A Correlation Study of Formulations of Incremental Deformation and Stability of Continuous Bodies

In the past a number of different linearized mathematical formulations of the infinitesimal incremental deformations of continuous bodies under initial stress have been proposed. The best-known formulations are reviewed, tabulated, and subjected to a comparative study. It is demonstrated that they can be derived as special cases of a unified general formulation, and are all correct and mutually equivalent. In each formulation, the incremental elasticity constants and the incremental material stress tensor have a different significance. Their mutual relationships are established. Thus the analysis of a problem which has already been solved according to one formulation need not be repeated for another formulation. Furthermore, the connections to the various definitions of the objective stress rate are shown. The arbitrariness of choice between the infinitely many possible forms of incremental equilibrium equations corresponds to the arbitrariness in the definitions of (a) the finite strain tensor, (b) the material stress tensor, (c) the objective stress rates, (d) the stability criterion, and (e) the elastic material in finite strain. For demonstration of the differences, the problems of surface buckling of an orthotropic half space and a column with shear are studied. It is shown that the predicted buckling stresses can differ almost by a ratio of 1:2 if the proper distinction between various formulations is not made.


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In equation (23) and the first line of column e in Table 1 the fraction $\frac{3}{2}$ should be changed to 2.

In the 4th line after equation (28), following the reference brackets, the last bracket should be followed by $F_{kl}F'_{rl} = \delta_{kr} + 2\epsilon^a_{kr}$.

In the 5th line after equation (22d) following $\delta t$, the phrase "if coordinates $x_i$ are used," should be inserted.

In equation (31) $F_{pq}$ should be replaced by $E^a_{pq}$.

In the 3rd line of footnote 7, equation (7) should be (13).

In equation (22b), the last minus sign should be changed to a plus sign.
\( i = 1, 2, 3 \) and the coordinates of the same particle after the incremental displacement \( u_i \) will be designated by \( x_i' = x_i + u_i \).

No formal distinction will be made between the subscripts referring to the initial and the final configurations. Subscripts preceded by a comma will denote partial derivatives with respect to \( x_i \) (not \( x_i' \), e.g., \( u_i, u_{ij} = \partial u_i/\partial x_j \)). Doubly repeated subscripts will imply summation over the range 1, 2, 3. To avoid lengthy discussions, attention will be restricted to the case of "dead" loads, i.e., loads which do not change direction and magnitude during the incremental deformation.

The best-known formulations of equilibrium equations for small incremental deformations which will be discussed here can all be written in the form

\[
\tau_{ij} + \rho_0 f_i = 0 \quad \text{(in the whole volume)} \tag{1}
\]

with the following expressions for \( \tau_{ij} \):

\[
\tau_{ij} = \sigma_{ij}' + \tau_{ij}^0 \tag{2a}
\]

\[
\tau_{ij} = \sigma_{ij}' + \tau_{ij}^0 + \frac{1}{2} \tau_{ij}^0 \tag{2b}
\]

\[
\tau_{ij} = \sigma_{ij}' + \tau_{ij}^0 \tag{2c}
\]

\[
\tau_{ij} = \sigma_{ij}' + \tau_{ij}^0 \tag{2d}
\]

Here \( \rho_0 \) is the initial mass density; \( n_1 \) is the unit outward normal in the initial state; \( f_i \) and \( p_i \) are the increments of mass forces and surface loads (considered as "dead" loads); \( e_{ij} \) and \( \omega_{ij} \) are the linearized incremental tensors of strain and rotation, i.e.,

\[
e_{ij} = \frac{1}{2}(u_{ij} + u_{ji}) \quad \omega_{ij} = \frac{1}{2}(u_{ij} - u_{ji}) \tag{3}
\]

\( \sigma_{ij}' \) is the stress tensor (of Cauchy) in the initial state; \( \tau_{ij} \) is a certain nonsymmetric incremental stress tensor whose significance will be explained after equation (12). Tensors \( \sigma_{ij}^0 \), \( \sigma_{ij}^0 \), \( \sigma_{ij}^0 \) represent certain incremental material stress tensors. They are labeled by different superscripts because each of them has a different significance as will be demonstrated later by equations (14), (15), and (16).

Expression (2a) can be traced back to Trefftz [34, equation (29)]. The stability theories presented by Goodier and Plass [1, 2, 6, 10, 16, 21, 36] are all based on expression (2a).

\( T_{ij}^0 \) and \( T_{ij}^0 \) are labeled by different superscripts because each of them has a different significance as will be demonstrated later by equations (14, 15), Pearson [27], Hill [13], Prager [29], Truesdell and Noll [35], Green and Adkins [15], Kappus [10], Novozhilov [28], Koiter [19], Nemst-Nasser [23], and others [1, 2, 6, 10, 16, 21, 36] are all based on expression (2a).

A special type of stability criterion (for bodies with prescribed boundary displacements) presented by Hadamard [12] also implies equation (2a).

