INTRODUCTION

Finite element methods make structural analysis with sophisticated constitutive relations feasible, and to obtain realistic results their use is in fact necessary. Basically, two types of constitutive relations may be distinguished: (1) Those where the relationship between stress and strain increments is linear; and (2) those where it is nonlinear (7). The former type includes the total strain theory (deformation theory) as well as hypoelasticity and incremental plasticity. The latter type includes the endochronic theory. The best available representatives of incrementally linear plastic models are those of William, et al. (1,2,41) and of Chen, et al. (16). Among hypoelastic models we can name that of Bathe and Ramaswamy (5) and among total-strain models those of Kupfer and Gerstle (28) and Cedolin, Crutzen, and dei Poli (15). All these models are simpler than the recent endochronic models (7-12). The latter ones, however, allow a much more realistic and comprehensive representation of various aspects of inelastic behavior; in particular, they give much better fits of the group of uniaxial, biaxial, and triaxial tests, provide failure conditions and strain-softening branches as part of the constitutive relation, give correct lateral strains and, most importantly, correct volume changes in all these tests and represent unloading and cyclic loading.

The present work is inspired by the previous endochronic models. In a recent comparative study of endochronic and plastic theories (8), a method of tangential linearization of the endochronic theory for proportional loading has been indicated. The resulting incrementally linearized formulation is always equivalent to some as part of the constitutive relation, give correct lateral strains and, most importantly, correct volume changes in all these tests and represent unloading and cyclic loading.

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INCREMENTALLY LINEAR INELASTIC CONSTITUTIVE LAWS

With the exception of endochronic theory, all the existing theories of time-independent inelastic behavior can be represented in the (hypoelastic) form

\[ \Delta \sigma_{ij} = C_{ijklm}(q,\xi) \Delta \varepsilon_{klm} \]  

in which \( \Delta \sigma_{ij} \) = components of strain tensor \( q \) and stress tensor \( \sigma \), referred to cartesian coordinates \( x_i (i = 1, 2, 3) \); and \( C_{ijklm} \) = tensor of tangential moduli. Concrete is assumed to be isotropic in the initial state, but moduli \( C_{ijklm} \) must exhibit stress-induced or strain-induced anisotropy. The strains may be assumed to be small.

Plastic Deformation.—Moduli \( C_{ijklm}(q,\xi) \) represent too many unknown material functions. In incremental plasticity, the number of unknown functions is tremendously reduced through the concept of plastic loading surface (or plastic potential), defined by \( F(\sigma_{lm},H_n) = 0 \) in which \( H_n(n = 1, 2, ..., ) \) are some state parameters, called hardening parameters. Let us begin by briefly outlining the well-known basic relations (22,30,32,34). Choosing \( \partial F/\partial H_n \) \( dH_n \) to be negative for loading, and noting that \( \partial F/\partial \sigma_{lm} \) \( d\sigma_{lm} \) \( + \) \( \partial F/\partial H_n \) \( dH_n \) = 0 (in which repeated indices imply summation), we have the loading criterion

\[ \frac{\partial F}{\partial \sigma_{lm}} > 0 \]  

which represents a condition of inelastic straining. So, the plastic strain increment,

\[ d\varepsilon_{lm} = \varepsilon_{lm}^{\text{pl}} \]  

must be a function of this expression. Assuming linearity, we may, therefore, set \( d\varepsilon_{lm}^{\text{pl}} = g_{lm} \frac{\partial F}{\partial \sigma_{lm}} d\sigma_{lm} \) in which \( g_{ij} \) is some coefficient tensor.

