STABILITY CONDITIONS FOR PROPAGATION OF A SYSTEM OF CRACKS IN A BRITTLE SOLID

Zdeněk P. Bažant and Hideomi Ohtsubo*
Department of Civil Engineering, Northwestern University,
Evanston, Illinois 60201

(Received 11 February 1977; accepted for print 26 August 1977.)

Introduction

In one proposed geothermal energy scheme [1,2], a large vertical main crack is produced in a hot dry rock mass by hydraulic fracture. To be able to remove heat from rock mass which is remote from the crack face, it is necessary to induce by cooling a secondary crack system normal to the wall of the main crack. Significant heat removal is possible only if the opening of secondary cracks is sufficient to allow rapid water circulation in them. The crack opening is wider, the larger is the spacing of cracks. The rate $H$ of heat removal from secondary cracks by non-turbulent water circulation is roughly proportional to $w^3/h$, where $w =$ width of cracks and $h =$ their spacing; $w$ is, in turn, proportional to $h$, and so $H \sim h^3$. Likewise, crack spacing is of importance when dealing with shrinkage cracks in reinforced concrete, for the opening of such cracks has a decisive effect on the rate of corrosion of the embedded steel reinforcement and on the shear transfer capability of aggregate interlock on rough crack surfaces. Other problems in which crack spacing is of interest include the vertical cracking of lava beds extruded and solidified at ocean floor [3], as well as cracking of mud flats and permafrost soils caused by drying [4,3].

Cooling of a homogeneous brittle elastic halfspace may be expected to produce a system of equally long parallel equidistant cracks normal to the surface. However, crack spacing is not unique according to the Griffith criterion, and also other equilibrium solutions in which the length alternates from one crack to another are possible. This suggests investigation of uniqueness and stability. It seems that stability questions have so far been considered only with regard to the propagation of a single crack and its direction of propagation [5-7] (Sih's criterion of maximum strain energy density, criterion of maximum energy release rate). This paper attempts to lay down foundations of stability analysis of a system of cracks for each of which the propagation direction is known. This problem is much less difficult than the problem of crack direction.

*On leave from Department of Naval Architecture, University of Tokyo, Bunkyo-Ku, Tokyo, Japan

Scientific Communication
Consider first the general case of a brittle elastic body which contains a number of cracks of arbitrary shape (Fig. 1a). For the sake of simplicity, assume that the body is in a state of either plane strain or plane stress, and that propagation of the cracks is governed only by Mode I (opening mode) stress intensity factors, $K_i$ [6], where subscript $i$ refers to the $i$th crack tip, $i = 1, 2, \ldots N$. Also, assume that the cracks do not branch and that they propagate in given directions along straight or curved trajectories. Let $a_i$ denote the length of crack up to its tip (Fig. 1a).

It is well known that the condition of stability of equilibrium of a single critical crack of length $a_i$ is

$$\frac{\partial K_i}{\partial a_i} < 0 \quad (i = 1, 2, \ldots N) \quad (1)$$

This holds for a general elastic body, and because a body with many cracks of which only one extends is a special case of a general elastic body with one extending crack, this condition also represents a necessary condition of stability of a crack system. It is not at all clear, however, whether Eq. (1) represents a sufficient condition, i.e., whether there are other conditions that have to be satisfied to assure stability.

To investigate equilibrium and stability, it is necessary to consider the work, $W$ (more precisely, Helmholtz free energy), that would have to be supplied to the body in order to extend the cracks;

$$W = U(a_1, a_2, \ldots, a_N; D) + \sum_{i=1}^{N} \int_0^{a_i} 2\gamma_i \, da_i \quad (2)$$

Here $U =$ elastic strain energy of the body, $2\gamma_i =$ specific energy of extension of the $i$th crack; and $D =$ loading parameter. In particular, $D$ will represent here the penetration depth of cooling. If yielding and microcracking near the advancing crack tip were absent and the crack surfaces were not rough but perfectly plane, $\gamma_i$ would equal the surface energy of the material. But these effects are always present and often they dissipate much energy; then $\gamma_i$ is a constant which is much higher than the surface energy.