The equilibrium equations corresponding to expression (2a) are due to Biot [4, 5, equation (I.7.32)]. They were also used by Neuber [21, equations (12), (1), (3), and (15)]. It can be easily verified that for a uniform initial stress field, that is for \( T_{ij}^0 = 0 \), these equilibrium equations may also be written in the form

\[
(\sigma_{ij}^0 + T_{ij}^0 \omega_{ij} + T_{ij}^0 \omega_{ij}) + \rho_0 f_i = 0 \tag{3a}
\]

which was used by Neuber [21, equation (16)]. A special form of equation (3a) in the coordinate system whose axes \( x_i \) coincide with the principal directions of initial stress was presented already by Southwell [31, equation (16)]. (It should be noted that the term in parentheses in equation (3a) does not have the significance of \( \tau_{ij} \) as will be later discussed, and may not be substituted for \( \tau_{ij} \) in boundary condition (1a).)

### Incremental Stresses and General Form of Incremental Equilibrium Equations

The work per unit initial volume which is done at the incremental deformation will be denoted by \( W \).

\[
\delta W = S_{ij} \delta e_{ij} \quad \text{where} \quad S_{ij} = T_{ij}^0 + \sigma_{ij} \tag{4a}
\]

\[
\delta W = T_{ij}^0 \delta u_{ij} \quad \text{where} \quad T_{ij}^0 = T_{ij}^0 + \tau_{ij} \tag{4b}
\]

in which \( \delta e_{ij} \) is certain incremental finite strain tensor — a symmetric tensorial functional of the displacement gradient \( u_{ij} \), such that \( \delta W = 0 \) when \( u_{ij} \) expresses a rotation. Equations (4a) and (4b) represent definitions of the incremental stress tensors \( \sigma_{ij} \) and \( \tau_{ij} \).

Tensor \( \sigma_{ij} \) may be assumed as symmetric since \( e_{ij} = e_{ji} \). It vanishes for a rigid-body rotation because in this case \( \delta e_{ij} = 0 \), and \( \delta W = 0 \).
For small incremental deformations, \( J^{-1} = 1 - u_{ik} \), so that
\[
\Delta T_{ij} = T_{ij} - T_{0ij} = \tau_{ij} + T_{0k}u_{ij,k} - T_{0k}u_{ik,k}
\]
It should be noted that the relationship between \( \Delta T_{ij} \) and \( \tau_{ij} \) is independent of the choice of the form of finite strain tensor \( \varepsilon_{ij} \), while the relationship between \( \sigma_{ij} \) and \( \tau_{ij} \) (or \( \sigma_{ij} \) and \( \Delta T_{ij} \)) depends on this choice.

The equilibrium conditions may be obtained from the principle of virtual work. For the equilibrium state after the incremental deformation, this principle yields the equation:
\[
\int_T \delta W dV = \int_T \rho \delta u_i dV + \int_T \left( p_i + \pi_i \right) \delta u_i dS
\]
which must be satisfied for any variation \( \delta u_i \) compatible with the boundary conditions of place; \( f_i \) and \( p_i \) are the incremental forces per unit mass and the incremental surface loads (dead loads); \( V \) = volume, \( S \) = surface of the body in the initial state. The assumption that the initial state is an equilibrium state may be expressed by the principle of virtual work as follows:
\[
\int_T T_{0i} \delta u_i dV = \int_T \rho f_i \delta u_i dV + \int_T p_i \delta u_i dS
\]
Here the left-hand side may also be written in the form
\[
\int_T \frac{\partial (\delta u_i)}{\partial x_j} dV \quad \text{or} \quad \int_T \frac{\partial}{\partial x_j} (\tau_{ij} \delta u_i) dV - \int_T \tau_{ij} \delta u_i dV
\]
Applying the Gauss theorem \([29, 16]\) to the first integral in the last expression, it is found that equation (10) is equivalent to the following condition:
\[
\int_{S'} n_i \tau_{ij} \delta u_i dS - \int_{S'} \tau_{ij} \delta u_i dV = \int_{S'} \rho f_i \delta u_i dV + \int_{S} p_i \delta u_i dS
\]
where \( n_i \) is the unit outward normal at the surface. To satisfy this condition for any \( \delta u_i \) it is necessary and sufficient that
\[
\tau_{ij} + \rho f_i = 0 \quad \text{(in volume V)}
\]
\[
n_i \tau_{ij} = p_i \quad \text{(on stress boundary)}
\]
This is the general form of the differential equilibrium equations for incremental deformations. These equations also corroborate the physical significance of \( \tau_{ij} \) as explained after (4b).

It should be noted that equations (4a)–(13a) are valid irrespective of the material properties as well as the choice of the form of \( \varepsilon_{ij} \).

Special Forms of Incremental Equilibrium Equations

The common basis from which various special formulations may be obtained is equation (6), expressing the incremental mixed (Piola-Kirchhoff) tensor (of the first kind) as a function of the general incremental finite strain tensor \( \varepsilon_{ij} \). Some of the infinitely many admissible forms of \( \varepsilon_{ij} \) will now be considered.

1 Substituting the classical Lagrangian (Green's) finite strain tensor
\[ e_{ij} = e_{ij} + \frac{1}{2} u_{ik} u_{kj} \]  
\( \sigma_{ij} = \sigma_{ij} + T_{ij} u_{ik} \quad (14a) \)

Superscript a is used here and in the sequel to denote quantities which are conjugated with \( \sigma_{ij} \). The relationship between the objective material stress increment \( \sigma_{ij}^o \) and the increment \( \Delta T_{ij} \) of the true stress follows from equation (8):
\[ \sigma_{ij}^o = \Delta T_{ij} - T_{ij} u_{ik} + T_{ij} u_{ik} \quad (14b) \]

It is seen that equation (14a) is identical with (2a). The formulation of stability theory used by Trefftz, Pearson, Hill, etc., is thus connected with the classical Lagrangian finite strain (or Cauchy-Green) tensor. In this work, it is noted that, according to equation (14b) (of (14a)), the tensor
\[ \Delta_{ij}^o = T_{ij}^o + \sigma_{ij}^o \quad (14c) \]

appears to be identical with the Piola-Kirchhoff tensor of the second kind \([35, 16, 10, 29]\) except that it is referred to the initial stressed state under consideration rather than a natural unstressed state.

