NUMERICAL ANALYSIS OF CREEP OF REINFORCEDPLATES

Z. P. BAŽANT*

Approximating the hereditary integrals (generally of non-convolution type) by finite sums, the integral-type creep problem is converted to a sequence of elasticity problems with initial strains. In this manner a highly accurate, fourth-order method of time integration is set up and applied to an orthotropic layered plate, or a plate reinforced by an orthotropic system of bars or fibres. Applying a well-known method for elastic problems with initial strains, it is shown how the inelastic strains in a layered plate can be replaced by an equivalent lateral distributed load. The method was verified by means of a numerical example of a rectangular plate. For the special case of a degenerate memory function, a modification, reducing substantially the requirements for computer storage and time, is derived.

Symbols

- $d_b$ = thickness of isotropic layer $b$ of plate
- $h_1, h_2$ = curvature parameters given by (9)
- $q$ = distributed lateral load of plate, in direction of $z$
- $q_1$ = fictitious $q$ equivalent to inelastic strains
- $t$ = time
- $w$ = deflection of plate, in direction of $z$
- $w_{xy}$ = $\partial w/\partial x\partial y$
- $w^1$ = deflections due to $q^1$
- $x, y$ = deflections of plate, in direction of $z$
- $z$ = lateral coordinate throughout the thickness of plate
- $A_{1x}, B_{1x}$ = rectangular coordinates in the plane of plate
- $E_b, E_{1b}$ = Young's modulus of layer $b$, and modulus given by (10)
- $D_b$ = cylindrical stiffness of layer $b$ (2)
- $D_{2x}, D_{2y}, D_{2xy}$ = total bending and torsional stiffness of the layered plate (2)
- $D_{xy} = D_{xy}^{1}, D_{xy}^{2}$ = bending and torsional stiffnesses of the orthotropic layer (reinforcement)
- $G_b$ = shear modulus of isotropic layer $b$
- $I_b = d_b/12$
- $L_{1b}, L_{2b}$ = memory functions of isotropic layer $b$, corresponding to $E_1$ and $G_1$ (11, 12)
- $M_{xzb}, M_{yzb}, M_{xzb}$ = bending moments and torsional moment in isotropic layer $b$
- $M_{xzb} = M_{xzb}^1 + M_{xzb}^2$, $M_{yzb} = M_{yzb}^1 - M_{yzb}^2$
- $M_{xzb}, M_{yzb}$ = fictitious prestress moments equivalent to inelastic strains (16)
- $M_{xzb}^1, M_{yzb}^1, M_{xzb}^2$ = values of $M_{xzb}, M_{yzb}$ due to $q^1$
- $S_{1x}, S_{2y}, S_{xy}$ = given by (21), (23)
- $\epsilon_{1x}, \epsilon_{2y}, \epsilon_{xy}$ = normal and shear components of strain tensor in layer $b$
- $\epsilon_1 = \epsilon_{1x} + \epsilon_2$, $\epsilon_2 = \epsilon_{2y} - \epsilon_1$
- $\epsilon_{1x}, \epsilon_{2y}, \epsilon_{xy}$ = inelastic strains in the sense of $\epsilon_1$, $\epsilon_2$, $\epsilon_{xy}$ (13, 14)
- $\nu_b$ = Poisson ratio for layer $b$
- $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$ = normal stresses and shear stress in layer $b$

* Zdenek P. BAŽANT, Associate Professor of Civil Engineering, The Technological Institute, Northwestern University, Evanston, Illinois 60201, USA
forced concrete plates or layered plates represent practically impor­
tures whose creep can usually be solved only numerically. For
problems at small strains the numerical method is, in general, well­
il is based on Theorem 1 given in the Appendix. This theorem was
for volumetric inelastic strains already in 1838 by Duhamel [6] and
widely utilized in thermo elasticity. For deviatoric strains and an
material, this theorem was first deduced by Reissner in 1931 [6].
t approaches it was later independently derived by Eschelby [6] in
anisotropic material) by Bažant [1, 2, 4]. For the finite element
an equivalent technique of solution of the effect of initial (inelastic)
been developed separately [7]. First application to a two-dimension­
in creep of homogeneous plates (non-linear creep of the rate type)
made by Lin [5] and by Bažant [1, 2].
his paper* a method of application of Theorem 1 to a reinforced
plate will be presented. In addition, a highly accurate algorithm
ution of memory-type creep problems will be shown and verified by
a numerical example.

