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REVUE ROUMAINE DES SCIENCES TECHNIQUES

TIRAGE À PART

SÉRIE DE

MÉCANIQUE APPLIQUÉE

TOME 17 N° 6 1972

ÉDITIONS DE L'ACADÉMIE DE LA RÉPUBLIQUE SOCIALISTE DE ROUMANIE

STATIC STRUCTURE — SOIL INTERACTION

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The conventional methods of structural analysis, i.e. the force and displacement methods, are extended to soil-structure interaction problems in which the foundation settlements for chosen time-constant reactions values can be approximately determined by the methods of soil mechanics. This
for structures supported on isolated footings. The nonlinear reaction-settlement diagrams are linearized by tangent compressibilities and fictitious initial settlements. The additional flexibilities and stiffnesses due to the foundation are expressed. Introducing creep operators for the structural material (concrete) and integral operators for consolidation of clay layers, the algebraic equations for time-independent interaction are transformed to a system of Volterra's integral (or integro-differential) equations, governing the interaction when a delayed response is exhibited by the structure as well as the soil. In an example this system is solved by a well-known numerical method.

INTRODUCTION

One of the basic problems of soil mechanics is the determination of settlements under statically indeterminate structures, and the criteria for their maximum tolerable values. A simplified approach often used consists of analysing the reactions of the statically indeterminate structure with the assumption that the foundations are absolutely rigid. The settlements are then calculated for the reaction so obtained. This approach is acceptable only for very flexible structures on very rigid foundations, a condition which is not always met. Often the settlements may affect largely the stresses in the structure, so that the structure with the subsoil must be analysed as a single system. For the analysis of the long-term interaction, creep in the structure and long-term consolidation in the soil must be taken into account, for they may alter considerably the effects of differential settlements. In addi-

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Rev. Roum. Sci. Techn.-Méc. Appl., Tome 17, N° 6, p. 1341—1361, Bucarest, 1972

tion, the nonlinearity of the reaction-settlement diagram usually cannot be neglected.

The empirical criteria for tolerable settlements in soil mechanics, such as a certain maximum permissible value of the settlement difference per unit length, or the settlement curvature along the soil surface, can be established only for a very specific type of structure, loading, etc., but have no general validity. In a rational approach, the stresses caused by differential settlements must be compared with the strength of the structure.

A general discussion of this problem and empirical criteria have already been presented in numerous publications [1] — [6]. In the earlier investigations, the foundation of simple structures was often treated as a continuous, isotropic, linearly elastic halfspace [6], [7], [8]. More accurate methods of soil mechanics have been used for the calculation of settlements under simple structures [6], [9], [10]. Elastic frames on a consolidating subsoil have been solved by the slope-deflection method, replacing the foundation with a spring-dashpot model [11]. Structures exhibiting creep and resting on a consolidating clay foundation have been solved by the author, assuming a simplified creep law (Eq. 17) for concrete and neglecting mutual interaction between the footings. The latter effect was included in a study of complex frames couples with shear walls (with openings) [15], [16] but the time effects were neglected. Various types of idealized foundations have been introduced in studies of beams and plates [17] — [20]. Recently a number of practical design problems have been solved by the finite element method subdividing into finite elements both the structure and the subsoil. This approach is generally applicable but often unnecessarily elaborate (e. g. for the bridge in Fig. 2), because in many instances the response of the subsoil can be determined with a practically sufficient accuracy according to the standard methods of soil mechanics for the calculation of settlements. This is particularly true for structures supported on isolated footings, and primarily such structures will be kept in mind in this paper.

The intent of this paper is to show how the response of the subsoil can be incorporated into the conventional structural analysis with or without presence of creep.

The dynamic structure-soil interaction, important for shock-wave and earthquake resistance, will be excluded from our considerations.

ASSUMPTIONS

1. The structure is connected with the subsoil at a finite number of discrete points, the footings.

2. The response of soil under time-constant reactions of footings is known. (This relationship must be provided by the soil mechanics theory).

3. The structure is not too stiff so that the changes of reactions in time are not too large. Then the reaction-settlement relationship can be linearized.

4. The consolidation of clay layers is governed by linear relationships.

5. The structure may be idealized as a system of bars or finite elements.

6. The deformations of the structure are small and the stress-strain law of the structural material may be assumed as linear.

INSTANTANEOUS SOIL RESPONSE

With regard to the duration of service of many structures, the deformations appearing within a few days after the load application may be regarded as instantaneous. According to the standard procedure in soil mechanics, the instantaneous settlements may be determined as follows. One subdivides the subsoil in several fictitious layers and determines, for a selected reaction value, the stresses in each of them. (For this purpose one can approximately assume the same stress distribution as in an elastic half-space). According to the data from consolidation tests of soil samples, the settlements of all layers are then determined and added. Repeating the procedure for various reaction values, the complete instantaneous reaction-settlement diagram may be obtained. Usually this diagram is nonlinear (Fig. 1). For the analysis of interaction a suitable linearization is necessary. Conventionally it has been assumed that $w_k = C_{kk} P_k$ where C_{kk} = secant compressibility of the foundation, P_k = reaction, w_k = settlement. This relationship is represented by the secant line in Fig. 1. This linearization, however, is not the best one because usually the value of reaction can be estimated in advance with

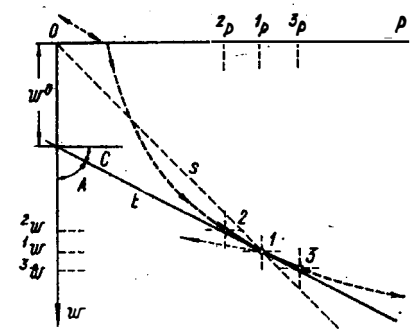

Fig. 1

an error less than about 30%. The time changes of reaction usually also do not exceed this limit. Then, obviously, the tangent line in Fig. 1. represents a better linearization of the reaction-settlement diagram. Thus

$$w_k = C_{kk} P_k + w_{kk}^0 \tag{1}$$

where

$$C_{kk} = ({}^3w_k - {}^2w_k)/({}^3P_k - {}^2P_k), \tag{1a}$$

$$w_{kk}^0 = {}^1w_k - C_{kk}\, {}^2P_k. \tag{1b}$$

Here C_{kk} is the incremental (tangent) compressibility of the foundation; ${}^1P_k$ is the estimated value of reaction P_k; ${}^2P_k$ and ${}^3P_k$ are some adjacent values of reaction (e.g. 0.9 ${}^1P_k$ and 1.1 ${}^1P_k$) used for the determination of C_{kk}; ${}^1w_k$, ${}^2w_k$, ${}^3w_k$ are the corresponding settlements. Alternatively, the relationship $w_{kk}^0 = {}^2w_k - C_{kk}\,{}^2P_k$ can be also assumed, making the computation of ${}^1w_k$ unnecessary.