2 Considering the expression (4a) for work, it may be convenient to look for such a formulation that the work \( T^o_{ij} e_{ij} \) be expressed as the work done by the components of \( T^o_{ij} \) on the displacements \( u_{ij} \), as if \( T^o_{ij} \) were forces kept constant during the incremental deformation. In the special case of a symmetric transformation \( u_{ij} = u_{ij} \), called pure deformation, this work equals exactly \( T^o_{ij} u_{ij} \) (per unit initial volume).

\[ u_{ij} \]

Consequently, \( e_{ij} \) may be obtained when the transformation \( u_{ij} \) is decomposed into a pure deformation \( e_{ij} \) (at which the work equals \( T^o_{ij} e_{ij} \)) followed by a rotation (at which the work is zero). This decomposition is called polar decomposition \([35, 35, 32]\). Up to the second-order terms in \( e_{ij} \), the finite strain tensor \( e^o_{ij} \) defined by this decomposition is \[ e^o_{ij} = e_{ij} + \frac{1}{2} u_{ik} u_{kj} - \frac{1}{2} u_{ik} u_{kj} \]  
\( e^o_{ij} \) (15)

The incremental elastic moduli \( C_{ijkl} \) and the objective material stress increment \( \sigma_{ij} \) which is conjugate to \( e^o_{ij} \) will be denoted by \( C_{ijkl}^o \) and \( \sigma_{ij}^o \). Substituting \( e_{ij} = e^o_{ij} \) in equations (6) and (8), it can be obtained
\[ \tau_{ij} = \sigma_{ij}^o + T_{ij} u_{ik} - \frac{1}{2} (T_{ij}^o u_{ik} + T_{ij}^o u_{ik}) = \sigma_{ij}^o + \Delta T_{ij} - T_{ij} u_{ik} + T_{ij} u_{ik} \quad (16a) \]
\[ \sigma_{ij}^o = \Delta T_{ij} - T_{ij} u_{ik} + T_{ij} u_{ik} + \frac{1}{2} (T_{ij}^o u_{ik} + T_{ij}^o u_{ik}) = \frac{1}{2} (T_{ij}^o u_{ik} + T_{ij}^o u_{ik}) \quad (16b) \]

Obviously, equation (16a) coincides with equation (2b), obtained first by Biot [4]. Also it should be noticed that the objective material stress tensor
\[ \sigma_{ij}^o = T_{ij}^o + \sigma_{ij}^o \quad (16c) \]

\[ \sigma_{ij}^o = \sigma^o_{ij} + \frac{1}{2} u_{ik} u_{kj} \]

Note that this is not equal \( T_{ij}^o e^o_{ij} \) e.g. for a uniaxial extension, \( e^o_{ij} = u_{ij} + \frac{1}{2} s_i \) while (assuming \( T_{ij}^o \) constant) the work equals \( T_{ij}^o u_{ij} \).

* This second-order approximation may be simply obtained as follows. The transformation of any vector \( d x_{ij} \) at the incremental deformation \( u_{ij} \), and its transformation at the pure deformation \( e^o_{ij} \) defined by the foregoing decomposition, are given by the relationships
\[ d x_{ij} = (s_i + u_{ij}) d x_{ij}, \quad d x^o_{ij} = (s_i + e^o_{ij}) d x_{ij} \]

where \( s_i \) is Kronecker delta. For rotation \( d x_{ij}^o d x_{ij} = d x_{ij}^o d x_{ij}^o \). Thus the following must hold
\[ (s_i + u_{ij}) d x_{ij} (s_i + u_{ij}) d x_{ij} = (s_i + e^o_{ij}) d x_{ij} (s_i + e^o_{ij}) d x_{ij} \]
\[ e^o_{ij} = e_{ij} + \frac{1}{4} u_{ik} u_{kj} \quad (e) \]

Replacing \( e^o_{ij} \) by \( e_{ij} \) in the second-order term, equation (15) is obtained.

\[ \Delta T_{ij} = T_{ij}^o + \sigma_{ij}^o \quad (16c) \]

is different from the Piola-Kirchhoff tensor of the second kind \([35, 16, 10, 29]\) given by equation (14c).

3 In general, any tensor polynomial in \( e_{ij} \) or \( e^o_{ij} \) whose first-order term equals \( e^o_{ij} \) might be adopted for representation of \( e_{ij} \). Thus the tensor
\[ e_{ij} = e^o_{ij} + \frac{1}{2} u_{ik} u_{kj} + a e_{ik} e_{kj} + b e_{ik} e_{kj} + \delta_{ij} e^o_{kl} e^o_{kl} \quad (17) \]

where \( a, b, c, d \) are arbitrary constants and \( \delta_{ij} \) is Kronecker delta, is the most general expression for an admissible second-order approximation to a certain finite strain tensor \( e_{ij} \).