2. Basic relationships for elastic layered plates
r to stepping into the proper subject of creep, it is necessary to
; some well-known relationships for elastic plates [7]. Assuming
ormal to the middle surface of plate remain straight and perpendiculars
w of a plate must satisfy the equation:
\[ D_x \frac{\partial^4 w}{\partial x^4} + 2D_{xy} \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q \]  
(1)
y = rectangular coordinates; \( q = q(x, y) \) = distributed load; \( D_x \), \( D_y \), \( D_{xy} \) rigidity constants. Consider that the plate consists of two layers
which layer \( b \) is isotropic and layer \( a \) is orthotropic. Then

where \( E_b, \nu_b = \) Young’s modulus and Poisson ratio for the isotropic layer \( b \);
\( d_b = \) thickness of layer \( b \); \( D_{xa}, D_{ya}, D_{xya} = \) bending and torsional rigidities
of the orthotropic layer \( a \); subscripts \( a \) or \( b \) refer to the layers \( a \) or \( b \). Actually
the plate may consist even of more layers if these layers have the same elastic
properties; then the constants \( D_b, D_{xa}, D_{ya}, D_{xya} \) must express the sum of
bending and torsional rigidities (with respect to the same middle surface) of
all the layers with the same properties. A system of reinforcing bars (or fibres)
in directions \( x \) and \( y \) may be viewed as a special case of orthotropic layer \( a 
\), such that \( D_{xya} \approx 0 \); \( D_{xa} \) and \( D_{ya} \) = total bending rigidities of all reinforce-
ment in directions \( x \) and \( y \). For the sake of simplicity it is assumed that every
layer is symmetrical with respect to the middle plane of plate; this implies,
e.g., that the reinforcing bars are distributed symmetrically. Later also the
following relationships will be needed:
\[ \epsilon_1 = -k_1 z, \quad \epsilon_2 = -k_2 z, \quad \epsilon_{xy} = -\frac{w_{xy}}{2} \]  
(3)
\[ \sigma_1 = M_{xb}/I_b, \quad \sigma_2 = M_{yb}/I_b, \quad \tau_{xy} = M_{xyb}/I_b \]  
(4)
\[ M_{xb} = -E_{1b} I_b k_1, \quad M_{yb} = 2G_b I_b k_2, \quad M_{xyb} = -2G_b I_b w_{xy} \]  
(5)
where \( \epsilon_1 = \epsilon_x + \epsilon_y, \quad \epsilon_2 = \epsilon_x - \epsilon_y \)
\[ \sigma_1 = \sigma_x + \sigma_y = E_{1b} \epsilon_1, \quad \sigma_2 = \sigma_x - \sigma_y = 2G_b \epsilon_2, \quad \tau_{xy} = 2G_b \epsilon_{xy} \]  
(7)
\[ M_{xb} = M_{xb} + M_{yb}, \quad M_{yb} = M_{xb} - M_{yb} \]  
(8)
\[ k_1 = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \quad k_2 = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}, \quad w_{xy} = \frac{\partial^2 w}{\partial x \partial y} \]  
(9)
\[ E_{1b} = \frac{E_b}{1 - \nu_b}, \quad G_b = \frac{E_b}{2(1 + \nu_b)} \]  
(10)
Here \( z \) = normal coordinate; \( \sigma_x, \sigma_y, \tau_{xy} \) = normal stresses and shear stress in
layer \( b \) (\( \sigma_z = 0 \)); \( \epsilon_x, \epsilon_y, \epsilon_{xy} \) = normal strains and shear strain \( (\epsilon_{xz} = \epsilon_{yz} = 0, \epsilon_z \) is generally nonzero); \( M_{xb}, M_{yb} \) = bending moments in layer \( b \), \( M_{xyb} \) =
torsional moment, \( G_b \) = shear modulus.