Consider now that the crack tips number $i = 1, \ldots m$ extend ($\delta a_i > 0$), the cracks numbered $m+1, \ldots n$ close and shorten ($\delta a_j < 0$), and the crack tips numbered $n+1, \ldots N$ remain stationary ($\delta a_j = 0$); $0 \leq m \leq n \leq N$. This includes the case $m = n$ when no crack closes, and the case $n = N$ when no crack remains
immobile. The work, $\Delta W$, that would have to be supplied in order to change the crack lengths by $\delta a_i$ (at constant loading parameter $D$ and for applied loads doing no work) is a function of $\delta a_i$. This function must admit Taylor series expansion, i.e.,

$$\Delta W = \delta W + \delta^2 W + \ldots; \delta W = \sum_{i=1}^{m} \left( \frac{\partial U}{\partial a_i} + 2\gamma_i \right) \delta a_i + \sum_{j=m+1}^{n} \frac{\partial U}{\partial a_j} \delta a_j \quad (3a)$$

$$\delta^2 W = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 U}{\partial a_i \partial a_j} \delta a_i \delta a_j + \sum_{i=1}^{m} \frac{\partial \gamma_i}{\partial a_i} (\delta a_i)^2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} W_{ij} \delta a_i \delta a_j \quad (3b)$$

in which $\delta W$ and $\delta^2 W$ are the first and second variations; and

$$W_{ij} = W_{ji} = \frac{\partial^2 U}{\partial a_i \partial a_j} + 2 \frac{\partial \gamma_i}{\partial a_i} \delta_{ij} H(\delta a_i) \quad (4)$$

where $\delta_{ij} = $ Kronecker delta and $H = $ Heaviside step function, i.e., $H(\delta a_i) = 1$ when $\delta a_i > 0$ and $H(\delta a_i) = 0$ when $\delta a_i < 0$. Usually the fracture properties of the body can be considered homogeneous, and then $\partial \gamma_i / \partial a_i = 0$.

For the cracks to change their length in an equilibrium manner, $\delta W$ must vanish for any $\delta a_i$. It is necessary to distinguish whether a crack extends ($\delta a_i > 0$) or closes ($\delta a_i < 0$). According to Eq. (2), $\delta W = 0$ occurs if, and only if

$$\text{for } \delta a_i > 0: \quad \frac{\partial U}{\partial a_i} = 2\gamma_i; \quad \text{for } \delta a_i < 0: \quad \frac{\partial U}{\partial a_i} = 0. \quad (5)$$

Eq. (5) includes the well-known Griffith fracture criterion. Note that the strain energy release rate is $-\partial U / \partial a_i$.

An equivalent form of Eq. (5) can be given in terms of the stress intensity factor, $K_i = \lim (c/\sqrt{2\pi r})$ for $r \to 0$ where $r = $ distance from the crack tip and $\sigma = $ transverse normal stress straight ahead of the crack. It is well known [6] that for plane strain $\partial U / \partial a_i = -K_i^2 / E'$ with $E' = E/(1-\nu^2)$ where $E = $ Young's modulus, $\nu = $ Poisson ratio. Thus, Eq. (5) is equivalent to

$$\text{for } \delta a_i > 0: \quad K_i = K_c; \quad \text{for } \delta a_i < 0: \quad K_i = 0 \quad (6)$$

where $K_c_i = (2\gamma_i E')^{1/2} = $ critical value of the stress intensity factor = fracture toughness of the material. Using $\partial U / \partial a_i = -K_i^2 / E'$, one may write

$$\frac{E'}{2} \frac{\partial^2 U}{\partial a_i \partial a_j} = -K_i \frac{\partial K_i}{\partial a_i} = -K_j \frac{\partial K_i}{\partial a_j} \quad (7)$$

Having stated the conditions of equilibrium, it is natural to ask whether the equilibrium configuration is stable. The crack system is said to be
FIG. 1 General and Special Crack Systems Investigated (a–e), Some Numerical Results (f,g) with Grid Used (h; 2h = 1m), and Bifurcation of Equilibrium Path (i,j)
stable if and only if no \( a_i \) can change without changing the loads (or \( D \)).

Thus, stability is ensured if and only if the work \( \Delta W \) that is done at any admissible crack length increments \( \delta a_i \) is positive, for if this work is not done \( \delta a_i \) cannot occur. On the other hand, if \( \Delta W < 0 \) for some \( \delta a_i \), energy is released, and when a release of energy is possible, changes \( \delta a_i \) will occur spontaneously, \( \Delta W \) being transformed into kinetic energy and ultimately dissipated as heat (which follows from the second law of the thermodynamics).