Now, following Drucker (19,20), we require that the second-order work done upon applying and removing any stress increment, \( d\sigma_{ij} \), be non-negative. This yields

\[ \Delta W = \frac{1}{2} d\sigma_{lm} d\varepsilon_{lm}^{\text{pl}} > 0 \]  

in which \( \varepsilon_{lm}^{\text{in}} = \) the inelastic strains = \( \varepsilon_{lm}^{\text{pl}} \); and \( \Delta W \) is shown as an area in Fig. 1(a). Eq. 2 is known as Drucker’s stability postulate. Comparing Eq. 2 with Eq. 1, we conclude that \( d\varepsilon_{lm}^{\text{pl}} \) must be proportional to \( \partial F/\partial \sigma_{lm} \), and by further comparison with the aforementioned expression for \( d\varepsilon_{lm}^{\text{pl}} \) we find that the proportionality coefficient must itself be proportional to \( \partial F/\partial \sigma_{lm} \); so
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The shape of the loading surface in the deviatoric section will be assumed to be of the von Mises type. This is no doubt a simplification, but consideration of other effects (such as fracturing strains in the sequel, etc.) appears to be much more important than playing with the shape of the loading surface. With a view toward unloading behavior, we will adopt kinematic hardening, renaming those parameters \( H_1 \) that give the center of the loading surface in the deviatoric and volumetric sections of the stress space as \( \alpha_1 \) and \( \alpha_v \). (For monotonic loading, though, it will be possible to use \( \alpha_1 = \alpha_v = 0 \).)

The loading function must further reflect the large effect of confining pressure \((-\sigma)\), which may be accomplished by choosing Drucker-Prager-type loading surface (21):

\[
F(\sigma_{ij}, H_1) = \bar{\tau}^* + g(\sigma^*) - H_1 = 0
\]

in which \( \bar{\tau}^* = (s_{1m}^* s_{km}^*/2)^{1/2}; s_{1m}^* = s_m - \alpha_{sm}; \sigma^* = \sigma - \alpha_r; s_{km} = \alpha_{km} - \delta_{km} \sigma; \delta_{km} = \text{Kronecker delta}; \text{and} \sigma = \sigma_{sk}/3 = \text{mean stress}. \quad \text{If} \quad \alpha_1 = 0, \text{then} \quad \bar{\tau}^* = \bar{\tau} = \text{the stress intensity. Noting that} \quad \partial F/\partial \sigma_{ij} = \partial F/\partial s_{ij} + \delta_{ij} \partial F/\partial \sigma_{kk} \text{and} \sigma = 3\sigma, \text{and carrying out these differentiations, one finds from Eq. 3 that}
\]

\[
de_{ij} = \frac{\partial \mu}{\partial \sigma_{ij}} = \frac{2}{3} \beta \mu
\]

in which \( d\bar{\tau}^* = s_m^* ds_{1m}/2 \bar{\tau}^*; \beta' = \partial g(\sigma^*)/\partial \sigma = \text{internal friction coefficient}; \beta = \beta'; \epsilon_{pl}^p = \epsilon_{pl}^p/3 = \text{volumetric (mean) plastic strain}; \text{and} \epsilon_{pl}^p = \epsilon_{pl}^p - \delta_{ij} \epsilon = \text{plastic strain deviator. If} \ F \text{is chosen to have the dimension of stress, coefficient} \ h \text{may now be shown to have the meaning of plastic hardening modulus, and} \ \beta \text{may be called the plastic dilatancy factor, for it relates the plastic volume change to the plastic deviatoric strain. The inequality in Eq. 6 ensues from the loading criterion, Eq. 1.}

Eqs. 5 and 6 follow from Eq. 4 when Drucker's postulate or the normality rule is used. However, this postulate is known to lose validity when frictional deformations are present (7,19,20,33). In concrete and, similarly, in all geologic materials, the hydrostatic pressure, \(-\sigma\), develops a frictional effect on inelastic shear. Consequently, the normality rule must be relaxed as far as the ratio of the volumetric deformations to the deviatoric deformations is concerned. So, the plastic dilatancy factor, \( \beta' \), i.e., the restriction \( \beta' = \beta \) following from the normality rule must be relaxed. Then, of course, the normality rule is violated and \( \Delta W \) in Eq. 3 can be negative, which, however, does not necessarily imply instability if the negativity of \( \Delta W \) is caused by friction (6,7,33).