3. Numerical integration of the creep problem
It will be assumed that the isotropic layer \( b \) exhibits linear creep and the
layer \( a \) does not creep at all.
he case of plane stress, assumed to exist in the plate, the linear creep
tegral-type, satisfying the conditions of isotropy, may be written in
form:

$$
\varepsilon_1(t) = \frac{\sigma_1(t)}{E_{1b}(t)} + \int_0^t \frac{\sigma_1(\tau)}{E_{1b}(\tau)} \, L_{1b}(t, \tau) \, d\tau = E_{1b}^{-1} \sigma_1(t)
$$

$$
\varepsilon_2(t) = \frac{\sigma_2(t)}{2 G_b(t)} + \int_0^t \frac{\sigma_2(\tau)}{2 G_b(\tau)} \, L_{2b}(t, \tau) \, d\tau = E_{2b}^{-1} \sigma_2(t)
$$

$$
\varepsilon_{xy}(t) = \frac{\tau_{xy}(t)}{2 G_b(t)} + \int_0^t \frac{\tau_{xy}(\tau)}{2 G_b(\tau)} \, L_{2b}(t, \tau) \, d\tau = G_b^{-1} \tau_{xy}(t)
$$

and $\tau = \text{time, or age of concrete}; \ E_{1b}^{-1}$ and $G_b^{-1}$ are creep operators, ling to the elastic constants $E_{1b}^{-1}$ and $G_b^{-1}; \ E_{1b}^{-1}$ and $G_b^{-1}$ have the form Volterra’s integral operators whose kernels
ntal conditions;

$\varepsilon(t)$ may be approximated by the finite sums

$$
\varepsilon_1(t) = \sum_{s=0}^t c_s L_{1b}(t, t_s) \sigma_1(t_s)/E_{1b}(t_s)
$$

$$
\varepsilon_2(t) = \sum_{s=0}^t c_s L_{2b}(t, t_s) \sigma_2(t_s)/(2 G_b(t_s))
$$

$$
\varepsilon_{xy}(t) = \sum_{s=0}^t c_s L_{2b}(t, t_s) \tau_{xy}(t_s)/(2 G_b(t_s))
$$

are constants and subscript $(r)$ pertains to time $t_r$, e.g. $\sigma_1(t_r) = \varepsilon_1\sigma_1(t_r)$ in the creep law (11) takes the form

$$
\varepsilon_1(t_r) = \frac{\sigma_1(t_r)}{E_{1b}(t_r)} + \sum_{s=0}^{t-r} c_s L_{1b}(t_r, t_s) \sigma_1(t_s)/E_{1b}(t_s)
$$

$$
\varepsilon_2(t_r) = \frac{\sigma_2(t_r)}{2 G_b(t_r)} + \sum_{s=0}^{t-r} c_s L_{2b}(t_r, t_s) \sigma_2(t_s)/(2 G_b(t_s))
$$

$$
\varepsilon_{xy}(t_r) = \frac{\tau_{xy}(t_r)}{2 G_b(t_r)} + \sum_{s=0}^{t-r} c_s L_{2b}(t_r, t_s) \tau_{xy}(t_s)/(2 G_b(t_s))
$$