One well-known unstable situation arises when \( K_i > K_{ci} \) (or \( -\delta U/\delta a_i > 2\nu_i \)) for \( \delta a_i > 0 \). Indeed, \( \delta W < 0 \) for \( \delta a_i > 0 \), and so \( K_i > K_{ci} \) is impossible. Similarly, the case \( K_i < 0 \) (or \( -\delta U/\delta a_i < 0 \)) for \( \delta a_i < 0 \) is also unstable. Therefore, with regard to non-negativeness of \( \delta W \), it is necessary that \( 0 \leq -\delta U/\delta a_i \leq 2\nu_i \) or \( 0 \leq K_i \leq K_{ci} \) at all times. Combining the foregoing conditions and Eq. (6), it follows that with regard to the first variation, \( \delta W \), only the following variations \( \delta a_i \) are admissible:

\[
\begin{align*}
&\text{for } K_i = K_{ci}: \delta a_i \geq 0; \text{ for } K_i = 0: \delta a_i \leq 0 \\
&\text{for } 0 < K_i < K_{ci}: \delta a_i = 0
\end{align*}
\]

(8a)

(8b)

If \( \delta W = 0 \), stability will be ensured if

\[
2\delta^2 W = \sum_{i=1}^{n} \sum_{j=1}^{n} W_{ij} \delta a_i \delta a_j > 0 \text{ for any admissible } \delta a_i. 
\]

(9)

Conversely, instability occurs if \( \delta^2 W < 0 \) for some admissible choice of \( \delta a_i \). The admissible increments \( \delta a_i \) are given by Eq. (8). If matrix \( W_{ij} \) is positive definite, stability is assured. However, if \( W_{ij} \) is not positive definite, the crack configuration may or may not be unstable. It is unstable if \( \delta^2 W < 0 \) at \( K_i = K_{ci} \) or \( K_i = 0 \) for some admissible \( \delta a_i \). Critical state occurs when \( \delta^2 W = 0 \) at \( K_i = K_{ci} \) or \( K_i = 0 \) for some admissible \( \delta a_i \).

Array of Parallel Cooling Cracks Penetrating a Halfspace

Consider now a homogeneous isotropic elastic halfspace which is initially (at time \( t = 0 \)) at constant temperature, \( T = T_0 \), and is then cooled at the surface \( x = 0 \) to temperature \( T_s \). This produces an array of straight parallel equidistant cracks normal to the surface (Fig. 1 b-d). The temperature field is assumed to have the form \( T-T_0 = f(x/D) \left( T_s-T_0 \right) \) where \( D = D(t) = \text{penetration depth of cooling. If all heat is transferred by conduction through the solid,} \)
one has \( f(\xi) = \text{erfc} \frac{\xi}{\sqrt{2}} = \frac{2}{\sqrt{\pi}} \int_{\xi}^{\infty} \exp(-\eta^2) \, d\eta \), \( \xi = \sqrt{2T} \alpha x/D \), \( D = \sqrt{12\pi c t} \), \( c \) = heat diffusivity. Because \( T \) is constant along lines parallel to the surface, it is logical to assume a periodic pattern of crack lengths. Accordingly, consider that every other crack has one length, \( a_2 \), and the cracks inbetween have another length, \( a_1 \); \( a_2 \geq a_1 \) (Fig. 1c). Cracks of equal lengths \( (a_2 = a_1) \) represent one possible equilibrium state. These states (not all necessarily stable) are plotted in Fig. 1f on the basis of finite element calculations for \( T_o - T_s = 100^\circ C \) (with error function \( T \)-profile), \( \alpha = 8 \times 10^{-6} \) per \( ^\circ C \) (linear thermal expansion coefficient), \( E = 37600 \) MN/m\(^2\), and \( \nu = 0.305 \) (2h=1m).