If the footings are spaced closely, the mutual influence between them must not be neglected. A load P_l applied in the l-th footing contributes also to the settlement w_k in the k-th footing. This settlement may be determined similarly as w_l and again the tangent linearized relationship may be assumed, i.e.

$$w_k = C_{kl}\,P_l + w_{kl}^0 \tag{2}$$

where $C_{kl} = C_{lk}$. If the contributions of all reactions are summed and the notation $w_k^0 = \sum_l w_{kl}^0$ is made, then

$$w_k = \sum_{l=1}^{n_k} C_{kl}\,P_l + w_k^0. \tag{3}$$

Similar equations may be also set up for the relationship between the horizontal reactions and the horizontal displacements of footings, as well as for the relationship between the reaction moments and the rotations of footings. If the vertical reactions are numbered as P_1, P_4, P_7, ... the horizontal reactions as P_2, P_5, P_8, ... and the reaction moments as P_3, P_6, P_9, ... (plane problem), all possible relationships between them have the form of Eq. (3). For C_{kl}, the more general term "compliance" should then be used.

For the determination of the compliances C_{kl} due to horizontal forces or reaction moments, soil mechanics presently gives little information [7], [21] — [25]. Approximately, the compliances for horizontal forces can be determined, assuming that the ratio between the vertical and the horizontal displacements is the same as for an elastic half-space. Thus it may be found that for square footings the C_{kl} and C_{kk} — values for a horizontal force and a horizontal displacement equal about the double of the values for a vertical force and a vertical displacement, respectively, while in a plane problem (cross-section of an infinite foundation strip) these values are approximately equal [16].
to the coupling between the vertical reaction of one square footing and the settlement in another square footing is approximately [16]: $C_{kk}\,b_k/d_{kl}$ where b_k is the distance of the characteristic point [25]
center and d_{kl} is the distance between the footings.

It is necessary to emphasize that all of the foregoing equations apply only for increasing loads. If some reactions begin to decrease, much lower values of the compliances C_{kl} (about five times smaller) must be considered.

The accuracy of the above linearization of the reaction-settlement diagrams depends on the preliminary estimate of the reaction value. To achieve more accurate results, the analysis of the whole problem discussed

later may be performed twice, introducing for the second analysis an improved linearization based on the reaction values obtained in the first analysis.

INSTANTANEOUS INTERACTION BY FORCE METHOD

The force method [26],
the complementary virtual work of the system which may be written as follows

$$\delta W^* = \int_V \underline{\sigma}^T\,\underset{\sim}{E}^{-1}\,\delta\underline{\sigma}\,\mathrm{d}V + \int_V \underline{\varepsilon}^{0T}\,\delta\underline{\sigma}\,\mathrm{d}V + \sum_k w_k\,\delta P_k \tag{4}$$

where $\underline{\sigma}$, $\underline{\varepsilon}$ = column vectors of stresses and strains in the structure; upper-script T stands for the transpose; $\underset{\sim}{E}$ = square matrix of the elastic constants defined by the elastic law $\underline{\sigma} = \underset{\sim}{E}\,(\underline{\varepsilon} - \underline{\varepsilon}^0)$, in which $\underline{\varepsilon}^0$ is the column vector of the inelastic strains (e. g. shrinkage or thermal dilatation), V = volume of the structure; $\delta\underline{\sigma}$ and δP_k = any self-equilibrating (virtual) system of stresses and reactions. All possible equilibrium states may be expressed as follows

$$\underline{\sigma} = \underline{\sigma}^{(0)} + \sum_{i=1}^{N} X_i\underline{\sigma}^{(i)}, \qquad P_k = P_k^{(0)} + \sum_{i=1}^{N} X_i\,P_k^{(i)} \tag{5}$$

where $X_1, \ldots X_N$ = chosen statically indeterminate forces; $\underline{\sigma}^{(i)}$ and $P_k^{(i)}$ = (virtual) equilibrium state of stresses and footing reactions for $X_i = 1$ and $X_j = 0$ for all $j \neq i$; $\underline{\sigma}^{(0)}$ and $P_k^{(0)}$ = stresses and reactions due to the given external loads for all X_i equal 0. The values $P_k^{(i)}$ form a matrix which is characteristic for the geometrical relationship between the chosen statically indetermin t forces and the reactions. By a standard, well-known procedure [27] can be shown that to satisfy the condition $\delta W^* = 0$ for any admissible $\delta\underline{\sigma}$ and δP_k, it is necessary and sufficient that the following system of linear algebraic equations is fulfilled

$$\sum_{j=1}^{N} (f_{ij} + f_{ij}^F)\,X_j + a_i + a_i^F = 0 \quad (i = 1, \ldots N) \tag{6}$$

Here

$$f_{ij} = \int_V \underline{\sigma}^{(i)T}\,\underset{\sim}{E}^{-1}\,\underline{\sigma}^{(j)}\,\mathrm{d}V, \tag{7a}$$

$$f_{ij}^F = \sum_k \sum_l C_{kl}\,P_k^{(i)}\,P_l^{(j)}, \tag{7b}$$

$$a_i = \int_V \underline{\sigma}^{(0)T}\underset{\sim}{E}^{-1}\underline{\sigma}^{(i)}\,\mathrm{d}V + \int_V \underline{\varepsilon}^{0T}\underline{\sigma}^{(i)}\,\mathrm{d}V + a_i^g, \tag{7c}$$

$$a_i^F = \sum_k \sum_l C_{kl}\,P_k^{(0)}\,P_l^{(i)} + \sum_k P_k^{(i)}\,w_k^0. \tag{7d}$$

In Eq. (7c) the term a_i^e, representing a possible enforced displacement in the direction of X_i has been added.