Assume that the stresses have already been calculated up to the time $t_{r-1}$ and that the values of the stresses $\sigma_1(t_r), \sigma_2(t_r), \sigma_{xy}(t_r)$ have been estimated. The most simple estimate is $\sigma_1(t_r) \approx \sigma_1(t_{r-1})$ obtained by extrapolation, e.g. by the formula $\sigma_1(t_r) = \sigma_1(t_{r-1}) + 3\sigma_1(t_{r-2}) - \sigma_1(t_{r-3})$ whose error is of order $\Delta t^4$. Then the values $\varepsilon_1(t_r), \varepsilon_2(t_r), \varepsilon_{xy}(t_r)$ may be computed, using eqs (13), and represent thus known quantities in eqs (14). Therefore eqs (14) may be formally regarded as a fictitious elastic stress-strain law with prescribed initial (inelastic) strains. Solving the elasticity problem with these initial strains, and given applied loads and prescribed displacements in time $t_r$, new values for the stresses $\sigma_1(t_r), \sigma_2(t_r), \sigma_{xy}(t_r)$ are obtained.

The basic feature of the numerical algorithm outlined is that the time integration of a creep problem is converted to a sequence of elasticity problems with prescribed initial strain. Each of these problems may be converted to a problem without initial strains according to Theorem 1 in the Appendix. How this may be implemented will be explained now.

4. Effect of inelastic strains in layered elastic plates

Because of the linearity of creep law, the distributions of $\varepsilon_1(t_r), \varepsilon_2(t_r), \varepsilon_{xy}(t_r)$ or $\sigma_1(t_r), \sigma_2(t_r), \sigma_{xy}(t_r)$ across the viscoelastic layer must be linear. Denoting the resultants of $\sigma_1(t_r), \sigma_2(t_r), \sigma_{xy}(t_r)$ over the viscoelastic layer by $M_{1b}(t_r), M_{2b}(t_r), M_{xyb}(t_r)$ and putting $M_{1b}(t_r) = M_{1b}(t_r) + M_{2b}(t_r), M_{b}(t_r) = M_{b}(t_r) - M_{xyb}(t_r)$, the following holds true:

$$
\sigma_1(t_r) = E_{1b}(t_r) \varepsilon_1(t_r) = M_{1b}(t_r)/I_b
$$

$$
\sigma_2(t_r) = 2 G_b(t_r) \varepsilon_2(t_r) = M_{2b}(t_r)/I_b
$$

$$
\tau_{xy}(t_r) = 2 G_b(t_r) \varepsilon_{xy}(t_r) = M_{xyb}(t_r)/I_b
$$

Expressing $\sigma_1(t_r), \ldots, \sigma_1(t_r)$ from eq. (4), substituting into (14) and taking into account eqs (15), it follows that:

$$
M_{1b}(t_0) = E_{1b}(t_0) \sum_{s=0}^t c_s L_{1b}(t_0, t_s) M_{1b}(t_s)/E_{1b}(t_s)
$$

$$
M_{2b}(t_0) = 2 G_b(t_0) \sum_{s=0}^t c_s L_{2b}(t_0, t_s) M_{2b}(t_s)/(2 G_b(t_s))
$$

$$
M_{xyb}(t_0) = 2 G_b(t_0) \sum_{s=0}^t c_s L_{2b}(t_0, t_s) M_{xyb}(t_s)/(2 G_b(t_s))
$$

The loading state designated by $F^4$ in Theorem 1 is represented by a distributed load $q_0$ which is in equilibrium with the prestresses $\sigma_0(t_r), \sigma_0(t_r), \sigma_{xy0}(t_r)$.
also in equilibrium with $M_{1b(r)}$, $M_{2b(r)}$, $M_{x+y}(r)$ and is determined according to a differential equation of equilibrium of plate. Thus

$$q_i^{(r)} = -\frac{1}{2} \frac{\partial^2 x}{\partial x^2} (M_{1b(r)} + M_{2b(r)}) - \frac{1}{2} \frac{\partial^2 x}{\partial y^2} (M_{1b(r)} - M_{2b(r)}) - 2 \frac{\partial^2 x}{\partial x \partial y} M_{x+y}(r).$$

(17)

raly, at the boundaries additional loads may be required to balance these differential equations of equilibrium of plate. Thus the deflections $w_i^{(r)}$ due to the load $q_i^{(r)}$ can be solved from eq. (1) with appropriate boundary conditions. The corresponding internal forces may be obtained from eqs (9), (5), (8). Finally, according to Theorem 1,

$$w_i^{(r)} = w_i^{(r-1)} + w_i^{(r)},$$

$$M_{1b(r)} = M_{1b(r)} - M_{1b(r)} + M_{1b(r)}.$$

(18)

$w_i^{(r)}$, $M_{1b(r)}$, ... is the elastic solution, due to the given applied loads in $t_i$, alone.