The total deviatoric and volumetric strains may be expressed as

\[
de_{ij} = \frac{ds_{ij}}{2G} + de_{ij}^v; \quad d\epsilon = \frac{d\sigma}{3K} + d\epsilon^v
\]

in which \( G \text{ = elastic shear modulus; and} K \text{ = elastic bulk modulus. Substituting Eqs. 5 and 6,} \ de_{ij}^v = de_{ij}^p \text{ and} \ d\epsilon^v = d\epsilon^p, \text{ we now see that} ds_{ij} \text{ is involved...}
in both equations linearly, i.e., $d\sigma_j$ and $de$ are expressed as linear functions of $d\sigma$. Later, we will need an inverted form of these relations. To this end, $d\mu$ must be expressed in terms of $d\epsilon_j$ rather than $d\sigma_j$. To get it, Eqs. 6 and 7 may be used to calculate $d\sigma_\beta = 2d\mu - \beta' d\sigma = 2d\mu - K\beta'(3d\epsilon - 2\beta d\mu)$. Then, from Eqs. 7 and 5, $d\mu = G(d\epsilon_j - s^{\mu} d\mu/\tau^\mu)$, therefore, $d\sigma_\beta = s^{\mu} d\epsilon_j/\tau^\mu$. Equating both expressions for $d\sigma_\beta$, one obtains an equation that yields

$$d\mu = \frac{G_s^\mu d\epsilon_j + \tau^\beta \beta' \mu^\epsilon}{2\tau^\beta (h + G + K\beta')}$$

Moreover, rewriting Eq. 7 as $d\epsilon_j = 2G d\sigma_\beta - d\epsilon_j^\mu$; $d\sigma = 3K d\epsilon - d\epsilon_j^\mu$

in which $d\epsilon_j^\mu = 2G d\sigma_\beta$; $d\epsilon_j^\mu = 3K d\epsilon - d\epsilon_j^\mu$.

are called the plastic stress decrements and, substituting Eq. 5 where $d\mu$ is given by Eq. 8, one can express $d\epsilon_j$ and $d\sigma$ in the form of a linear function of $d\epsilon_j$.

Fracturing Deformation.—Plastic slip does not lead to strain softening, i.e., the stress decrease at increasing strain. The only mechanism that can explain it is the microcracking or fracturing. This was realized by Dougill (18), who proceeded to make an important contribution by formulating a pure fracturing. Noting that for the strain-softening branch we have $\epsilon_j > 0$ while $A \epsilon_j < 0$. This makes the fracturing theory more suitable for strain softening than plasticity. The $\Delta \Pi$ is shown as an area in Fig. 1(b), and the stress-strain curve is shown as a sequence of elastic and inelastic stress increments.

Because Eq. 12 indicates the occurrence of fracturing, it is necessary that $\partial \sigma \partial \epsilon_j d\epsilon_j = -G_s \partial \epsilon \partial \epsilon_j$ in which $g_{ij}$ is some coefficient tensor. Now, comparing Eq. 12 to Eq. 10, we conclude that $\partial \sigma \partial \epsilon_j$ must be proportional to $\partial \epsilon \partial \epsilon_j$ and by further comparison with the last expression for $\partial \sigma \partial \epsilon_j$ we find the scalar proportionality coefficient, $2\frac{\partial \sigma \partial \epsilon_j}{\partial \epsilon_j}$; this yields

$$\partial \sigma \partial \epsilon_j = -\frac{\partial \epsilon \partial \epsilon_j}{2d\epsilon_j}$$

For unloading and reloading it will be useful to assume, similarly to plastic response, that the fracturing also generally exhibits kinematic hardening. The center of fracturing surface $A$ in the deviatoric and volumetric sections of the strain space will be denoted as $J_3$, and $J_3$. (for monotonic increasing we will use $J_3$).