This method of approach, i.e. the force method, is best suited for framed structures, in which the stress state may be considered as uniaxial. Then the vectors $\underline{\sigma}$ and $\underline{\varepsilon}$ may be replaced by scalars σ and ε, and $\underset{\sim}{E}$ by the Young's modulus E. The flexibilities f_{ij}, a_i which are due to the structure can, of course, be actually computed as a sum of the flexibilities of the elements or parts of structure, as it is customarily done in structural analysis [27], [39]. Values f_{ij}^F represent the additional flexibilities and displacements due to the foundation. They are distinguished by upperscript F. The first term in Eq. (7c) represents the deformation in the sense of X_i, due to the applied loads, and the second term in (7c) represents the deformation in the sense of X_i, due to initial (inelastic) strains in the structure. It is well known [39] how these terms can be expressed as a sum of contributions of the various parts or elements of the structure, and therefore these details will not be described here.

It is just for the sake of brevity that Eqs. (5) and (7a-d), as well as equations (10) and (12a-d) given in the sequel, are not written in terms of the properties of the elements or structural parts.

INSTANTANEOUS INTERACTION BY DISPLACEMENT METHOD

For this method [26] it is convenient to transform Eq. (3) into the form

$$P_k = \sum_{l=1}^{n_k} A_{kl} w_l - P_k^0 \tag{8}$$

where $[A_{kl}] = [C_{kl}]^{-1}$ = matrix of the subsoil stiffnesses, $P_k^0 = \sum A_{kl} w_l^0$.

The state of deformation will be characterized by a finite number of generalized displacements $u_1, \ldots u_N$. The virtual work of the system is expressed as follows

$$\delta W = \int_V \underline{\varepsilon}^T \underset{\sim}{E} \delta \underline{\varepsilon} \, dV + \sum_k P_k \delta w_k - \sum_i F_i \, \delta u_i \tag{9}$$

where $F_1, F_2 \ldots$ are the applied loads in the sense of $F_1, F_2, \ldots$. All kinematically admissible (compatible) states of deformation have the form

$$\underline{\varepsilon} = \sum_{i=1}^{N} u_i \, \underline{\varepsilon}^{(i)}, \quad w_k = \sum_{i=1}^{N} u_i \, w_k^{(i)} \tag{10}$$

where $\underline{\varepsilon}^{(i)}$, $w_k^{(i)}$ = (assumed) state of deformation for $u_i = 1$ and $u_j = 0$ for all $j \neq i$. To satisfy the condition $\delta W \neq 0$ for any admissible δu_i, it is necessary and sufficient that the following system of linear algebraic equations hold true

$$\sum_{j=1}^{N} (k_{ij} + k_{ij}^F) \, u_i = b_i + b_i^F + F_i \qquad (i = 1, \ldots N) \tag{11}$$

in which

$$k_{ij} = \int_V \underline{\varepsilon}^{(j)T} \underset{\sim}{E} \, \underline{\varepsilon}^{(i)} \, dV, \tag{12a}$$

$$k_{ij}^F = \sum_k \sum_l A_{kl} \, w_k^{(i)} \, w_l^{(j)}, \tag{12b}$$

$$b_i = \int_V \underline{\varepsilon}^{0T} \underset{\sim}{E} \, \underline{\varepsilon}^{(i)} \, dV, \tag{12c}$$

$$b_i^F = \sum_k P_k^0 \, w_k^{(i)}. \tag{12d}$$

Here k_{ij} represents the stiffness due to the structure alone, b_i represents the force in the sense of u_i, which is equivalent to the given initial (inelastic) strains $\underline{\varepsilon}^0$. The method of expressing k_{ij} and b_i as a sum of contributions of the individual elements of parts of the structure is well known [39], [27] and therefore these details will not be depicted. The quantities k_{ij}^F represent additional stiffnesses due to the foundation, and b_i^F are the generalized forces in the sense of u_i which are equivalent to the initial settlements P_k^0.

LONG-TIME CONSOLIDATION OF THE SOIL

In a subsoil that contains clay layers, the settlement may proceed for many weeks, months or years. For a time constant reaction P_k, the resulting relationship of the linear theory of consolidation may always be written in the form

$$w_k(t) = U(t) \, C_{kk}^{\infty} P_k + C_{kk} P_k \tag{13}$$

in which t = time elapsed from load application, C_{kk}^{∞} = limiting foundation compliance for $t \to \infty$, $U(t)$ = degree of consolidation, by definition $U(0) = 0$, $U(\infty) = 1$. Function $U(t)$ depends on many characteristic of the soil, on the type of loading, on the kind of drainage at the faces of the clay layer , etc. Theory of calculation of $U(t)$ is well-known and for some typical situations the function $U(t)$ has been tabulated. For a greater number of reactions, all of which are constant in time, Eq. (13) may be generalized, in analogy with Eq. (3), as follows (for $w_k^0 = 0$)

$$w_k(t) = \sum_{l=1}^{n_k} [U_{kl}(t) \, C_{kl}^{\infty} P_l + C_{kl} P_l] \tag{14}$$

where $U_{kl}(t)$ are the degrees of consolidation which express, for $k \neq l$, the mutual effects between the footings; again $U_{kl}(0) = 0$, $U_{kl}(\infty) = 1$.

It should be noted that the nonlinear secondary consolidation due to deviatoric creep of clay is neglected in the above expressions. The present

knowledge of this phenomenon is rather limited and does not allow prediction of settlements.

The instantaneous settlement interpreted by C_{kk} and C_{kl} in Eqs. (13) and (14) expresses primarily the settlement due to other than clay layers in the subsoil. The instantaneous settlement of clay layers, which is due to the elastic deformation of clay particles and pore water, is relatively small and can usually be neglected. (The instantaneous deformation which occurs in one-dimensional consolidation tests as a result of entrapped air may not be included in C_{kk} and C_{kl}, of course).