It is interesting to consider the finite strain tensor
\[ e_{ij} = e^o_{ij} + \frac{1}{2} u_{ik} u_{kj} - e_{ik} e_{kj} \quad (17a) \]

which represents the so-called logarithmic strain because \( e^o_{ij} = u_{ij} - \frac{1}{2} u_{ij} = \) the second-order approximation to \( \ln (1 + e_{ij}) \). The substitution of \( e_{ij}^o \) for \( e_{ij} \) in equations (8) and (13a) leads to the relationships
\[ \tau_{ij} = \sigma_{ij}^o + T_{ij}^o u_{ik} - T_{ij}^o u_{ik} - T_{ij}^o u_{ik} \quad (17b) \]
\[ \sigma_{ij}^o = \Delta T_{ij} - T_{ij}^o u_{ik} - T_{ij}^o u_{ik} + T_{ij}^o u_{ik} \quad (17c) \]

In this manner one could obtain infinitely many forms of equilibrium equations, in each of which \( \sigma_{ij}^o \) would be defined in a different manner. It may be verified, however, that no admissible expression for \( e_{ij} \) leads to equation (2d) used by Biot and Neuber (as well as to equation (2a)). The closest admissible expression is (2c) which differs from (2d) by the term \( T_{ij}^o u_{ik} \). For materials of small volume compressibility (nonporous materials) this difference is, of course, negligible.

Relationships Between Various Definitions of Incremental Elastic Moduli

All of the equations introduced so far are valid without regard to the material properties. Let us now consider that the material behaves in the infinitesimal incremental deformation as elastic (while the initial stressed state may still include other than elastic strains, e.g., plastic strains). Then
\[ \sigma_{ij}^o = C_{ijkl}^o u_{ij} \quad (18) \]

where \( C_{ijkl}^o \) are the incremental elastic moduli associated with \( \sigma_{ij}^o \), etc. It is easy to verify by substitution that equations (14), (16a), (17b), and (2d) (or equation (4a) for the various forms of \( \tau_{ij} \)) are mutually equivalent and represent the same material if the following relationships between the incremental elastic moduli hold:
\[ C_{ijkl}^o = C^o_{ijkl} - \frac{1}{2} (\delta_{ij} T_{kl}^o + \delta_{ij} T_{kl}^o + \delta_{ij} T_{kl}^o + \delta_{ij} T_{kl}^o) \]
\[ C_{ijkl}^o = C^o_{ijkl} + \frac{1}{2} (\delta_{ij} T_{kl}^o + \delta_{ij} T_{kl}^o + \delta_{ij} T_{kl}^o + \delta_{ij} T_{kl}^o) \]
\[ C_{ijkl}^o = C^o_{ijkl} - \delta_{ij} T_{kl}^o \]

Another relationship which correlates \( C_{ijkl}^o \) and \( C^o_{ijkl} \) has been shown by Biot [5, equation (11.4.25)].

Obviously all of these relationships satisfy the necessary conditions of symmetry for the incremental elastic moduli, that is, \( C_{ijkl}^o = C_{ijlk}^o \). In addition, relationships (19a) and (19b) also preserve the symmetry condition \( C_{ijkl}^o = C_{ijlk}^o \) which must be fulfilled by \( C_{ijkl}^o \), \( C^o_{ijkl} \) and \( C_{ijkl}^o \) if a potential for infinitesimal incremental deformations exists.

Relationship (19c), however, does not satisfy the latter symmetry condition, and if the potential exists
\[ C_{ijkl}^o \neq C^o_{ijkl} \]

This asymmetry, which is an inconvenient formal feature of equa-
tion (2d) devised by Biot and Neuber (or equation (3a) obtained by Southwell), disappears only for an incompressible material.

In the sequel an expression for the incremental material stress tensor \( \sigma_{ij} \), corresponding to equation (2d), will be also needed. Equations (2d) and (8) yield

\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl} = \Delta T_{ij} - T_{kjl} \varepsilon_{jk} - T_{ijkl} \varepsilon_{jk} \quad (20a)
\]

The relationship of the stress material increments \( \sigma_{ij} \) and \( \sigma_{ijkl} \) is, according to (16b) and (20a),

\[
\sigma_{ij} = \sigma_{ij}^{(0)} + \frac{1}{2} (T_{ijkl} \varepsilon_{jk} + T_{ijk} \varepsilon_{jkl}) - T_{ijkl} \varepsilon_{jk} \quad (20b)
\]

which is a relationship known from Biot's theory [15, equation (11.2.22)]

With the help of boundary conditions (1a) or (11a) it is possible to determine how the incremental elastic moduli can be measured. This topic has been dealt with in detail by Biot [5] and a relationship of moduli \( C_{ijkl} \) and \( C_{ijkl} \) to the "measurable slide moduli" [3] has been derived. Using this relationship and equations (19a, b, c), the moduli \( C_{ijkl} \) and \( C_{ijkl} \) may be also determined from measurements.