In the case of kinematic hardening, to the center of fracturing surface must be expressed in both equations linearly, i.e.,

$$d\sigma_\beta = 2G d\sigma_\beta - d\sigma_j$$

in which $\epsilon_j = \epsilon_j - \beta_j$; $\epsilon_j = \epsilon - \beta_j$; and $d\epsilon_j^\mu$ and $d\epsilon_j^\mu = d\epsilon_j^\mu - \delta_i d\epsilon_j^\mu$ may be called the volumetric and deviatoric fracturing stress increments (actually decrements or relaxations). Noting that for $G = G(\epsilon_j, \sigma)$ we have $dG =\epsilon_j^\mu$

$$\frac{\partial}{\partial \epsilon_j} d\epsilon_j = \frac{\partial}{\partial \epsilon_j} d\epsilon_j$$

$$\frac{\partial}{\partial \epsilon_j} d\epsilon_j = \frac{\partial}{\partial \epsilon_j} d\epsilon_j$$

$\epsilon_j^\mu = \epsilon - \epsilon_j$, $\epsilon = \epsilon_{ij}/3$ = volumetric (mean) strain; $\epsilon_j = \epsilon_{ij}$
that is equal to the current secant modulus and is much less than the initial mcor.rect (8), the fracture theory gives for this straining direction a stiffness increment. The dilatancy factor, $\alpha$, thus obtained does not generally obey the normality rule, of course.

Differentiating, we get

$$\frac{dG}{dfK(v)} = fK(v), \quad G/G_0$$

$\alpha = 4 - \gamma \cdot dG$.

Eliminating $dK$ from Eq. 17 we may express the fracturing dilatancy factor as

$$\alpha = \frac{9 \varepsilon^* \frac{dK}{dG}}{4 \gamma^* \frac{dG}{dfK(v)}}.$$ (18)

Eq. 18 allows us to exploit recent important theoretical results of Budianski and O'Connell (14). Using the self-consistent method for composites, they calculated the approximate macroscopic $K$ and $G$ for a perfectly elastic solid containing a random isotropic array of identical elliptical cracks of any aspect ratio $a/b$. Their results indicate a decrease of $K$ and $G$ as well as Poisson ratio $v$ with the crack concentration, and they are presented in the form $K/K_0 = f_K(v)$, $G/G_0 = f_G(v)$ in which $f_K$ and $f_G$ are certain monotonic functions listed in Appendix I (Eq. 29). Differentiating, we get $dK = K_0 dv df_K(v)/dv$, $dG = G_0 dv df_G(v)/dv$, and substituting in Eq. 18 we obtain

$$\frac{df_K(v)}{dG} = \frac{9 \varepsilon^* K_0}{4 \gamma^* G_0 \frac{df_G(v)}{dfK(v)}} \quad \text{with} \quad v = \frac{3K - 2G}{2(3K + G)}.$$ (19)

The dilatancy factor, $\alpha$, thus obtained does not generally obey the normality rule, of course.

The fracturing theory is more realistic than plasticity in the case of strain increments that are parallel to the current loading surface. Whereas plasticity gives in this case a purely elastic response, which is too stiff and definitely incorrect (8), the fracturing theory gives for this straining direction a stiffness that is equal to the current secant modulus and is much less than the initial elastic modulus. (One can check it by considering an increment, $de_{23}$, from an initial state with $s_{11}$ and $e_{11}$ nonzero, all other $s_{ij}$ and $e_{ij}$ being zero.) For concrete, good data are lacking, but the equality of the tangent modulus for this load direction to the secant modulus is known to be about correct for various other materials (8). This is definitely more reasonable (and safer) than the plastic theory, which yields for the tangential strain increments the initial elastic modulus.

**COMBINED PLASTIC-FRACTURING CONSTITUTIVE LAW**

The mechanism of inelastic strain in concrete consists of both microcracking and plastic slip. The former prevails at low confining pressure and in the later stages of the uniaxial compression test. The latter dominates at high confining pressure and is also pronounced on the rising branch of the uniaxial compression test. This is shown in an idealized, exaggerated manner in Fig. 1(d).

The separate contributions of microcracking and plastic slip can be estimated by observing the unloading slopes. If the unloading slope is about the same as the initial loading slope, the inelastic strain is essentially plastic and is given by the offset of the unloading diagram at $\sigma = 0$ (see Fig. 1(d)). If the unloading slope declines, which is typical throughout the strain-softening branch (40), the inelastic strain due to the slope decline must be attributed essentially to microcracking or fracturing, and may be defined according to Figs. 1(e) and 1(f). The fracturing behavior is obviously suitable for representing the strain softening, whereas plasticity is unsuitable for this purpose (even for obtaining a peak on the response curve), and, if used, then Drucker's postulate is violated.