Because of the linearity of the clay consolidation theory, the principle of superposition applies. This allows to express the response to time-variable reactions $P_k(t)$ by summing the responses to the time-increments dP_k. Thus, according to Eq. (14)

$$w_k(t) = \sum_{l=1}^{n_k} \left[C_{kl} P_l(t_0) + U_{kl}(t - t_0) C_{kl}^{\infty} P(t_0) + \right.$$
$$\left. + \int_{t_0}^{t} U_{kl}(t - \tau) C_{kl}^{\infty} \frac{dP_l(\tau)}{d\tau} d\tau \right] \tag{15}$$

where t_0 = time of the first load application, τ = integration variable. (Note that $t - \tau$ is not a multiplicator but an argument). When $w_k(t)$ is given, Eqs. (15) represent a system of linear Volterra's integral equations. For certain special forms of kernel function, namely for $U_{kl}(t) = \sum_i c_i \, e^{t/t_i}$ where c_i, t_i is constant, Eqs. (15) can be converted to a system of ordinary differential equations. But generally this is impossible. It should also be pointed out than much lower values of C_{kl}/C_{kl}^{∞} must be used, if $P_k(t)$ begins to decrease at a certain time.

CREEP OF CONCRETE

Let us now summerize briefly the formulation of creep law of concrete used in structural analysis. In the range of working stresses the creep of concrete may be assumed to be linear with respect to stress and obeying the principle of superposition in time. For constant environmental conditions, creep of concrete may be characterized by creep function $C(t,\tau)$, representing the creep strain in time τ caused by a constant unit stress applied in time t. Because of aging of concrete, $C(t,\tau)$ is not simply a function of the time lag $t - \tau$ as in the case of U. A suitable approximate expression for function $C(t,\tau)$ is defined by Eq. (30).

When the stress σ is time-variable, the principle of superposition in time may be applied. Summing the responses due to the increments $d\sigma(t)$ [14], [29], [30], [31], [41], it follows that

$$\varepsilon(t) = \sigma(t_0)/E(t_0) + \int_{t_0}^{t} I_C(t, t_0) \frac{d\sigma(\tau)}{d\tau} d\tau + \varepsilon^0(t) \tag{16}$$

where ε° = shrinkage and I_C = creep compliance

$$I_C(t,\tau) = t/E(\tau) + C(t,\tau). \tag{16a}$$

For a very young concrete, it is possible to consider that approximately

$$C(t,\tau) = (\varphi(t) - \varphi(\tau))/E_0 \tag{17}$$

where $\varphi(t) = C(t, t_0)$ monotonous increasing function of time, t_0 = time of first loading, $E = E(t_0)$. For this form of creep function, Eq. (16a) may be converted by differentiation to the followed differential equation

$$\frac{d\varepsilon}{d\varphi} = \frac{1}{E(t)} \frac{d\sigma}{d\varphi} + \frac{\sigma}{E_0} + \frac{d\varepsilon^\circ}{d\varphi} \tag{17a}$$

which is usually referred to as the rate-of-creep theory [29], [13], [31], [41] and φ is called creep coefficient.

In reinforced concrete structures the effect of reinforcement on the average creep of cross-section may be taken into account by reducing φ or C in a certain ratio [40], [41].

For the sake of simplicity, the foregoing equations were written only for uniaxial stress σ and the corresponding strain ε. Equations for the multiaxial stress are analogous [29], [31], [41].

For a detailed discussion of concrete creep the reader is referred to the survey article [31] and especially to the handbook [41].

LONG-TIME STRUCTURE SOIL INTERACTION

As far as the problem is linear, the equations for the case when both the creep in structure and the clay consolidation are present, may be obtained using the analogy of creep with elasticity [32], [31], [13], [41]. For this purpose it is useful to define the creep operator E^{*-1}, writing Eq. (16) in the form

$$\varepsilon(t) = E^{*-1}\sigma(t). \tag{18}$$

Likewise, integral operators C_{kl}^* for clay consolidation may be defined, writing Eq. (15) in the form

$$w_k(t) = \sum_{l=1}^{n_k} C_{kl}^* P_l(t), \quad (k = 1, \ldots n_k). \tag{19}$$

Equations govering both creep in the structure and consolidation in the soil can be obtained by replacing E^{-1} and C_k in the compatibility equations (6) for the corresponding elastic problem with the operators E^{*-1} and C_{kl}, respectively. (Before carrying out this replacement, expressions (8) containing E^{-1} and C_{kl} must be substituted in Eq. (6)). For the case of a uniaxial state of stress in the structure (and $w_k^0 = 0$), the following

equations, representing conditions of compatibility for the changes of deformations in X_i-directions from t_0 to t, can thus be obtained

$$\sum_j \int_{t_0}^{t} [\tilde{f}_{ij}(t,\tau) + \tilde{f}_{ij}^{F}(t-\tau)\frac{\mathrm{d}X_j(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau + \tilde{a}_i + \tilde{a}_i^{F} = 0, \tag{20}$$

where

$$\tilde{f}_{ij}(t,\tau) = \int_V \sigma^{(i)} I_C(t,\tau)\,\sigma^{(j)}\mathrm{d}V, \tag{21a}$$

$$\tilde{f}_{ij}^{F}(t-\tau) = \sum_k \sum_l C_{kl}^{\infty} P_k^{(i)} P_l^{(j)} U_{kl}(t-\tau), \tag{21b}$$

$$\tilde{a}_i = \int_{t_0}^{t} \int_V I_C(t,\tau)\,\sigma^{(i)}\,\frac{\mathrm{d}\sigma^{(0)}(\tau)}{\mathrm{d}\tau}\,\mathrm{d}V\,\mathrm{d}\tau + \tilde{a}_i^0 \tag{21c}$$

$$\tilde{a}_i^{F} = \int_{t_0}^{t} \sum_k \sum_l C_{kl}^{\infty} U_{kl}(t-\tau)\,P_l^{(i)}\frac{\mathrm{d}P^{(0)}(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau \tag{21d}$$

where $\tilde{a}_i^0$ represents deformation in the direction of X_i from t_0 to t due to shrinkage of the structure. The structure of the expressions (21a) and (21c) is analogous to Eqs. (7a) and (7c) for f_{ij} and a_i. Coefficients f_{ij} may be regarded as the elastic flexibilities of a structure having elastic moduli $1/I_C(t,\tau)$ instead of $E(t)$; $\tilde{a}_i$ may be viewed as the elastic deformations from t_0 to t in the direction of X_i due to given applied loads in time t for these values of elastic moduli. The well-known method [39] of expressing f_{ij} and a_i as a sum of contributions of the individual parts or elements of the structure could thus be directly applied to the calculation of $\tilde{f}_{ij}$ and $\tilde{a}_i$.