5. Algorithm of numerical integration

The algorithm of time integration as outlined after eq. (14) can be made efficient by essentially the same refinements as those used in solving a system of integral equations [6]. Thus, when the final values of $M_{1b(r)}$, $M_{2b(r)}$, $f_{x+y}(r)$ for the $r$-th step have been found, their accuracy may be improved by essentially the same refinements as those used in solving a system of integral equations. The algorithm of time integration just described is represented in Fig. 1.

It is necessary to note that this higher-order integration method may be utilized only in the time intervals, in which all the applied loads evolve as a continuous functions with continuous first three derivatives. Otherwise it makes sense to use only the simple algorithm, involving no extrapolation (i.e. starting with the values $M_{1b(r)} = M_{1b(r)} = M_{1b(r)} = M_{1b(r)}$ etc. Then the values of $M_{1b(r)}$, $M_{2b(r)}$, $f_{x+y}(r)$ (eqs 16) and the solutions of the elastic plate.

For the evaluation of the hereditary integrals according to eqs (16), the values $c_i^{(r)}$ cannot be selected according to the Simpson’s rule because of the presence of subintervals between $t_0$ and $t_r$ is alternately even and odd. A suitable integration formula without this limitation and with the same order of approximation is:

$$\int_{t_0}^{t_r} f(t) dt = \frac{At}{24} [9(f_{t_0}) + f_{t_0} + 19(f_{t_1}) + f_{t_1} + 5(f_{t_2}) + f_{t_2} + 2(f_{t_3}) + f_{t_3} + \sum_{r}^{2} (-f_{t_{r-1}} + 13f_{t_r} + 13f_{t_{r+1}} - f_{t_{r+2}})].$$

(19)

Numerical example

As a test example the solution of a rectangular plate was programmed. The edges $x = 0$ and $z = 0$ were considered as fixed, and the edges $y = 0$ and $z = b$ as simply supported. A constant, uniform load $q$ applied in time $t_0$ was considered. The numerical data were: $a = b = 400$ cm, $D_c = 4 \times 10^{11}$ kp/cm² (kp = force, kilogram), $D_{xy} = D_{xy} = D_{xy} = D_{xy} = 0$, $\rho = 0.15$, $t_0 = 60$ days, $t_1 = 180$ days. For the solution of the elasticity problem, the finite difference method with a square grid of mesh size $\Delta x = \Delta y = a/16$ was adopted; the functions $M_{1b}$, $M_{2b}$, ... were represented by the arrays of their nodal values, and the partial derivatives in eqs (17), (1), (9) were replaced by the finite difference expressions. The elastic analysis of the plate was thus reduced to a system of algebraic equations. The results of analysis (on IBM 7094) according to the flow chart in Fig. 1 are given in Table I for different numbers of sub-intervals, $n$, of the time interval $(t_0, t_1)$. Some of the results are graphically represented in Figs 2a, b, c. It is noteworthy that the time changes in the distribution of the relative (not the absolute) values of stresses in the reinforcement are very small (Fig. 2c) while the shifts in the
The computer results indicated that more than two iterations per step bring hardly any improvement in accuracy. It should be noted that the problem discussed is practically relevant only for the stresses due to lateral loads in symmetrically prestressed concrete plates. This analysis cannot be applied to non-prestressed concrete plates because the phenomenon of cracking has not been accounted for. Another area of application are the plates of fibre-reinforced plastics and laminated plates.