**Objective Stress Rates**

If the increment of deformation is associated with interval \( \Delta t \) of time \( t \), then

\[
\lim_{\Delta t \to 0} \left( \frac{\varepsilon_{ij}}{\Delta t} \right) = \text{objective stress rate of } T_{ij},
\]

\[
\lim_{\Delta t \to 0} \left( \frac{\varepsilon_{ijkl}}{\Delta t} \right) = \text{rotation rate of } \varepsilon_{ijkl}.
\]

The various time rates obtained from \( \sigma_{ij}^{(0)}, \sigma_{ij}^{(0)}, \sigma_{ijkl}^{(0)}, \sigma_{ijkl}^{(0)} \) represent the objective tensor rates (stress fluxes), i.e., tensors which are invariant at any observer transformation [35, 10, 29, 16]. The objective rates of the stress tensor \( T_{ij} \) obtained from the expressions (14b), (16b), (17c), and (20a), are as follows:

for \( \sigma_{ij} \:
\]

\[
\frac{\partial T_{ij}}{\partial \theta} = T_{ij} - T_{kjl} \varepsilon_{jk} - T_{ijkl} \varepsilon_{jk} + T_{ijkl} \varepsilon_{jk} - T_{ijkl} \varepsilon_{jk} - T_{ijkl} \varepsilon_{jk} \quad (22a)
\]

for \( \sigma_{ijkl} \:
\]

\[
\frac{\partial T_{ijkl}}{\partial \theta} = T_{ijkl} - T_{kjl} \varepsilon_{jk} - T_{ijkl} \varepsilon_{jk} + T_{ijkl} \varepsilon_{jk} - T_{ijkl} \varepsilon_{jk} - T_{ijkl} \varepsilon_{jk} \quad (22b)
\]

for \( \sigma_{ijkl} \:
\]

\[
\frac{\partial T_{ijkl}}{\partial \theta} = T_{ijkl} - T_{kjl} \varepsilon_{jk} - T_{ijkl} \varepsilon_{jk} + T_{ijkl} \varepsilon_{jk} - T_{ijkl} \varepsilon_{jk} - T_{ijkl} \varepsilon_{jk} \quad (22c)
\]

where

\[
T_{ij} = \lim_{\Delta t \to 0} \Delta T_{ij}/\Delta t
\]

Because \( \Delta T_{ij} \) is the change of stress (of Cauchy) in a given particle, \( T_{ij} \) must represent the material rate which is expressed as

\[
T_{ij} = T_{ijkl} \varepsilon_{jk} + \Delta T_{ijkl}/\Delta t [29].
\]

(If the material properties are defined by equation (18) for \( \Delta t \to 0 \), the material is called hyperelastic.)

It is readily recognized that expression (22a) is the Truesdell's stress rate [29, 10, 22] and expression (22c) is the corotational stress rate, due to Jaumann [29, 10, 35, 22]. The stress rates (22b) and (22c) have probably not been used so far.

Thus it may be concluded that the Truesdell's stress rate corresponds to the use of the classical Lagrangian strain tensor \( e_{ij} \) in the theory of incremental deformations or stability, and to Trefftz's equations of neutral equilibrium. The correspondence between the Truesdell's stress rate and the stability theory leading to Trefftz's expression (2a) has already been shown by Maoer [22].

The corotational (Jaumann's) stress rate corresponds to neutral equilibrium equation (2d) or (17b), due to Biot [5] and Neuber [25] but has no corresponding form of finite strain tensor unless the material is incompressible. The fact that for the material \( T_{ijkl} = C_{ijkl} \varepsilon_{ijkl} \), the incremental potential exists for certain moduli such that \( C_{ijkl} \neq C_{ijkl} \), and is nonexistent if \( C_{ijkl} \neq C_{ijkl} \), is to be regarded as an inconvenience of the corotational (Jaumann's) stress rate (22d).

This feature disappears for the stress rate (22c) which corresponds to the finite strain tensor (17a) (logarithmic strain) and the Biezeno and Hencky's neutral equilibrium equations given by (2c) and (17c). The difference between (22c) and (22d) can be important, of course, only for materials of high volume compressibility such as porous materials.

Another often used rate is the convected stress rate, due to Cotter and Rivlin [29, 10, 35]:

\[
T_{ij} = T_{ij} + T_{ijkl} \varepsilon_{jk} + T_{ijkl} \varepsilon_{jk} \quad (22c)
\]

(Which represents a second-order approximation to \( e_{ij}/(1 + e_{ij}) \) were used in equation (8), the corresponding stress rate would equal expression (22c) plus the term \( T_{ijkl} \varepsilon_{jk} \). But no finite strain tensor could lead to equation (22c) itself. Hence, for strongly compressible materials the convected stress rate (22c) leads to the same kind of inconvenience as the corotational (Jaumann's) rate (22d).

Oldroyd's stress rate [10] equals Truesdell's rate (22a) if the term \( T_{ijkl} \varepsilon_{jk} \) is omitted. Therefore, for an incompressible material this rate also corresponds to the classical Lagrangian tensor \( e_{ij} \). But for compressible materials the corresponding incremental moduli are again nonsymmetric if the incremental potential exists.