The stress-strain curve may now be imagined as a sequence of elastic, plastic, and fracturing increments as shown in Fig. 1(c). To combine the plastic and fracturing constitutive laws, we could, in principle, proceed in two simple ways: either we superimpose $d\varepsilon^p$ and $d\varepsilon^f$ due to the same $d\sigma_{ij}$, or we superimpose $d\varepsilon^p_{ij}$ and $d\varepsilon^f_{ij}$ due to the same $d\varepsilon_{ij}$. However, in case of strain softening, only the latter approach [see Fig. 1(c)] is admissible because we cannot associate $d\varepsilon^p_{ij}$ with stress decrements $d\sigma_{ij}$. So superposing $d\varepsilon^p_{ij}$, $d\sigma^p$ (Eq. 9) and $d\varepsilon^f_{ij}$, $d\sigma^f$ (Eq. 15), we may set

$$d\sigma_{ij} = 2Gde_{ij} - 2G s_{ij}^* \frac{d\mu}{\gamma^*} - \frac{d\varepsilon_{ij}}{\gamma^*};$$

$$d\sigma = 3Kde - 2K \beta d\mu - \frac{2}{3} \alpha d\kappa.$$ (20)
sets for various concretes, searching simultaneously for the dependence of the deviatoric strains on the curve of small loading steps to integrate the constitutive relation, Eq. 19, for a specified endochronic theory (listed in Ref. 10) has been developed. The program uses the fact that $\alpha$, $\beta$, and $\gamma$ had to be determined empirically, by fitting test data. The selection as intuitive concepts (see Appendix I).

One of these functions, i.e., $\phi$, has been determined by a theoretical argument based on a micromechanics model (Eq. 19). However, for the other five functions involves, aside from the initial elastic constants, $G_0$ and $K_0$, six independent scalar coefficients, $h$, $\phi$, $\beta$, $\phi_0$, $h_0$, and $\alpha$, characterizing the material. These coefficients are functions of the invariants of stress and strain (see Appendix I). One of these functions, i.e., $\alpha$, has been determined by a theoretical argument based on a micromechanics model (Eq. 19). However, for the other five functions, no micromechanics models are available at present, and, therefore, $h$, $\phi$, $\beta$, $\phi_0$, $h_0$, and $\alpha$ had to be determined empirically, by fitting test data. The selection of the form of these functions has been guided by various physical as well as intuitive concepts (see Appendix I).

For data fitting, a computer program similar to that used previously for the endochronic theory (listed in Ref. 10) has been developed. The program uses small loading steps to integrate the constitutive relation. Eq. 19, for a specified form of functions $h$, $\phi$, $\beta$, $\phi_0$, and $h_0$. The numerical algorithm is analogous to that in Ref. 9, and the simulation of various types of tests (uniaxial, biaxial, triaxial, shear compression, etc.) is done in the same manner. The response curves are automatically plotted by a Calcprop plotter, and a sum of the square deviations from the characteristic data points (the objective function to be minimized) is evaluated. The data fitting is done collectively for various data sets for various concretes, searching simultaneously for the dependence of the material parameters on the strength of concrete. (For details see Ref. 10.)
When $dW$ changes from $0$ to $\geq 0$, set $\alpha = \sigma$; $\beta = \varepsilon$ .... (27a)

When $dW'$ changes from $0$ to $\geq 0$, set $\alpha = \sigma$; $\beta = \varepsilon$ .... (27b)

When $dW$ changes from $< 0$ to $\geq 0$, set $\alpha = \sigma$; $\beta = \varepsilon$ .... (27c)

When $dW'$ changes from $< 0$ to $\geq 0$, set $\alpha = \sigma$; $\beta = \varepsilon$ .... (27d)

