linear creep laws, however, have a weakness which consists in the fact that a critical load does not depend upon the age of concrete $t_0$ at the instant the load is applied. For these reasons it is necessary (for linear creep law) to go on from the problem of stability to consider the problem of buckling strength.

This would not, of course, be necessary if a non-linear creep law could be considered from the instant when the stress passes beyond the range of linear creep. Then a finite critical time would be obtained

and the concept of critical time could be introduced in the problem of bearing capacity.

NOTE: A long-term critical load considerably depends upon the percentage of reinforcement of a column. Let us assume, in order to illustrate the problem, a cross-section where $E_a/E = 7$ and $i_a^2/i_b^2 = 1{\cdot}5$ for radii of inertia of the respective parts of the section, this being a reasonably typical example. Let us further assume that $\varphi_\infty(\infty) = 1$. For a low percentage of reinforcement by sectional area (0·6%), $\rho_a = 0{\cdot}06 \times 7 \times 1{\cdot}5 = 0{\cdot}063$, so that, according to equation 33a, the critical load is $P_{cr} = P(1 + 0{\cdot}063)/(1 + 1) = 0{\cdot}532P$. For a high percentage of reinforcement (3%), $\rho_a = 0{\cdot}03 \times 7 \times 1{\cdot}5 = 0{\cdot}315$, so that $P_{cr} = P(1 + 0{\cdot}315)/(1 + 1) = 0{\cdot}657P$, which is 24% more than for the low percentage of reinforcement.

## Conversion of a long-term buckling strength into a short-term buckling strength

It is expedient to relate the creep (long-term) buckling to the elastic (short-term) buckling which has already been thoroughly investigated, even as regards the experimental data. If the value of stress in concrete or in steel were chosen as a criterion for a comparison, there would result relatively complex relationships. Meanwhile, from the bearing capacity aspect, it is the values of the total bending moment and of the normal force that count rather than the stresses (the calculation of which is, anyway, rather problematic in view of the considerable influence upon creep of the dimensions and of the distance from the surface[8]). As a criterion for comparison, therefore, the value of a total bending moment, $-Py$, is suggested here.

It is pertinent to express the influence of creep, as of physical characteristics of the material, through a reduction of the modulus of elasticity. In view of the dominant influence of the first sine component in equation 5, it is possible to confine our considerations to a column with a sinusoidal initial curvature. We establish, consequently; which reduced modulus of elasticity $\bar{E}$ a given column would have to possess in order that a load $P$, under elastic deformation, should produce a bending moment in it, the ratio of which to the permissible bending moment $M_{\text{perm, p}}$ for a short-term load at a time $t$ is equal to the ratio of the actual bending moment $-Py$ for a long-term load at a time $t$ to a permissible bending moment $M_{\text{perm, c}}$ for a long-term load. This condition may additionally be extended, even to the case of the combination of a long-term load $P$ and a short-term load $P_p$ (with the eventual dynamic coefficient included). Instead of the additional application of a short-term load $P_p$ at a time $t$, it is possible to assume first an elastic unloading of the column at time $t$ by which the ordinate of deflexion is reduced at a ratio of $1 - P/P_E$ and, subsequently, the application of a combined load $P + P_p$ to the column, whereby the said ordinate will be increased again at a ratio of $[1 - (P + P_p)/P_E]^{-1}$. The above condition is formulated as follows:

$$(P + P_p)\frac{a_1\eta \sin \pi x/l}{1 - P(t_0)/P_E(t_0)}\,\frac{1 - P/P_E}{1 - (P + P_p)/P_E} : M_{\text{perm, c}}$$

$$= (P + P_p)\frac{a_1 \sin \pi x/l}{1 - (P + P_p)/\bar{P}_E} : M_{\text{perm, p}} \quad \ldots\ldots\ldots (38)$$

where $\bar{P}_E$ is the Euler critical load corresponding to a reduced modulus $\bar{E}$, i.e. $\bar{P}_E = (\bar{E}I + E_aI_a)\pi^2/l^2$. From equation 38 there results a formula which determines $\bar{P}_E$:

$$\frac{P + P_p}{\bar{P}_E} = 1 - \left(1 - \frac{P + P_p}{P_E}\right)\frac{1}{\eta''} \ldots\ldots (39)$$

where

$$\eta'' = \eta'\,\frac{1 - P(t)/P_E(t)}{1 - P(t_0)/P_E(t_0)}\,,\quad \eta' = \eta(t)\,\frac{M_{\text{perm, p}}}{M_{\text{perm, c}}} \ldots (39a)$$

For $P$ = constant and $E$ = constant, $\eta'' = \eta'$. The ratio $M_{\text{perm, p}}/M_{\text{perm, c}}$ depends upon the statistical distributions of strength for short- and long-term loads (including the changes of the stress distribution in the section) and upon the distributions of the values of these loads. Approximately $\eta' = \eta$.