Table I

<table>
<thead>
<tr>
<th>$x = a/2$</th>
<th>$x = 3a/16$</th>
<th>$y = b/2$</th>
<th>$y = 4a/3$</th>
<th>$y = a/2$</th>
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</thead>
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<td>$z$</td>
<td>$M_{1b}$</td>
<td>$M_{2b}$</td>
<td>$M_{1s}$</td>
<td>$M_{2s}$</td>
</tr>
<tr>
<td>-----------</td>
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<td>-----------</td>
</tr>
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<td>-168.18</td>
<td>-406.94</td>
<td>-48.472</td>
</tr>
</tbody>
</table>

Fig. 2. Lines of equal relative values in the right lower quadrant of plate. (Dashed lines pertain to the initial state at time $t_0$, continuous lines to time $t_1$; $S$ denotes the simply supported edge, $F$ the fixed edge and $O$ the center of plate.) a) Maximum bending moments in the isotropic layer $b$. (In time $t_0$, $O$ corresponds to the stress value $-192.8, 10$ to $629, 3$; in time $t_1$, $O$ corresponds to $-168.2, 10$ to $416.8$.) b) Minimum bending moments in the isotropic layer $b$. (In time $t_0$, $O$ corresponds to $-312.8, 10$ to $94.4$; in time $t_1$, $O$ corresponds to $-231.0, 10$ to $166.3$.) c) Bending moments in the reinforcement (layer $a$) in $x$ direction. (In time $t_0$, $O$ corresponds to $-290.4, 10$ to $629.3$; in time $t_1$, $O$ corresponds to $-406.9, 10$ to $875.2$.)
equation of order $n_0$ [3]. In such a case the requirements for computer storage are known to be much lower. The same result can be, however, achieved without abandoning the integral-type creep law and the numerical algorithm described. Indeed, substituting (20) into (11), it may be found that (13) may be replaced by the following equations

$$
\varepsilon_0(t) = \sum_b B_{1b}(t) S_{1b}(t), \quad \varepsilon_2(t) = \sum_b B_{2b}(t) S_{2b}(t), \quad \varepsilon_x(t) = \sum_b B_{3b}(t) S_{xy}(t)
$$

where

$$
S_{1b}(t) = \int_0^t A_{1b}(\tau) \sigma_1(\tau) \, d\tau, \quad S_{2b}(t) = \int_0^t A_{2b}(\tau) \sigma_2(\tau) \, d\tau,
$$

$$
S_{xy}(t) = \int_0^t A_{xy}(\tau) \tau_{xy}(\tau) \, d\tau.
$$

Then, replacing these integrals with finite sums according to formula (19), it is possible to obtain the recurrent equations which follow

$$
S_{1b}(t) = S_{1b}(t-1) + \frac{At}{24} (B_{f1}(t) + 3B_{f1}(t) - 11B_{f1}(t-1) + 6B_{f1}(t-2)) + 5f_{r-3} - f_{r-4}), \quad S_{2b}(t) = \ldots, \quad S_{xy}(t) = \ldots
$$

where $f_r = A_{1r}(t) G_{1r}(t)$. They are applicable for $r > 3$ and the starting values are:

$$
S_{1b}(0) = \frac{3At}{8} (f_{10} + 3f_{11} + f_{12} + f_{13}), \ldots
$$

It us seen that the storage in computer of the entire history of stresses $\sigma_1$, $\sigma_2$, $\sigma_{xy}$ is not needed; only the history over the last four steps $At$ needs to be stored. It may be verified that the amount of storage required is the same as for the corresponding rate-type creep law and the numerical method of the fourth order. The amount of computer time for evaluation of $\varepsilon_0^0$, $\varepsilon_2^0$, $\varepsilon_x^0$ becomes also substantially reduced. The general form of the flow chart in Fig. 1 remains unchanged.