Various interesting properties of the parallel crack system can be analyzed even without numerical solutions of \( K_i \) and \( W_{ij} \). Since \( K_c \) is a constant, a sufficient condition of stability is the positive definiteness of the matrix \( W_{ij} = \delta^2 U/\delta a_i \delta a_j \), which requires that

\[
W_{22} = W_{11} > 0 \quad \text{and} \quad \begin{vmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{vmatrix} > 0 \quad (10)
\]

A critical state, corresponding to a bifurcation point on the basic equilibrium path \( a_1 = a_2 \), would arise if \( \delta^2 W = \frac{1}{2} \sum_j \sum_k W_{ij} \delta a_j \delta a_k = 0 \) for some admissible \( \delta a_i \). This condition is satisfied if \( \sum_j W_{ij} \delta a_j = 0 \) or

\[
\begin{align*}
W_{11} \delta a_1 + W_{12} \delta a_2 &= 0 \\
W_{21} \delta a_1 + W_{22} \delta a_2 &= 0
\end{align*} \quad (11)
\]

in which \( W_{11} = W_{22} \) and \( W_{12} = W_{21} \). Setting the determinant zero, one has

\[
W_{11} \ W_{22} - W_{12}^2 = 0,
\]

and using Eq. (7) this becomes \((-K_2 \delta K_2/\delta a_2)^2 - (-K_2 \delta K_2/\delta a_1)^2 = 0\). Noting that \( W_{11} = W_{22} \) and \( K_2 = K_1 \) for \( a_2 = a_1 \), one has \( W_{22} - W_{12} = 0 \), yielding \((\delta K_2/\delta a_2)^2 - (\delta K_2/\delta a_1)^2 = 0\) as a condition of possible critical state. Admissibility of the corresponding eigenvector \((\delta a_1, \delta a_2)\) must be checked, though. Since \( W_{11} = W_{22}, W_{12} = W_{21}, \) and at the same time \( W_{22} = \pm W_{12} \), Eq. 11 suggests

\[
\delta a_1/\delta a_2 = \pm 1 \quad (12)
\]

as possible critical states.

The plus sign in Eq. (12) yields \( \delta a_1 = \delta a_2 \). In this case Eq. (11) reduces to \((W_{11} + W_{12}) \delta a = 0\) where \( \delta a = \delta a_1 = \delta a_2 \). Noting that, by virtue of the chain rule of differentiation, \( W_{11} + W_{12} = (\partial W_1/\partial a_1) \delta a_1/\delta a + (\partial W_1/\partial a_2) \delta a_2 \)
/\alpha = \partial W_1/\partial \alpha with W_1 = \partial W/\partial \alpha = -K_1^2/E', one concludes that Eq. (11) degen-
rates into the condition (\partial K_1^2/\partial \alpha) \delta \alpha = 0 or \partial K_1/\partial \alpha = 0, which is a condi-
tion of instability of the basic equilibrium path, a_1 = a_2. The condition
of stability of this path is \partial K_1/\partial \alpha < 0 or \partial W_1/\partial \alpha > 0, which is analogous to
the well-known stability condition for a single crack (Eq. 1).

Bifurcation of the equilibrium path would be obtained if Eq. (12) admitted
the minus sign, i.e., \delta a_1 = -\delta a_2. For Eq. (11) to allow this, W_{11} and W_{12}, and
thus also \partial K_2/\partial a_2 and \partial K_2/\partial a_1, would have to be of the same sign. Of these,
\partial K_2/\partial a_2 must be negative, or else a critical state of another type, associated
with the first condition in Eq. (10) would precede this bifurcation. As far
as \partial K_2/\partial a_1 is concerned, the finite element calculations described in the
sequel indicated that for the present crack system with a_1 = a_2 it is always
negative, which also agrees with some intuitive considerations.

Since in the present case both \partial K_2/\partial a_2 and \partial K_2/\partial a_1 are negative, it ap-
ppears that Eq. (11) would indeed admit the minus sign. However, this means
that either \delta a_1 or \delta a_2 must be negative, and this violates condition (8)
because K_1 = K_2 = K_c. Hence, a bifurcation of the type given by Eqs. (11)
and (12) is seen to be impossible.

The remaining possible critical state according to Eq. (10) is given by
the condition W_{22} = W_{11} = 0 or

\[ \left[ \partial K_2/\partial a_2 \right]_{a_1} = \text{const.} = 0. \]  

(13)

The associated second variation is \delta^2 W = \frac{1}{2} W_{22} (\delta a_2)^2 with \delta a_1 = 0, and the
bifurcation ("instability") mode is

\[ \delta a_2 > 0, \delta a_1 = 0. \]  

(14)

This mode also represents a bifurcation point on the basic equilibrium path
a_1 = a_2 (Fig. 11). According to Eq. (8b), K_1 does not have to remain equal to
K_c but it may decrease, i.e., the tip of the crack a_1 may be unloading. In
fact K_1 ought to decrease after bifurcation since extension of crack a_2
should have a non zero effect on K_1. It might be also of interest to note
that if \partial K_2/\partial a_2 > 0 (or W_{11} = W_{22} < 0), then W_{11} W_{22} - W_{12}^2 or det (W_{ij}) is
always negative.