Let us investigate now $\bar{P}_E$ and $\bar{E}$ in dependence on $P$ for $P_p = 0$ (Figure 3). For $P \to 0$, through a determination of the limits of equation 15 or 19, it is found that $\lim \eta = 1$ ($\lim \dot{\eta}(t_0) = 0$). From equation 39,

$$\lim_{P \to 0} \bar{P}_E = P_E[1 + \lim (\eta - 1)P_E/P]^{-1}$$

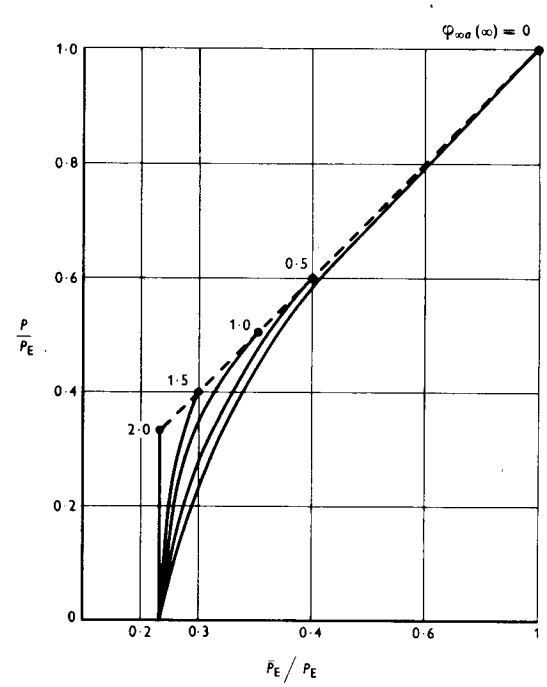



*Figure 3: Graph showing the reduced critical load $P_E$ for $\varphi_{\infty a}(t_0) = 2$, $\sigma_a = 0$ and $\eta'' = \eta$ and for different values of $\varphi_{\infty a}(\infty)$.*

For $\rho_a = 0$, it is possible to determine

$$\lim \bar{P}_E = P_E/[1 + EC(t, t_0)] = P_{E_{eff}}, \lim \bar{E} = E_{eff}$$

For $\rho_a > 0$, $\lim \bar{P}_E$ is slightly larger because

$$\lim (\eta - 1)P_E/P$$

$$= (1 - \rho_a)C(t, t_0) \int_{t_0}^{t} \exp\left[-\gamma\rho_a \int_{t_0}^{\theta} \varphi_\infty(t)dtd\theta\right]$$

For $P \to P_{cr}$, $\eta(\infty) \to \infty$ which yields $\lim \bar{P}_E = P_{cr}$. According to the law of an effective deformation modulus, for all values of $P < P_{cr}$, $P_E = P_{E_{eff}}$ and $E = E_{eff}$.

The correct values of $E$ are higher than these values but—as can be verified—they are lower than the $\bar{E}$ values according to the Dischinger-Whitney law. Consequently, the error inherent in the law of an effective creep modulus works in favour of safety, whilst the error inherent in the Dischinger-Whitney law works against it. These errors are the higher, the higher is $P/P_E$, and come near to zero for small values (when $P \to 0$). A graph of the $P_E/P$ values for $\varphi_\infty(t_0) = 2$, $t \to \infty$, $\eta' = \eta$, $\rho_a = 0$ and $P_p = 0$ is shown in Figure 3. The deviation of approximate theories from the actual behaviour is considerably smaller for $\bar{E}$ than for a deflexion, but they are nevertheless still large enough.

Instead of a reduced modulus $\bar{E}$, there might also be introduced an increased long-term load $\bar{P}$, producing with $P_p$ in the elastic strut the same (relative) bending moment as the load $P$ and $P_p$ in the actual strut. This yields for $\bar{P}$ the condition:

$$(P + P_p)\frac{a_1\eta \sin \pi x/l}{1 - P(t_0)/P_E(t_0)} \frac{1 - P/P_E}{1 - (P + P_p)/P_E} : M_{perm,\,c}$$

$$= (\bar{P} + P_p)\frac{a_1 \sin \pi x/l}{1 - (\bar{P} + P_p)/P_E} : M_{perm,\,p} \quad \ldots\ldots\ldots(40)$$

and, after modifications:

$$\frac{P_E}{\bar{P} + P_p} = 1 + \left(\frac{P_E}{P + P_p} - 1\right)\frac{1}{\eta''} \ldots\ldots(41)$$

For this method it is not necessary to know elastic buckling coefficients for different values of $E$.