Various details in the foregoing method for unloading and reloading are strictly empirical, obtained from the fitting of tests. Coefficient 2 in Eqs. 27c and 27d means that for reloading, the center of the fracturing surface is not placed into the last lower limit point but between this point and the origin. Eq. 26 means that during unloading, the elastic moduli are not calculated according to the pure fracturing material (which would require $dG = dK = 0$). At high confining pressures ($\sigma < 0$), Eq. 26 gives a much larger value of $K$ than that for the loading branch, which is needed mainly for fitting the steep unloading branch under hydrostatic loading [Fig. 3(b)]. Physically, this is not unreasonable because high confining pressures should keep all voids and microcracks closed, resulting in high $K$ and $G$. During unloading, $|\alpha|$ decreases, which makes $K$ and $G$ again smaller and provides the characteristic decline of slope of the unloading branch during unloading [Fig. 3(b)]. The fracturing dilatancy factor, $\delta$, is determined, however, on the basis of elastic moduli $G_p$ and $K_p$ for the last lower peak point all throughout the unloading branch.
In the basic relations, Eqs. 20, 8, 16, and 17, material functions \( h, \phi, \alpha, \beta, \) and \( \beta' \) can, in general, be introduced in forms that differ for loading, unloading, and reloading. Only a simple change in these functions is considered herein as indicated by coefficient \( C_1' \). This coefficient models the fact that for unloading, \( C_1 = C_{u1} \) the inelastic strain is less than for reloading, \( C_1 = c_2 \), for which it is again less than for virgin loading, \( c_4 < c_3 < 1 \). It was proved (8) that

\[
\begin{align*}
0 &< c_4 < 2c_u, \\
&< c_3 < 1
\end{align*}
\]

Drucker's postulate \( \Delta W = 0 \), (8) is satisfied for an unload-reload cycle if \( c_4 < c_3 < 2c_u \), and this is fulfilled by the values of \( c_1 \) identified by data fitting \( c_4 = 0.5 \) and \( c_3 = 0.8 \).

The fits of typical test data for unloading and cyclic load (23,38,39,40) are shown in Fig. 2.

Alternatively, we could have used as "associated" loading-unloading criteria the conditions that \( d\mu \) or \( d\kappa \) would change their signs.

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The present model gives zero plastic (as well as fracturing) strain increment if the loading proceeds in the direction parallel to the current loading surface. Although experimental evidence is lacking, this response is no doubt too stiff (8). Some plastic strain should be produced by this loading, too. The endochronic

![FIG. 3.—Fits of Uniaxial and Biaxial Test Data: (a) Axial and Lateral Strains; (b) Volume Change; (c) Failure Strains; (d) Failure Envelopes; (e) Uniaxial Tests](image)

![FIG. 4.—Fits of Triaxial Test Data: (a,b,d) Standard Triaxial Tests; (b) Volume Change; (c) Failure Envelopes for Proportional Triaxial Loading](image)
the deviatoric loading surface yet at the same time cause no effect on loading that is normal to the loading surface is to insert at the right-hand side of Eq. 20 the term
\[
-c_0 \frac{\Phi}{2} \left( d \varepsilon_{ij} - e_{ij}^* d \tilde{\gamma}^* \right), \quad \text{with} \quad d \tilde{\gamma}^* = \frac{e_{lm}^* d \varepsilon_{lm}^*}{2\tilde{\gamma}^*} \quad (28)
\]
with some coefficient \(c_0\). This term vanishes for proportional loading, as can be checked by substituting \(d \varepsilon_{ij} = e_{ij}^* ds, d \varepsilon_{lm}^* = e_{lm}^* ds\) and is also zero for hydrostatic loading. Thus, the effect on the fits in Figs. 3–5 is small. For loading that substantially deviates from the \(e_{ij}^*\) direction, this term becomes significant, making the response softer. Geometrically, this term can be interpreted as a

**SUMMARY AND CONCLUSIONS**

Incremental plasticity and fracturing (microcracking) material theory are combined to obtain a nonlinear triaxial constitutive relation that is incrementally linear. A new hardening rule, called jump-kinematic hardening, is used for unloading, reloading, and cyclic loading. In the case of continuous tensile cracks, the theory applies only for the solid (albeit microcracked) concrete between the cracks. Because of friction and dilatancy due to shear, the tangential moduli are nonsymmetric.