Often the statical system undergoes changes during construction. Eq. (20) must then be generalized. Let t_0 denote the time in which the structure is given the definite statically indeterminate system and assume that the statical system prior to time t_0 is statically determinate, which is a typical case in bridge construction. Eq. (20) must then be extended to include the changes of deformations in the direction of X_i that occur during the time interval $(t_0, t)_1$ as a consequence of the stress history preceding the time t_0. This may be done by adding at the left-hand side of Eq. (20) the following term

$$\tilde{a}_i^1 = \int_V \sigma^{(i)} \int_0^{t_0} [I_C(t,\tau) - I_C(t_0,\tau)]\frac{\mathrm{d}\sigma(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau\,\mathrm{d}V + \sum_k \sum_l C_{kl}^{\infty} P_l^{(i)} \int_0^{t_0} [U_{kl}(t-\tau) - U_{kl}(t_0-\tau)\frac{\mathrm{d}P_k(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau \tag{21e}$$

in which the changes of stresses $\sigma(\tau)$ and reactions $P_k(\tau)$ preceding the time t_0 must be introduced.

Equations (20) are valid for the general case of nonhomogenous structures exhibiting different creep properties in various parts of the structure. (This may be caused by combination of concrete and steel parts, by combination of concrets of different age, different humidity or different cross-section dimensions). In this case $E(t)$ and $I_C(t,\tau)$, occurring in Eqs. (21a) and (21c), are space-variable. However, if the structure can be assumed as homogenous and all U_{kl} as equal, equations (8) and (21) are considerably simplified because $E(t)$ and $I_C(t,\tau)$ can be brought before the volume integrals.

A steel structure supported on a consolidation clay foundation is governed by a special case of Eq. (20) for $I_C(t,\tau) = 0$.

Let us now show the special form in Eqs. (20) for the case of rate-of-creep theory. Substituting (17) into (20), and differentiating Eqs. (20) with respect to the creep coefficient φ, the following system of integro-differential equations results

$$\sum_j (f_{ij} + f_{ij}^{F})\frac{\mathrm{d}X_j}{\mathrm{d}\varphi} + \sum_j f'_{ij}X_j + \frac{\mathrm{d}}{\mathrm{d}\varphi} + \sum_j \int_{t_0}^{t} \tilde{f}_{ij}^{F}(t-\tau)\,X_j(\tau)\,\mathrm{d}\tau + \frac{\mathrm{d}a_i}{\mathrm{d}\varphi} + \frac{\mathrm{d}a_i^{F}}{\mathrm{d}\varphi} + \frac{\mathrm{d}\tilde{a}_i^{F}}{\mathrm{d}\varphi} + a'_i \tag{22}$$

in which

$$f'_{ij} = \int_V \sigma^{(i)} E_0^{-1} \sigma_{(j)}\mathrm{d}V, \tag{23a}$$

$$a'_i = \int_V \sigma^{(0)}(t)\,E_0^{-1}\sigma^{(i)}\mathrm{d}V. \tag{23b}$$

Equation (20) or (22) can be solved numerically, using the well-known methods for the Volterra's integral equations of the second kind [33], [34], [38]. The most straightforward method is based on the approximation of the hereditary integrals by finite sums; in the case of integro-differential equations (22) also the derivatives must be approximated by the finite difference expressions. In this manner equations (20) or (22) are converted to a system of algebraic equations which may be solved by a step-by-step procedure.

It would be a mere exercise in formalism to write the general form of expressions (21a), (21c) for a structure with multiaxial state of stress. (In Eq. (21a) the term $\sigma^{(i)}\underset{\sim}{E}^{-1}(\tau)\sigma^{(j)}$ would then appear, and in Eq. (21c), in addition, a square matrix of the creep functions analogous to $I_C(t,\tau)$ would have to be introduced.)

Equations for the displacement method could be derived in a similar manner. However, their use is not as advantageous as in the time-independent problems because of the difficulties connected with the determination of the inverses [38] to the operators E^{*-1} and C_{kl}^{*}.

NUMERICAL EXAMPLE

For practical verification an example of a three-span continuous bridge girder (Fig. 2) standing on a subsoil with a clay layer has been analysed, using a digital computer. As statically indeterminates, the bending moments X_1, X_2 above the intermediate supports were chosen. It was assumed that concrete in the part which is right of the center of the middle span is 120 days younger than concrete in the left part; time t was defined as the age of the left part. The structure was assumed to start acting as statically indeterminate and carrying the constant permanent load at the instant $t_0 = 180$ days.

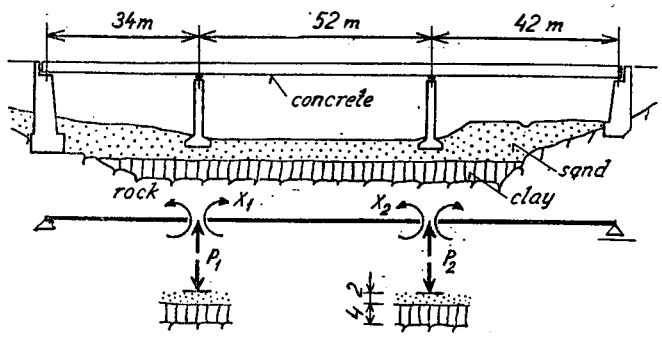

Fig. 2

Prior to time t_0 the structure rests on formwork and the stresses are considered approximately as zero, while at the same time the reactions have already non-zero, time constant values. For the parameters of Eq. (20) the following values have been determined (or assumed)

$$f_{ij} = f^l_{ij} + f^r_{ij} \text{ where } f^l_{11} = 12.8\times10^{-3},\ \delta^l_{12} = 2.1\times10^{-6},\ \delta^l_{22} = 1.1\times10^{-6},$$
$$f^r_{11} = 1.1\times10^{-6},\ f^r_{12} = 2.1\times10^{-6}, \delta^r_{22} = 14.2\times10^{-6}; \tag{24}$$