In the case of degenerate memory functions, the storage requirements are in fact proportional to the desired order of error. If one is contented with the trapezoidal rule for the evaluation of the hereditary integrals in (11), then

$$
S_{1b}(t) = S_{1b}(t-1) + \frac{At}{2} (A_{1b}(t-1) \sigma_1(t-1) + A_{1b}(t) \sigma_1(t)), \quad S_{2b}(t) = \ldots, \quad S_{xy}(t) = \ldots
$$

so that besides the current values of stresses only the current values and the last preceding values of $S_{1b}$, $S_{2b}$, $S_{xy}$ ($z = 1, \ldots, n_0$) need to be stored. The
art for this case is represented in Fig. 3 in which the notation \( M_{ib} = t(\tau) \), \( M_{ib} = M_{ib}(t_{n-1}) \) is used.

\[
\begin{align*}
M_{1b} &= 0, M_{2b} = 0, M_{3b} = 0, \\
M_{4b} &= 0, M_{5b} = 0, M_{6b} = 0, \\
M_{7b} &= 0, M_{8b} = 0, M_{9b} = 0, \\
M_{10b} &= 0, M_{11b} = 0, \\
M_{12b} &= 0, M_{13b} = 0, M_{14b} = 0, \\
S_{1b} &= S_{2b} = S_{3b} = 0, \\
S_{4b} &= S_{5b} = S_{6b} = 0, \\
S_{7b} &= S_{8b} = S_{9b} = 0, \\
S_{10b} &= S_{11b} = S_{12b} = 0.
\end{align*}
\]

is worth noting that the Arutyunian’s and Maslov’s formula \([2, 3]\)

\[
L_{12}(t, \tau) E_{12}^{-1}(\tau) = - \frac{\partial}{\partial t} \left[ E_{12}^{-1}(\tau) + \sum_{\alpha=1}^{n_{\tau}} \varphi_{\alpha}(\tau)(1-e^{-\alpha(t-\tau)}) \right] \tag{26}
\]

4. If the plate has a simple shape, the finite difference method in space coordinates is suitable.

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Appendix

**Theorem 1.** — Let the constitutive equation for small strains be

\[
\sigma = C (\varepsilon - \varepsilon^0) \tag{A1}
\]

where \( \sigma \) = stress tensor, \( \varepsilon \) = strain tensor, \( \varepsilon^0 \) = initial strain tensor, \( C \) = fourth rank tensor of elastic moduli; \( C \) and \( \varepsilon^0 \) may depend on space coordinates. Introduce the prestress tensor

\[
\sigma^0 = C \varepsilon^0 \tag{A2}
\]

and define \( F^1 \) as the state of volume and surface loads which equilibrate \( \sigma^0 \). Then in a given body under zero applied loads the stresses are \( \sigma^1 - \sigma^0 \), the (linearized) strains are \( \varepsilon^1 \) and the (small) displacements are \( u^1 \) where \( \varepsilon^1, \varepsilon^0 \), \( u^1 \) is the solution of the same body (with the given prescribed displacements) for loads \( F^1 \) and zero initial strains.

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Числовой анализ ползучести железобетонных плит (З. П. Базант). Приближение дегенерированных интегралов неспирального типа с помощью конечных количеств преобразует явление ползучести интегрального типа в ряд последующих друг за другом задач по упругости с начальной деформацией. Таким образом удалось разработать очень точный метод временного интегрирования четвертой степени и применить его для некоторой многослойной ортотропной пластини или для некоторой ортотропной пластины, армированной металлической или волокнистой системой. Применяя хорошо известный метод, учитывающий начальные деформации, для задач по упругости, видно, что каким образом можно заменить в некоторой многослойной пластине неупругие деформации эквивалентной распределенной системой сил, действующей в боковом направлении. Доказательство правильности метода производится на числовом примере прямоугольной четырехугольной пластины. Для специального случая некоторой дегенерированной мемориальной функции приведен вывод такого модифицированного метода, который значительно сокращает требования по накоплению и времени на вычислительной машине.