As a third method, an increased buckling length $l$ may be introduced. Here, however, it is problematic whether it is more correct to assume that a column of an increased buckling length has an equal absolute value $\bar{y}_0 = y_0$, or an equal relative value $\bar{y}_0/\bar{l} = y_0/l$ in regard to the buckling length etc. In the former case, the right-hand side of equation 38 would remain the same, except that we have to write $P_E l/\bar{l}^2$ instead of $\bar{P}_E$, so that the following relationship would exist:

$$\bar{l}/l = \sqrt{P_E/\bar{P}_E} \ldots\ldots\ldots\ldots\ldots(42a)$$

In the latter case, it would be necessary to write, on the right-hand side of equation 38, the expression $a_1\bar{l}/l$ instead of $a_1$. We should then obtain a quadratic equation for $\bar{l}/l$, the solution of which is:

$$\frac{\bar{l}}{l} = -A + \sqrt{A^2 + \frac{P_E}{P + P_p}}$$

$$A = \frac{1}{2\eta''}\left(\frac{P_E}{P + P_p} - 1\right) \ldots\ldots\ldots\ldots(42b)$$

which is a value smaller than that given by equation 42a.

## EXAMPLE

In the design of a particular prestressed concrete bridge constructed by the cantilever method, the question arose of how slender the columns could be designed in order to keep the effects of dilatation to a minimum. The procedure of analysis follows.

The buckling length is determined and the Euler critical load of a strut inside an elastic frame $P_E$ = 8,030 tonnes. The self-weight and the permanent load produce in the strut an axial compressive force $P$ = 2,810 tonnes, a live load $P_p$ = 605 tonnes. The load $P$ is assumed to be applied, in the mean, to a concrete of an age $t_0$ = 2 months and the factors of creep are assumed to be $\varphi_\infty(t_0) = 2{\cdot}10$, $\varphi_\infty(\infty) = 1{\cdot}05$. The percentage of reinforcement being 1·1, $\rho_a = 0{\cdot}10$ (equation 6a). First we calculate

$$P/P_E = 0{\cdot}35,$$

$$\varphi_{\infty a}(t_0) = 2{\cdot}10\left(1 - \frac{0{\cdot}10 \times 8{,}030}{2{,}810}\right) = 1{\cdot}50$$

$$\varphi_{\infty a}(\infty) = 1{\cdot}05\left(1 - \frac{0{\cdot}10 \times 8{,}030}{2{,}810}\right) = 0{\cdot}75$$

(equation 17)

and then from Figure 2c can determine $\eta_a = 2{\cdot}75$, so that

$$\eta(\infty) = 1 + 0{\cdot}9 \times 1{\cdot}75 = 2{\cdot}58.$$

According to equation 39, for $\eta'' = \eta$ we will then have

$$(P + P_p)/\bar{P}_E = 1 - \left(1 - \frac{3{,}415}{8{,}030}\right)/2{\cdot}58 = 0{\cdot}777$$

(equation 39)

and, as a final result,

$$P_E/\bar{P}_E = \frac{3{,}415}{0{\cdot}777 \times 8{,}030} = 0{\cdot}548$$

or

$$P_E/(\bar{P} + P_p) = 1 + \left(\frac{8{,}030}{3{,}415} - 1\right)/2{\cdot}58 = 1{\cdot}52$$

(equation 41)

$$\bar{P} + P_p = 5{,}270 \text{ tonnes}$$

For this value of $\bar{P}_E$ (or $\bar{P}$), the buckling coefficient should now be determined as for an elastic strut and the section then assessed.

The idea of an initial curvature has in principle been introduced only to make possible the establishment of a relationship between short-term (elastic) and long-term buckling (under the creep). Otherwise the magnitude of an initial curvature has no influence upon the result and the diversity of its forms comes into play to a very limited extent only. Consequently, an analysis of buckling for an elastic column with a reduced modulus of elasticity $\bar{E}$ or, alternatively, with an increased long-term load $\bar{P}$ (or with an increased length of buckling) can be accomplished by any method which need not even be based on the conception of an initial curvature. It should be noted that the solution presented here does not concern bearing capacity and applies for slender columns with low stress values and, consequently, a linear creep.

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**Contributions discussing the above paper should be in the hands of the Editor not later than 31 October 1968.**