**Stability Criterion**

The diversity of admissible forms of equations considered hitherto is projected in other theorems of the theory of stability and incremental deformations. As an example consider the criterion of infinitesimal stability [35, 32, first attempt 7]. To avoid lengthy discussions let us restrict attention to conservative systems with dead loads. Then a given stressed state is infinitesimally stable if the work done at any further infinitesimal incremental deformation compatible with the boundary conditions of place is greater than or equal the work of given initial surface and volume loads. Noting that according to (4a) \( W = T_{ijkl} \varepsilon_{ijkl} + \frac{1}{2} C_{ijkl} \varepsilon_{ijkl} \) (for small \( e_{ij} \)), this criterion may be expressed as follows:

\[
\int (T_{ijkl} \varepsilon_{ijkl} + \frac{1}{2} C_{ijkl} \varepsilon_{ijkl} ) dV \geq \int \left( \rho u_{ij,ij} dV + \int (p_{ij} u_{ij} dS) \right)
\]

Subtracting equation (9a) written without the sign \( \delta \), the sufficient condition of stability under dead loads is obtained in form of the inequality:

\[
\int (T_{ijkl} \varepsilon_{ijkl} + \frac{1}{2} C_{ijkl} \varepsilon_{ijkl} + T_{ijkl} (e_{ij} - e_{ij}) ) dV \geq 0
\]

which must be satisfied for any kinematically admissible incremental displacements \( u_{ij} \). The special forms of the general criterion (25) are obtained by selection of a certain finite strain tensor \( e_{ij} \). If the classical Lagrangian form \( e_{ij} \) is substituted and

\[\footnote{Biot's notation and terminology is: \( \varepsilon_{ij} = \sigma_{ij} = "\text{incremental stress}" \), \( \varepsilon_{ij} = \sigma_{ij} = "\text{alternative strain}" \).} \]

\[\footnote{From criterion (25) it can be concluded that the body is at the limit of stability when the variation of the (25) is zero. Using this condition, equation (7) for \( p_{ij} = \kappa = 0 \) can be deduced from criterion (25) \[1\].} \]
the equality $C^{p}_{ijkl} = C_{ijkl}$ is considered, the stability criterion (25) takes on the form

$$\int_{V} \left( C^{p}_{ijkl} + \delta_{ijkl} T_{i}^{0} - \delta_{ik} T_{jk}^{0} - \delta_{jk} T_{ij}^{0} - \delta_{ij} T_{kl}^{0} \right) u_{i} u_{k} dV \geq 0 \quad (26)$$

presented by Trefftz [34, equation (26)]. For the special case of an elastic body with displacements prescribed at the whole boundary, this criterion was obtained by Hadamard ([12], equations (VI.18, VI.20); [9]; cf. also [27]), Prager [29, equations (4.18, 4.22), p. 219], Truesdell and Noll [35, equations (686, 19)] and Goodier and Plass [14, equation (15)].

Substituting for $C^{p}_{ijkl}$ from equations (19a) and (19b), other forms of criterion (25) may be obtained, e.g.,

$$\int_{V} \left[ C^{p}_{ijkl} + \frac{1}{2}(\delta_{ik} T_{kl}^{0} - \delta_{ik} T_{jk}^{0} - \delta_{jk} T_{ik}^{0} - \delta_{ij} T_{kl}^{0}) \right] \times u_{i} u_{k} dV \geq 0 \quad (27a)$$

$$\int_{V} \left[ C^{p}_{ijkl} + \frac{1}{2}(\delta_{ik} T_{kl}^{0} - \delta_{ik} T_{jk}^{0} - \delta_{jk} T_{ik}^{0} - \delta_{ij} T_{kl}^{0}) \right] \times u_{i} u_{k} dV \geq 0 \quad (27b)$$

Elastic Materials in Finite Strain

In the preceding, only the incremental properties were assumed to be elastic, while the initial state could have been inelastic. Let us now examine the relationship between the definition of an elastic material and the formulation of incremental deformations. The elastic material may be defined by the condition that a certain material stress tensor be a tensorial function of a certain finite strain tensor. If the Piola-Kirchhoff stress tensor of the second kind and the classical Lagrangian strain tensor with respect to the unstressed state are used, the definition of the elastic material may be written as follows [35, 16]:

$$T_{ij} = F_{ij} E_{ij} = \delta_{ij} T_{ij}^{0} + \delta_{ij} T_{ij}^{0}$$

where $F_{ij} = \frac{\partial E_{ij}}{\partial X_{i}}$. For a stress-free state, $X_{i}$ and $x_{i}$ coordinates of a particle in the stress-free state and after deformation, respectively; $J_{s} = \det (F_{ij}) E_{kl} = \text{Cauchy-Green deformation tensor}$ [35, 16] = $F_{ijkl} E_{kl} = \delta_{ij} + e_{ij}$ where $e_{ij}$ is the classical Lagrangian finite strain tensor as referred to the stress-free state; $f_{ij}$ is the given tensorial function.

For incremental deformations, superposed on finite deformations,

$$\Delta T_{ij} = T_{ij} - T_{ij}^{0} = (\Delta T_{ij}/\Delta \alpha_{ik}) F_{ij}$$

(29) (Note that $\Delta T_{ij}$, defined by equation (13a), is not an objective stress increment.) Substitution of equation (28), rearrangement and omission of higher-order terms [16, equations (21.7-21.8)], [35, equations (68.16-17)] leads then to the relationship

$$C^{p}_{ijkl} = C_{ijkl} = C^{p}_{ijkl}$$

must be satisfied for any two vectors $X_{i}, \alpha_{i}$. This is a necessary (but not sufficient) local condition of infinitesimal stability. However, if displacements are prescribed on the whole boundary and the initial stress is homogeneous, equation (27c) is also a sufficient condition, irrespective of the shape of the body [35, equation (686.18)].