The theory combines the plastic stress decrements with the fracturing stress decrements, which reflect microcracking, and accounts for internal friction, pressure sensitivity, inelastic dilatancy due to microcracking, strainsoftening, degradation of elastic moduli due to microcracking, and the hydrostatic nonlinearity due to pore collapse. Failure envelopes are obtained from the constitutive law as a collection of the peak points of the stress-strain response curves. The jump-kinematic hardening allows for inelastic response during unloading, reloading, and cyclic loading and, at the same time, it does not in itself cause violation of Drucker's postulate. As a consequence of the incremental linearity, the plastic strain increments vanish for loading that is parallel to the loading surface; this response may be too stiff and questionable for material instability predictions.

Six scalar material functions are needed to fully define the monotonic response. One function, the dilatancy due to microcracking, is determined theoretically, based on Budianski and O'Connell's calculation of the effective elastic constants of a randomly microcracked elastic material by the self-consistent method for composites.

All material parameters are identified by fitting published test data, and their dependence on concrete strength is also given. The fits are as good as those for the previous endochronic theory, except for a high number of cycles and the time dependence. The theory is ready for use in finite element programs.

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Appendix I.—Material Functions and Data Fits

Degradation of Elastic Moduli.—According to Eqs. 36. 44′, and 42′ of Ref. 14

\[ f_k(v) = 1 - \frac{16}{9} \left( \frac{1 - v^2}{1 - 2v} \right) c; \quad f_\alpha(v) = 1 - \frac{8}{45} (10 - 7v) c, \]

with \( c = \frac{45}{8} \frac{\nu - v_0}{(1 + \nu)(10v_0 - \nu - 8v_0 \nu)} \) \hspace{1cm} (29)

in which \( v_0 \) = initial Poisson ratio. These expressions are precisely those obtained for long elliptical cracks (14). The expressions for circular cracks may be more pertinent but the crack shape does not affect \( f_k \) by more than a few percent, thus, the expressions for long cracks are preferred over those for circular cracks because they are simpler (14).

Inelastic Material Functions.—These have been identified as follows:

\[ h = \frac{a_0 - a_1 \tilde{\gamma} + |J_3 f'_c|^{1/4}(a_2 + a_3 J_1) + a_4 J_3}{J_2 - a_5 J_1}, \]

\[ \beta' = \frac{\bar{\beta}}{f'_c + I_1 - a_6 \tilde{\gamma}}, \]

\[ \phi = G \tilde{\gamma} \frac{b_0 + b_1 J_2 + b_2 J_3}{(f'_c + b_1 J_2)^2 + b_3 J_1 + I_3 (b_3 + b_6 \tilde{\gamma})} \]

\[ \alpha' = \alpha_0 + \frac{a_1 J_3 - a_7 \bar{\gamma}^{2/3}}{a_3 + f'_c(\alpha_4 + \alpha_5 J_1)} \hspace{1cm} (31) \]

\[ \beta = \left( \frac{\beta'' f'_c}{1 + \beta'' f'_c} \right)^2_1 \beta'' = \left( \frac{c_0 J_1 + c_1 J_3}{(c_2 + c_3 J_2) + (c_6 J_3 - c_5 J_1 J_2)^{-1}} \right) \]

in which \( J_1 = |J_3 I_1|; I_1, I_2, I_3 \) = first, second, and third invariants of \( \sigma_\alpha^\mu; J_2 = \) second and third invariant of stress deviator \( \sigma_\alpha^\mu; J_3 = \) second invariant of strain deviator \( \varepsilon_\alpha^\mu; I_1, I_2, I_3 \) are always taken in absolute value if negative, \( I_3 = |\det(\sigma_\alpha^\mu)|; J_2 = \tilde{\gamma}^{2/3}; J_3 = 1/2 s_\alpha^\mu s_\alpha^\mu; J_1 = I_3 - \sigma_\alpha^\mu J_2; J_2 = \tilde{\gamma}^{2/3} = 1/2 e_\alpha^\mu e_\alpha^\mu; \) and

\[ a_0 = \frac{f'_c}{90} \text{ psi}; \quad a_1 = \frac{f'_c}{150} \text{ psi}; \quad a_2 = \left( \frac{15 \times 10^7}{\tilde{\gamma}} \right)^{1/2} \]