Further light may be shed on the problem if the path of equilibrium
states is regarded as a function of parameter D (cooling penetration depth);
Denoting \( W_i = \frac{\partial W}{\partial a_i} \) where \( W \) is given by Eq. (2), the equilibrium path is distinguished by the conditions \( W_i = 0 \) (\( i = 1,2 \)). Derivatives \( W_i \) at equilibrium states are functions of \( a_1 \) and \( a_2 \). However, by contrast to \( W \) in Eq. (2), \( a_i \) are generally not independent of \( D \) because along the equilibrium path the crack lengths \( a_1 \) and \( a_2 \) depend on \( D \). Thus, \( W_i \) along the equilibrium path are implicit functions of \( D \); i.e.,

\[
\left[ \frac{\partial W}{\partial a_i} \right]_{D=\text{const.}} = W_i [a_1(D), a_2(D)] = 0 \quad (i = 1,2) \quad (15)
\]

Assume that on the basic equilibrium path \( a_2 = a_1 \) there is a critical point (bifurcation point) corresponding to \( D = D_0 \) (Fig. 1j). Functions \( W_i \) ought to admit Taylor series expansions at \( D = D_0 \). The cracks should also be in equilibrium at adjacent states with \( D \) sufficiently close to \( D_0 \). If both \( \delta a_1 \) and \( \delta a_2 \) are assumed to be positive, then \( W_i \) would have to be constant for all such \( D \)-values. Consequently, \( dW_i/dD = 0, d^2W_i/dD^2 = 0 \), etc., must be true at \( D = D_0 \). This yields

\[
\sum_{j=1}^{2} \left[ \frac{\partial W_i}{\partial a_j} \right]_{D_0} a_j' = 0 \quad (i = 1,2) \quad (16)
\]

\[
\sum_{j=1}^{2} \sum_{k=1}^{2} \left[ \frac{\partial^2 W_i}{\partial a_j \partial a_k} \right]_{D_0} a_j' a_k' + \sum_{j=1}^{2} \left[ \frac{\partial W_i}{\partial a_j} \right]_{D_0} a_j'' = 0 \quad (i = 1,2) \quad (17)
\]

where \( a_j' = da_j/dD \) along the equilibrium path at \( D = D_0 \). These equations represent conditions of continuing equilibrium, analogous to those which follow from the perturbation method of structural stability theory [8]. If they admit solution for \( D \to \text{const.} \), then a critical state is reached. Setting \( a_i' \sim \delta a_i \), the condition in Eq. (16) is obviously identical to Eq. (11), which is a bifurcation of a type that is inadmissible. However, there exists no reason why a higher-order bifurcation governed by Eq. (17) could not take place. Since Eq. (16) is not satisfied, such higher-order bifurcation would have to conform to \( a_i' = a_2' \), i.e., to the increments for the basic path \( (\delta a_1 = \delta a_2) \) and would have to take place at increasing \( D \) (i.e., at increasing cooling penetration depth). Therefore \( a_1' \) and \( a_2' \) in Eq. (17) must be equal, and a higher-order bifurcation, with the secondary path being tangent to the basic path \( (a_1 = a_2) \) at the bifurcation point, would occur if Eq. (17) admitted a solution with \( a_1'' \neq a_2'' \).