Equation (27c) may be also given alternate forms if moduli $C_{ijkl}^{p}$ or $C_{ijkl}^{p}$ are used.

The procedure leading from (29) to (30) is not given in detail because it may be found in a book by Jaumann [16, p. 492].
The resulting dependence of the buckling stress $P_n$ on the ratio $N^2/Q^4$ according to Biot’s formula [5, equation (IV.4.41), p. 211] is plotted in Fig. 1 as curve c.

If other equilibrium equations and other incremental moduli are used, the whole analysis need not be performed again, because at each stage of analysis relationships (26a, b) or (19a, b, c) must be satisfied (although the equations may look quite different).

To obtain the resulting dependence of the buckling stress $P_n$ as a function of $N^2/Q^4$ or $N^2/Q^4$ may be obtained from curve c in Fig. 1. They are represented by curves a and b. It is seen that the differences between various formulations of stability theory, i.e., between the various definitions of the incremental moduli, can be substantial. It is found, for instance, that the values of $P_n$ for the case $N^2/Q^4 = 0.75$ and the case $N^2/Q^4 = 0.75$ is almost in the ratio 1:2. Differences disappear for $N^2 = Q^4$, i.e., for an isotropic incompressible half space. It is especially noteworthy that the ratio between the values $N^2/Q^4, N^2/Q^4$ and $N^2/Q^4$ leading to the same $P_n$ approaches 1:1 as $N^2/Q^4$ tends to zero (or $P_n \to 0$), which is the case of a strongly orthotropic medium.

**Example of a Column in Flexure and Shear (Timoshenko Beam)**

The formal differences between various formulations of stability theory also disappear for thin bodies, such as shells, plates and bars, if the assumption is made that the cross sections (or normals) remain plane and perpendicular to the deflected middle line (or surface). However, differences are encountered if shear is considered, i.e., the cross sections do not remain perpendicular to the deflected middle line. (Shear must be taken into account, e.g., in orthotropic columns or built-up columns.)

As an example, consider now the buckling of a perfect pinned orthotropic column of rectangular cross section of area $A$ and moment of inertia $I$. Let axis $x$ be the longitudinal axis and consider that the column is initially under uniaxial stress $T_{11}$. The longitudinal displacements equal $u_1 = x \psi(x)$ where $\psi$ is the rotation of cross section. The shear angle is $\gamma = \psi + w_z$, where $w$ is the deflection in the sense of $z = z$; $u_2 = w(x)$ and $w_x = dw/dx$. (Equations $u_1 = x \psi(x), u_2 = w(x)$ may be viewed as assumed expressions for the first terms of Kantorovich’s direct variational method.) As a starting point, let us use the variational principle (10).

Obviously one can write $\int A \sigma_i \delta \epsilon_i \, dA = M \delta \psi_x + T \delta \gamma$ where $M = E \psi_x$ is bending moment; $T = GA \gamma$ is shearing force; $A = \text{area of cross section}; I = \int x^2 \, dA$; $E = \text{longitudinal Young’s modulus}; G = \text{shear modulus for directions } x, z$ (adjusted for the shear correction coefficient for the cross section considered); $\psi_x = \delta \psi/\delta x$. All components of $T_{ij}^0$ are zero except $T_{11}^0$. Furthermore, $u_{11} = \psi_{11}, u_{12} = w_z, u_{22} = \psi, u_{13} = \psi_{13} = \psi_{23} = \psi_{33} = 0 = 0$.

If the Green’s strain tensor $e_{ij}^G$, given by (14) is used and the relation $\varepsilon_{ij}^G = \frac{1}{2}(\psi_{ij}^G + w_{ij})$ is noted, equation (10) with $e_{ij} = \sigma_{ij}/\delta \epsilon_{ij}$ now reads:

\[ \int_0^L \left[ \int_A T_{11}^0 \delta (z \psi_{11}^G + w_z) + G^* A (\psi_{11} + w_z) \delta \psi_x \right. \]
\[ \left. + \delta w_x + E^* \psi_x + \delta w_x \right] \, dx = 0 \quad (38) \]

where $x = 0$ and $L$ are the ends of the column. Integrating by parts with respect to $\delta \psi_x$, and noting that $\psi = 0$ at the hinged ends of column, it follows, after rearrangement:

\[ \int_0^L \left[ T_{11}^0 A \psi_x, \delta w_x - (T_{11}^0 1 \psi_x, \delta \psi) + G^* A (\psi + w_z) \delta \psi_x \right. \]
\[ \left. + \delta w_x - (E^* \psi_x, \delta w_x) \right] \, dx = 0 \quad (39) \]

This equation is satisfied for any $\delta \psi_x$ and $\delta w_x$, if only if,

\[ -(T_{11}^0 1 \psi_x, \psi_x) + G^* A (\psi + w_z) = 0, \quad (39a) \]

If the second equation is subtracted from the first one and is integrated from 0 to $x$ (noting that $\psi = w = 0$ for $x = 0$), equations (39a) are brought to the form

\[ E^* \frac{d \psi}{dx} - P_i w + T_{11}^0 \frac{d \psi}{dx} = 0, \quad (40) \]

\[ G^* \left( \psi + \frac{d w}{dx} \right) + T_{11}^0 \frac{d w}{dx} = 0 \]