$$a_i = a^l_i + a^r_i \quad \text{where} \quad a^l_1 = 0.0108, \quad a^l_2 = 0.0092,$$
$$a^r_1 = 0.00073, \quad a^r_2 = 0.0133; \tag{25}$$

$$C_{11} = 1.1\times10^{-5}, \quad C_{22} = 0.8\times10^{-5};$$
$$C_{11\infty} = 8.6\times10^{-5}, \quad C_{22\infty} = 6.7\times10^{-5}; \tag{26}$$

$$\text{For } t \geqslant 180: \quad P^{(1)}_1 = -0.0486, \quad P^{(1)}_2 = 0.0192,$$
$$P^{(2)}_1 = P^{(1)}_2, \quad p^{(2)}_2 = -0.0430, \tag{27}$$
$$P^{(0)}_1 = 1408, \quad p^{(0)}_2 = 1552;$$

$$\text{For} \quad 120 \leqslant t < 180; \quad P^{(0)}_1 = 580, \quad P^{(0)}_2 = 610; \tag{28}$$
$$\text{For} \quad 0 \leqslant t < 120: \quad P^{(0)}_1 = 580, \quad P^{(0)}_2 = 0; \tag{29}$$

The creep function of concrete was assumed in the following form

$$C^l(t, \tau) = 0.13\left(1 + \frac{120}{\tau}\right)\ln(1+t-\tau), \quad C^r(t, \tau) = C^l(t-120, \tau-120). \tag{30}$$

As an approximation to the known diagrams of the degree of consolidation [28], it was assumed that

$$U_{11}(t) = 0.17 \ln(30 + t), \qquad U_{22}(t) = 0.16 \ln(20 + t). \tag{31}$$

All of the above values are given in days, meters (m) and megaponds ($= Mp = 1000$ force kilograms). Upperscripts l and r pertain to the left and right parts of the structure, respectively.

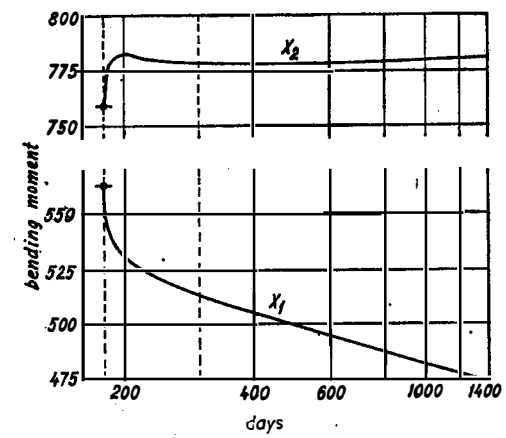

Fig. 3

The time-variation of the reactions X_1 and X_2 obtained by numerical (computer) solution of Eq. (22), are plotted in Fig. 3. A considerable change in the bending moment X_1 may be observed.

CONCLUSIONS AND SUMMARY

1. If the foundation settlements for time-constant reaction values can be determined with a sufficient accuracy, according to the approximate methods of soil mechanics, it is relatively simple to extend the conventional methods of structural analysis to the soil-structure interaction problems. The terms which have to be added to the flexibilities or stiffnesses of the structure alone, are given by Eqs. (7b, d, e) or Eqs. (12b, d). This approach is particularly suited for structures supported on isolated footings.

2. The nonlinear reaction settlement diagrams are best linearized by the tangent (not secant) compressibilities and fictitious initial settlements (Eq. 1a, b).

3. Introducing the creep operators for the structural material (concrete) and the integral operators for time consolidation of clay layers, the algebraic equations for the time-independent interaction are transformed to a system of Volterra's integral (or integro-differential) equations (Eqs. 20 or 22), which govern the interaction when a delayed response is exhibited by the structure as well as the soil. This system can be solved by well-known numerical methods.

APPENDIX

ADDITIONAL REMARKS

Some further comments, based on the previous work of the author (published in Czech), will be made.

A special case of Eqs. (22) has been derived in 1964 and applied in the design computations of a bridge similar as in Fig. 2 [12], [13]. The analysis yielded relatively small changes in the vertical reactions of piers from the time of erection to infinity, one reaction decreasing by only 2.8%, the other by 4.3%. But the increase of the maximum bending moment due to permanent load was 42% [12], [13].

For the magnitude of the changes of X_i in time, the difference between the rate of concrete creep and clay consolidation is of importance. The rate of consolidation can vary in a very wide range, depending upon the permeability and thickness of clay layer, and the drainage at its faces. For certain values of these parameters, the mean rate of consolidation can be about the same as the rate of concrete creep. Thus, for instance, if the consolidation coefficient [28] equals 2.8 ft²/month, and both the upper and the lower face of the layer are drained (by means of adjacent sand layers), the rates mentioned would be about the same if the layer thickness is 6 ft [12]. For other clays this thickness giving the same rate can vary up to 4 inches or 32 ft [12]. Usually, the slower the clay consolidation, the lower are the final stresses in the concrete structure due to the same final settlement because creep of concrete can reduce its effect. (An exception is a very slow consolidation which still proceeds when concrete is very old and creeps little.)