Eqs. (16) and (17) were written under the assumption that both \( \delta a_1 \) and \( \delta a_2 \) are positive (and \( K_1 = K_2 = K_c \)). Consider now that \( \delta a_2 > 0 \) and \( \delta a_1 = 0 \),
i.e., one crack stops growing. In this case Eqs. (16) - (17) still represent a possible condition of critical state, but in view of Eq. (8b) it is also possible that only $K_2$ remains at its critical value $K_c$ while $K_1$ decreases below $K_c$. In fact, since bifurcation given by Eq. (16) was shown to be inadmissible (according to Eq. 8), it is not possible that $K_1$ and $K_2$ remain equal $K_c$ during bifurcation. Hence, it is necessary that

$$\frac{\partial K_1}{\partial a_2} < 0$$

(18)

during bifurcation. Thus $\partial W_1/\partial D$ cannot be zero at $D = D_0$, and only the condition $\partial W_2/\partial D = 0$ applies as a condition of continuing equilibrium, yielding

$$\frac{\partial W_2}{\partial a_2} a_2' = 0$$

(19)

where $a_2' = da_2/dD$. Eq. (19) represents the condition of a critical state, provided that it holds true in the limit for $D \to \text{const}$. This case is identical with Eq. (13) derived from the condition $\delta^2 \bar{w} > 0$.

It is seen that the present variational stability analysis yields the condition in Eq. 13. This condition is identical to the elementary condition in Eq. 1, which is immediately obvious even without the variational analysis. It remains to be seen whether, for the particular crack system at hand, $\partial K_2/\partial a_2$ can indeed change its sign. Therefore, some finite element computations have been carried out. The grid in Fig. 1f, composed of four-node quadrilateral elements, formed by condensing a block of four constant-strain triangles, was used. The derivatives of potential energy ($\partial W/\partial a_1$, $\partial^2 W/\partial a_1 \partial a_2$, etc.) were calculated from their finite difference approximations, using the potential energy (Eq. 2) in the whole grid for various crack length $a_1$ and $a_2$. The temperature profile was approximated as parabolic, and the Young's modulus $E = 37,600$ MN/m$^2$, the Poisson ratio $\nu = 0.305$ and the linear thermal expansion coefficient $\alpha = 8 \times 10^{-6}$ per $^\circ\text{C}$ (all typical of granite) were used. Some of the results are shown in Fig. 1g, in which the intersections of curves $K_1$ and $K_2$ represent equilibrium states if $K_c = 22.8$ MNm$^{-3/2}$. In one of these intersections the slope of the curve of $K_2$ versus $a_2$ is positive, which violates Eq. 13 and indicates that the equilibrium is unstable. This proves that instability due to the violation of Eq. 13 is indeed possible. However, it is not at all clear from Fig. 1g that the instability governed by Eq. 13
must have a horizontal tangent at the bifurcation point; see (Fig. 1). Assuming that the trend remains unchanged, the segment before the bifurcation given by Eq. 13 and that in, therefore, controlled. On the other hand, S. Nemat-Nasser intuitively expected the elementary condition in Eq. 1 (or Eq. 13) to control. Following the present finite element calculations which proved that the bifurcation due to $\frac{\partial K_2}{\partial a_2}$ is possible, L. M. Keer et al. demonstrated by a complete analytical solution of the problem based on singular integral equations that the bifurcation due to $\frac{\partial K_2}{\partial a_2}$ turning zero is not merely a possibility but a phenomenon which does actually occur. Simultaneously, refined finite element calculations were being performed by the authors together with K. Aoh of University of Tokyo (to be reported separately) and these calculations led to the same conclusion. These calculations and the work of Keer et al. also indicated that (for granite and for a temperature drop of 100°C) the bifurcation is reached when $a_1$ and $a_2$ are roughly equal 1.8 times crack spacing.

Since $K_1$ was shown to decrease after bifurcation, the equilibrium path must have a straight horizontal segment of finite length after the bifurcation point; see (Fig. 1). Assuming that the trend remains unchanged, the segment would end by a state in which $K_1 = 0$, $K_2 = K_c$, and subsequently crack $a_1$ would begin to close, $\delta a_1 < 0$. The fact that the bifurcation for $\frac{\partial K_2}{\partial a_2} = 0$ occurs at constant $a_1$ means that in the plot of $D$ versus $a_2$ the equilibrium path must have a horizontal tangent at the bifurcation point; see (Fig. 1). If the path continued as a straight horizontal line beyond the bifurcation point, there would be infinitely many equilibrium crack lengths $a_2$ corresponding to the same $D$ and same $a_1$, and this would require the potential energy release rate to be independent of $D$. Obviously, this is impossible. Hence, the path of $D$ versus $a_2$ after the bifurcation point (Fig. 1) must curve either upward or downward. If it curved downward, it would mean that a longer crack $a_2$ corresponded to a smaller cooling penetration depth $D$ (at constant $a_1$), i.e.,