Solution of these equations satisfying the boundary conditions $w = \psi = 0$ at $x = 0$ and $x = L$ may be sought in the form $\psi = A \cos \pi x / L, w = B \sin \pi x / L$, with $A, B$ as arbitrary constants. Substitution in (40) yields the characteristic equation

\[ T_{11}^0 - T_{11}^0 (E^* + G^* + E^* G^* / T_{11}^0) = E^* G^* = 0 \quad (41) \]

and the smallest critical value of initial stress

\[ T_{11}^0 = \frac{1}{2} c - \frac{1}{\sqrt{c^2 - E^* G^*}} \quad \text{where} \quad c = E^* + G^* + E^* G^* / T_{11}^0 \]

Here $T_{11}^0 = E^* \pi^2 (L/L)^2 = \text{Euler’s critical stress for modulus } E^*$. It is easy to verify that for a very slender column, i.e., $T_{11}^0 / E^* \to 0$, $T_{11}^0 / G^* \to 0$, expression (42) has the limit $T_{11}^0$.

Solutions for the other formulations of incremental deformation may be obtained by substituting in equations (40) or (41) the relations:

\[ E^* = E^* - T_{11}^0, \quad G^* = G^* - \frac{1}{2} T_{11}^0 \quad (43a) \]

\[ E^* = E^* - 2 T_{11}^0, \quad G^* = G^* - \frac{1}{2} T_{11}^0 \quad (43b) \]

which represent a special case of (19a, b) for uniaxial stress. It can be verified that a direct derivation, similar as above but based on strain tensors $e_{ij}^G$ or $e_{ij}^G$, yields the same results.
Thus, for instance, the characteristic equation (41) can take on considerably different forms

\[ T_{11}^0 (10 + B^2 / T_{16}^0) - T_{11}^0 (5B^2 + 6B^2 + 4B^2 G^2 / T_{16}^0) + 4B^2 G^2 = 0 \]  
(44a)

\[ T_{11}^0 (9 + B^2 / T_{16}^0) - T_{11}^0 (3B^2 + 6B^2 + 2B^2 G^2 / T_{16}^0) + 2B^2 G^2 = 0 \]  
(44b)

where

\[ T_{16}^0 = ED^2 / (L^2 A), \quad T_{16}^0 = ED^2 / (L^2 A). \]

It can be verified that for slender columns (\( T_{11}^0 \ll B^2 \)) equations (40) and (42) are identical with the Engesser's equations for buckling with shear [33, p. 133]. These equations were originally derived by consideration of equilibrium from the assumption that the shear force equals \( P \omega \), which represents the component of initial axial force \( P = -T_{11} A \) along the cross section which is perpendicular to the deflected middle line.

Alternatively, the problem has also been solved assuming [33, p. 134] that the shear force equals \( P \psi \), which represents the component of \( P \) along the cross section that was perpendicular to the beam axis in the initial state. From the equilibrium condition the equation analogous to the second of equations (40) is then obtained (putting \( G = G' \)) in the form

\[ G' (\psi + \omega) = T_{11} \psi \]  
(45)

It is interesting to note that this formulation can be found to correspond to still another form of finite strain tensor, namely, \( \varepsilon' = \varepsilon_i - 2\varepsilon_i \psi \), and to the incremental moduli \( B' = B \) and \( G' = G + 4T_{11}^0 \).

In engineering literature the term \( T_{11}^0 / d\psi / d\tau \) in the first of equations (40) has been, as a rule, neglected. This is an admissible approximation for slender columns in which \( T_{11}^0 < B \) but is exact only for Biot's incremental moduli \( B' \) and \( G' \).

To illustrate the importance of distinguishing properly between \( B' \), \( B \), etc., consider four different (incrementally orthotropic) hypolicastic materials, for which \( B'/G' = 20 \) or \( B'/G' = 20 \) or \( B'/G' = 20 \) or \( B'/G' = 20 \) at any \( T_{11}^0 \). The plots of the smallest critical stress \( T_{11}^0 \) versus column slenderness \( L/r \) have been computed from equations (42), (44a), (44b), and are shown in Fig. 2.

### Conclusions

1 Expression (6) for the incremental mixed (Piola-Kirchhoff) stress tensor (of the first kind) is the basic relationship which provides a unified formulation of the incremental equilibrium equations and enables to determine the relationships between various special forms. In different formulations the incremental material stress tensors are not identical, as is exemplified by

### Table 1

<table>
<thead>
<tr>
<th>Form</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
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<tr>
<td>Finite strain tensor</td>
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<td>(Green's, classical Lagrangian)</td>
<td>equation (14)</td>
<td>equation (15)</td>
<td>equation (16)</td>
<td>equation (17a)</td>
<td>equation (17b)</td>
<td>otherwise nonexistent</td>
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<tr>
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<td>equation (2b) or (15a) (Biot)</td>
<td>equation (2c) or (15b) (Biezeno and Hencky)</td>
<td>equation (2d) and (2e) (Biot, Neuber, Southwell)</td>
<td>equation (20a) (Biot's incremental stress)</td>
<td></td>
</tr>
<tr>
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<td>equation (16b) (Biot's alternative stress)</td>
<td>equation (17c) (Jaumann's corotational rate)</td>
<td>and (2c) (Biot, Neuber, Southwell)</td>
<td>equation (20a) (Biot's incremental stress)</td>
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<tr>
<td>Incremental moduli if a potential exists</td>
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