Some design problems of complex buildings on compressible foundations were also solved, neglecting the time variation of stresses [15], [16] and using special forms of Eqs. (6) and (11). These buildings, assembled from precast panels are shown in Figs. 4 and 5. They represent a system of shear walls with openings, interconnected by horizontal slabs. On account of the high rigidity of the panels, this system is rather sensitive to differential settlements. The primary system for solution by the force method was introduced as statically indeterminate

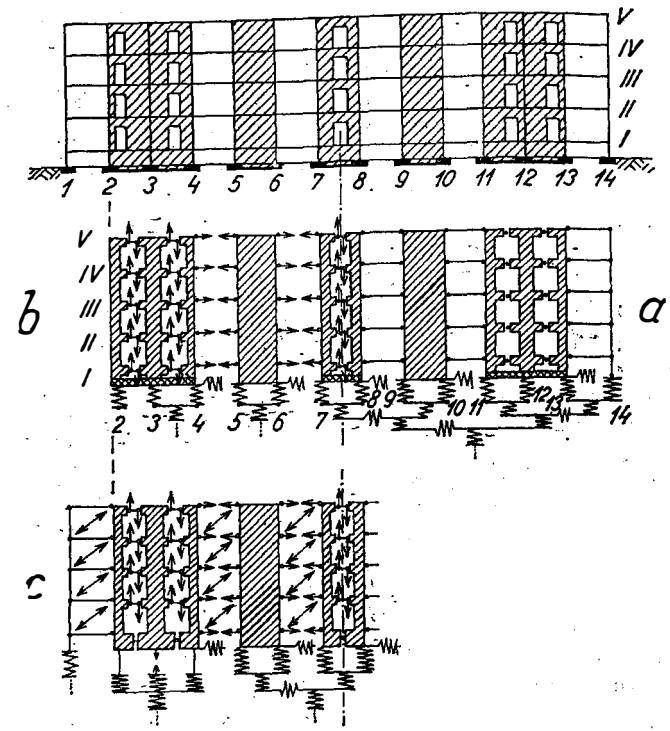

Fig. 4

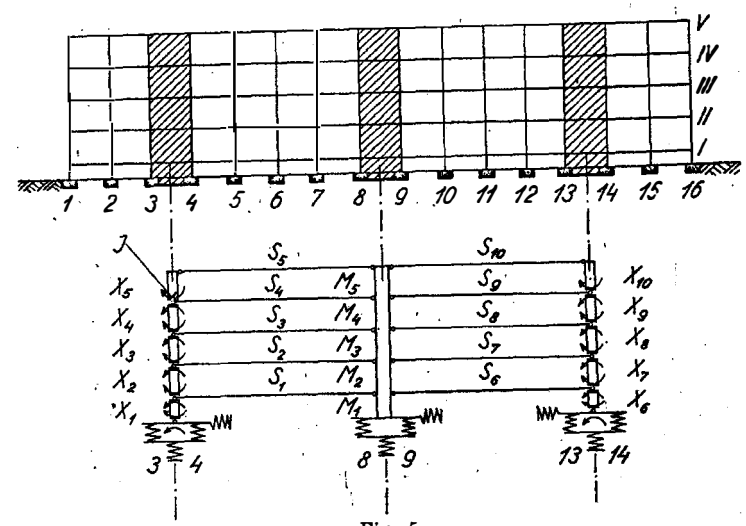

Fig. 5

according to Figs. 3b (with redundant horizontal forces in the lintels above openings). A more accurate system accounting for the small resistance of the refill panels [37] (non-bearing partition walls) against differential settlements is shown in Fig. 4c. Fig. 6 shows the displacement mode which has been used in a simplified analysis by the displacement method [16]. Some of the results of time-independent analysis for these buildings with four or eight storeys are shown in Figs. 7a, b, in which

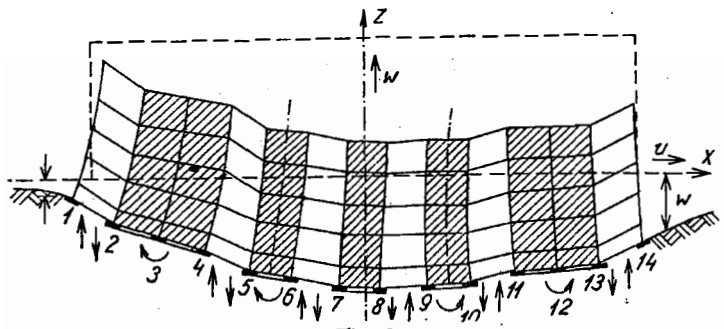

Fig. 6

the diagram of the bending moment in the shear wall for the structure on Fig. 5, and the distribution of forces in the horizontal slabs for the structure on Fig. 4 are plotted. It was found that when the height of building of this type exceeds a certain limit the effects caused by the same differential settlement are almost independent of the height. Fig. 8 shows a suitable primary system for the solution of another type of precast buildings, with longitudinal (rather than transversal) main bearing

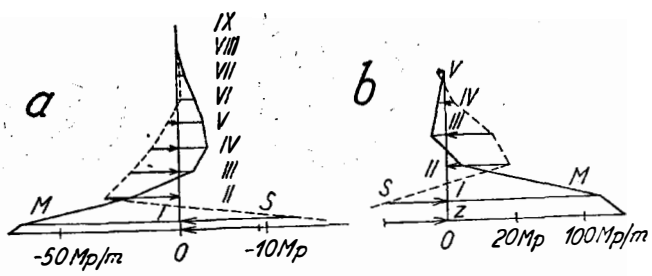

Fig. 7

walls. Another approach, which has been used to solve the design problems for this type of structures, is the semi-continuous model. The structure is regarded as a system of columns interconnected with a medium resistant to axial force, and eventually to shear (Fig. 9). This model, which is similar to the model widely used for wind loads [35], [36], gave basically the same results [16] as the analysis using models in

Figs. 3b, c. An important aspect in practical design of these buildings is the gradual construction, which may affect significantly the forces due to differential settlements. Then a separate analysis of force increments in each stage of construction has to be performed and the results superposed [15].

The analysis of buildings according to the models in Figs 4—9 appeared to be relatively simple, as compared with the more accurate methods such as the finite element method.

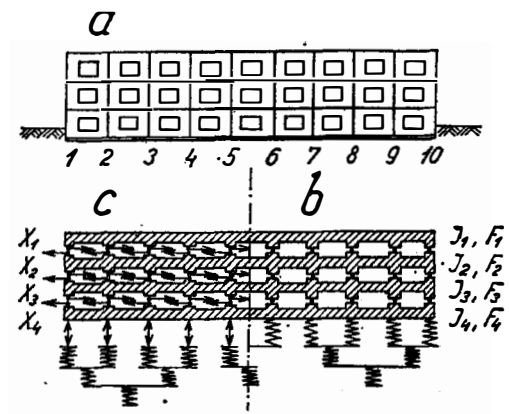

Fig. 8

In analysing complex structural systems, such as those just shown, particular attention to the correct evaluation of all stiffnesses is important. It must be realized that underestimation of the stiffness in soil

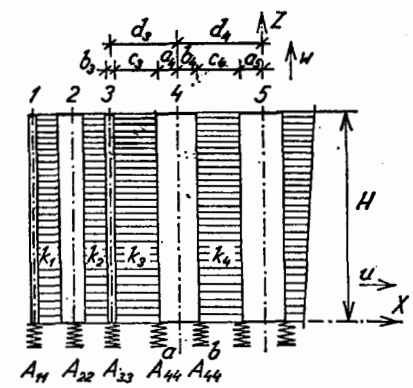

Fig. 9

interaction problems is unsafe because it results in smaller stresses (for a given differential settlement), while in the analysis of the effects of applied load (incl. wind) the designer underestimating the stiffness is usually on the safe side.