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equilibrium extension of cracks $a_2$ would require withdrawal rather than supply of energy. Therefore, if adjacent equilibrium states exist after the bifurcation point, their path in Fig. 1j must curve upward, i.e., with increasing penetration depth of cooling the leading cracks must get longer, not shorter, as may naturally be expected. It must be emphasized, however, that the shape of the post-bifurcation paths in Figs. 1i and j has been deduced here only qualitatively. Prior to formulating this qualitative deduction, the post-bifurcation paths of the type shown in Figs. 1i and j were obtained quantitatively by Keer et al.² by means of a singular integral equation approach.

The possibility that every other crack ($a_1$) might close is suggested by empirical observations of drying shrinkage cracks, e.g., in mud flats or in concrete. This was also suggested by the behavior of cracks in an experiment at Los Alamos Scientific Laboratory [9] in which a concrete slab was cooled by liquid nitrogen and hexagonal crack patterns at the surface were made easily observable by formation of nitrogen bubbles on evaporation from open cracks. The possibility of crack closing was also evidenced at the outset of the finite element work by the fact that for a sufficiently large value of $a_2/2h$ the normal stress $\sigma_y$ along the line of symmetry between two adjacent cracks ($a_1$ and $a_2$) became compressive up to a certain depth from the surface. This showed that on this line of symmetry it is possible to introduce a closed crack up to a certain depth without causing any change of the stress state in the entire elastic half-space. It follows that shorter closed cracks may exist between opened leading cracks and this suggests that every other crack (cracks $a_1$) may close after bifurcation. However, it does not follow theoretically that every other crack must close. Keer et al.² demonstrated by analytical solution of the problem that cracks $a_1$ must indeed close after bifurcation.³ So, it is certain that the spacing of the opened (leading) cracks doubles whenever the ratio of the opened cracks to their depth reaches a certain fixed value (about 1.8). This type of behavior, which has been suggested before on the basis of empirical observations [3, 4], is favorable for the afore-mentioned scheme for extracting geothermal heat, because it would mean that the width of the opened cracks is proportional to the penetration depth

²Ibid.
³The fact of closing is distinguished from the fact that $K_1$ must decrease after bifurcation, which is here established by Eq. (18).
of cooling and that the flux of water through the cracks is proportional to
the square of the crack depth. These crude projections may, however, be greatly
modified when the effect of water circulation on the temperature profile and
possible development of eddy currents in the cracks is taken into account.

Conclusions

1. A system of cracks is stable if and only if the work $\Delta W$ needed to produce
any admissible crack length increments is positive definite. This is
assured if the second variation $\delta^2 W$ of $W$ is positive definite for all
admissible crack length increments.

2. A system of identical parallel equidistant cooling cracks propagating
into a halfspace can exhibit instability. The critical state is indi­
cated by the vanishing of the derivative of the stress intensity factor
of cracks $a_2$ with regard to $a_2$ at constant $a_1$, which is the same as the
well-known stability condition for a single crack considered separately.
At the critical state every other crack, of length $a_1$, ceases to grow
($\delta a_1 = 0$) while the intermediate cracks of length $a_2 = a_1$ continue to
advance ($\delta a_2 > 0$) at constant temperature. The path of the equilibrium
states plotted in the space $(a_1, a_2)$ or in the space $(a_1, D)$ then
bifurcates ($D =$ penetration depth of cooling). After bifurcation, cracks
$a_1$ gradually close. The plot of $a_1$ versus $D$ has a horizontal tangent at
bifurcation point.

3. Without numerical results, it cannot be ruled out that a higher-order
bifurcation, in which the bifurcating path and the main path $a_1 = a_2$ have
a common tangent, might be also possible for a system of parallel cracks.

4. Vanishing of the determinant of the second derivatives of work $W$ with
respect to $a_1$ and $a_2$ does not cause bifurcation in a system of parallel
cracks because associated eigenvector $(\delta a_1, \delta a_2)$ indicates negative $\delta a_1$.