A major obstacle in practical problems is often the lack of knowledge concerning the compressibilities of soil, the input data in the analysis. As a rule their statistical dispersion is high so that it is realistic to consider in the design certain unfavorable extreme states, deduced by a probabilistic approach. For such states the most important characteristics are: the maximum average curvature of the diagram of soil compressibility along the building (Fig. 10b) and the maximum local difference between the compressibility values on a short length (Fig. 10c). (For the stability of very tall buildings, the maximum average slope of the compressibility diagram is also of importance). For the determination of these characteristics it would be useful to set up some statistically based rules, specifying e.g., the average distance between the points characterized by maximum-minimum-maximum compressibilities and the shortest distance between the points with maximum and minimum compressibilities (Fig. 11).

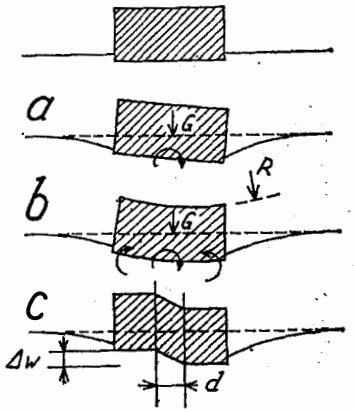

Fig. 10

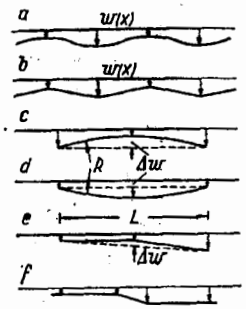

Fig. 11

BASIC NOTATIONS

a_i = displacements in the sense of X_i, due to the structure and caused by applied loads, inelastic strains and enforced displacements, for all X_j equal 0 (Eq. 7c);

a_i^F = displacement in the sense of X_i, due to foundation and caused by applied loads and w_{ki}^0, for all X_j equal 0 (Eq. 7d);

$\tilde{a}_i$ = same as a_i but due to creep in structure (Eq. 21c);

$\tilde{a}_i^F$ = same as a_i^F but due to time consolidation in soil (Eq. 21d);

a_i' = parameter analogous to $\tilde{a}_i$ defined by Eq. (23b);

b_i = force in the sense of u_i, due to inelastic strains in structure (Eq. 12c);

b_i^F = force in the sense of u_i, due to P_{ki}^0;

f_{ij}, f_{ij}^F = flexibilities due to the structure (Eq. 7a) or the foundation (Eq. 7b);

$\tilde{f}_{ij}$, $\tilde{f}_{ij}^F$ = creep flexibilities (Eq. 21a) or flexibilities associated with soil consolidation (Eq. 21b);

f_{ij}' = flexibilities defined by Eq. (23a);

k_{ij}, k_{ij}^F = stiffnesses due to the structure (Eq. 12a) or to the foundation (Eq. 12b);

t = time;

u_i = generalized displacements ($i = 1, \ldots, N$);

w_k = k — th component of settlement or displacement of footings ($k = 1, \ldots n_k$);

$w_k^{(i)}$ = settlements (footing displacements) for $u_i = 1$ and $u_j = 0$ for all $j \neq i$;

w_k^0 = fictitious initial settlements, characterizing the nonlinearity of the reaction settlement diagram (Eqs. 1, 2, Fig. 1);

A_{kl} = foundations stiffnesses = components of the inverse of matrix $[C_{kl}]$ (Eq. 8);

$C(t, \tau)$ = creep function of structural material (e.g. Eq. 16):

C_{kl} = foundation compliances (Eqs. 1—3);

C_{kl}^∞ = limiting foundation compliances for clay consolidation as $t \to \infty$(Eqs. 13, 14);

$\underset{\sim}{E}$, E = square matrix of elastic moduli of the structure, or Young's modulus;

F_i = applied load in the sense of u_i;

$I_0(t, \tau)$ = creep compliance (Eq. 16a);

P_k = k-th reaction component ($k = 1, \ldots n_k$);

$P_k^0 = \sum_l A_{kl} w_l^0$ = fictitious initial reaction due to w_k^0 (Eq. 8);

$P_k^{(i)}$ = footing reactions for $X_i = 1$, $X_j = 0$ for all $j \neq i$ and zero loads;

U_{kl}, U = degree of consolidation corresponding to C_{kl} or C_{kk} (Eq. 14);

$\dot{U}_{kl}$ = dU_{kl}/dt;

V = volume of the structure;

X_i = statically indeterminate forces ($i = 1, \ldots N$);

$\underline{\varepsilon}$, ε = column vector of strain components, or uniaxial strain;

$\underline{\varepsilon}^{(i)}$, $\varepsilon^{(i)}$ = strains for $u_i = 1$ and $u_j = 0$ for all $j \neq i$;

$\underline{\varepsilon}^0$, ε^0 = initial strains in the structures;

φ = creep coefficient (Eq. 17, 17a);

$\underline{\sigma}$, σ = column vector of stress components, or uniaxial stress;

$\underline{\sigma}^{(i)}$, $\sigma^{(i)}$ = stresses due to $X_i = 1$, $X_j = 0$ for all $j \neq i$ and zero loads;

$\underline{\sigma}^{(0)}$, $\sigma^{(0)}$ = stresses due to applied loads for all X_i zero;

τ = time as integration variable;

$\underline{\varepsilon}^T$, σ^T = transpose $\underline{\varepsilon}$, $\underline{\sigma}$;

Received September 13, 1971

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