Remark. - Equilibrium path bifurcation is characteristic of a perfect
crack system. An imperfect crack system, e.g., a system of cracks which are
almost but not exactly equidistant, would probably not exhibit bifurcation of
equilibrium path, just like an imperfect column does not. However, such a
case would be much more difficult to solve.
Acknowledgment

Support of the U. S. National Science Foundation under Grants AER 75-00187 and ENG 75-14848 to Northwestern University is gratefully acknowledged. The authors are grateful to Drs. S. Nemat-Nasser, L. M. Keer and K. S. Parihar for their many helpful comments and suggestions during the long period of gestation of this paper. The first author first communicated to them the present general variational analysis of stability early in 1976, and benefited substantially from the subsequent discussion with them. This led to the initial draft of this paper in July 1976, which contained the present conditions of stability and numerical results. Discussion of this draft with these colleagues greatly assisted in the preparation of the final manuscript.

References


Appendix

The fact that in a system of parallel cracks the bifurcation associated with the determinant condition in Eq. 10 is impossible is contingent upon (a) \( \partial K_2 / \partial a_1 \) being negative, and (b) both cracks being at the point of extension. Prior to completing the finite element calculations which confirmed that \( \partial K_2 / \partial a_1 \) is always negative, S. Nemat-Nasser intuitively suggested to the authors that it should always be so. Later it was thought that "it is generally true that an extension of a given crack accompanied by no increase in applied loads would result in a decrease of the stress intensity factor at other active cracks, because such an extension decreases the overall stiffness of the elastic body."\(^4\) Subsequently, however, an example of a cracked structure for which \( \partial K_2 / \partial a_1 \) is positive has been found; hence the sign of \( \partial K_2 / \partial a_1 \) is not certain in advance, for the general case.

To show it, consider a horizontal simply supported continuous beam of constant cross section (of depth H) and two equal spans (of length L), loaded

\(^4\) L.M. Keer et al., loc. cit.
in the middle of the left span by downward load $P_1$ and in the middle of the right span by equal but upward load $P_2 = -P_1$. The bending moments in the middles of the left and right spans are $M_1 = P_1 L/4$ and $M_2 = -M_1$. Assume further that there are two vertical cracks, one reaching upward from the bottom of the cross section in the middle of the left span to depth $a_1$ from the bottom, and the second reaching downward from the top of the cross section in the middle of the right span to depth $a_2 = a_1$ from top. Assume also that $K_1 = K_2 = K_c$. Let now the crack length $a_1$ be increased by $\delta a_1$ while keeping $a_2$ and the loads constant. Increase of $a_1$ will cause the left span to become less stiff, and it will cause the left span to deflect downward. In a continuous beam, this must cause the right span to deflect upward, and because $a_2$ is constant, $K_2$ must increase, i.e. $\partial K_2 / \partial a_1 > 0$. Alternatively, this may be also deduced by noting that the decrease of left span stiffness must cause the bending moments to redistribute so that $M_1$ would decrease and $|M_2|$ would increase; an increase of $|M_2|$ at constant $a_2$ must cause $K_2$ to increase.

Likewise, condition (b), namely that both cracks are on the verge of extension, does not have to always occur. Consider the same beam but with different loads $P_1$ and $P_2$ and with both cracks of lengths $a_1 = a_2$ emanating from the bottom of the cross section. Assume now that the beam is slender, so that $K_1$ and $K_2$ are proportional to $M_1$ and $M_2$, and that $a_1 \ll H$, $a_2 \ll H$, so that crack lengths have negligible effect on the stiffness of the spans. First, let loads $P_1 = P_2 = 1.0$ be applied. This causes equal moments, $M_1 = M_2 = L/4$, and assume that this creates equally long cracks $a_1 = a_2$ which are both critical, $K_1 = K_2 = K_c$. Subsequently, load $P_2$ is changed to $P_2 = 1.3$ and load $P_1$ is changed to $P_1 = 0.3$. This causes $M_1$ to become zero while $M_2$ remains unchanged ($M_2 = L/4$). So, $P_2 = 1.3$ and $P_1 = 0.3$ gives a state where $K_2 = K_c$ and $K_1 = 0$, crack $a_1$ being on the verge of extension and crack $a_2$ being on the verge of closing. For checking stability on this state, one must obviously consider $\delta a_2 \geq 0$ and $\delta a_1 \leq 0$ as the admissible $\delta a_i$.

In cases where condition (a) or (b) is reversed, the stability condition $\det (W_{ij}) > 0$ cannot be dismissed a priori and must be evaluated to see whether or not it is satisfied for all admissible $\delta a_i$. 