\[ a_3 = \left( \frac{35,000 \text{ psi}}{f'_c} \right)^3; \quad a_4 = \left( \frac{f'_c}{2,700 \text{ psi}} \right)^2; \quad a_5 = 1.95; \quad a_6 = 1.73 \hspace{1cm} (33a) \]

\[ b_0 = (2 \text{ psi})^{3/2}; \quad b_1 = 36,000; \quad b_2 = 13 \times 10^4; \quad b_3 = \frac{14,000 \text{ psi}}{f'_c} \]

\[ b_4 = 134; \quad b_5 = \left( \frac{45,000 \text{ psi}}{f'_c} \right)^{2.5}; \quad b_6 = \left( \frac{f'_c}{840 \text{ psi}} \right)^{4} \hspace{1cm} (33b) \]

Analysis of Material Functions.—They are basically introduced in the following form:

\[ h = \frac{a_0 + h_s(I_3)}{h_s(\tilde{\gamma})}; \quad \phi = \frac{\beta_s(\tilde{\gamma})}{\beta_s(\tilde{\gamma})} \]

\[ \beta' = \frac{\beta_s(\tilde{\gamma})}{\beta_s(\tilde{\gamma})} \hspace{1cm} (35) \]

The plastic hardening modulus, \( h \), must be finite for hydrostatic compression since there is no plastic volume change, and it must also tend to infinity at small stress [Figs. 1(g) and 1(j)], since plastic strains at small stresses are negligible. This indicates that the denominator of the expression for \( h \) must depend on \( \tilde{\gamma} \) and must vanish as \( \tilde{\gamma} \to 0 \). (The \( \tilde{\gamma} \) should not be used because plasticity declines as the strain softening advances.) The confining pressure restricts plastic strain, but mainly in triaxial loading and not much in biaxial loading. This suggests that function \( h \) should decrease with \( I_3 \) as the triaxial compression test progresses (\( I_3 = |\sigma_\alpha^\mu| \)).

The fracturing compliance, \( \phi \), must increase if the shear strain increases, which can be modeled by an increasing function, \( \phi_s(\tilde{\gamma}) \) [Fig. 1(g)]. Furthermore, \( \phi \) must vanish at hydrostatic compression loading, for the nonlinearity of the hydrostatic diagram stems from fracturing. Thus, since \( dK \) in Eq. 20 is divided by \( \tilde{\gamma} \), then \( \phi \) must be proportional to \( \tilde{\gamma} \) to counteract it (Eq. 35). On the other hand, fracturing must become rather limited at high confining pressure, so Eq. 35 for \( \phi \) must be divided by an increasing function of \( |\sigma| \), i.e., \( \phi_s(|\sigma|) \). Fracturing must get intensified as shear stress increases, so \( \phi \) must increase with \( \tilde{\gamma}/\alpha \). Furthermore, fracturing must be much more restricted under triaxial compression than under uniaxial compression. This can be introduced in Eq. 35 by function \( \phi_s(I_3) \), which increases in absolute value.

The volumetric fracturing compliance coefficient, \( \alpha' \), adjusts \( \phi \) so as to represent the volumetric nonlinearity, chiefly due to hydrostatic compression, \( -\sigma \) [Fig. 1(k)]. This effect is much more significant in hydrostatic compression.
tests than in biaxial or uniaxial tests. Thus, it should be governed by \( I_1 \) rather than \(-\sigma\). The hydrostatic-compression test first indicates a softening, due to pore collapse [Fig. 1(k)], which is indicated by function \( g \), growing from zero; however, later there is a stiffening again after the pores have closed, and this is modeled by an increasing function \( \alpha_s(I_1) \) in the denominator of \( \beta' \), Eq. 36. The last effect, however, should be delayed or eliminated by simultaneous shear; therefore, an increasing function, \( \alpha_s(\gamma) \), should multiply \( \beta_s(\gamma) \) in Eq. 36.

The plastic friction coefficient, \( \beta' \), may be approximately interpreted as the tangent slope of Mohr's failure envelope [Fig. 1(j)]. This slope decreases as the hydrostatic pressure, \(-\sigma\), increases, so the expression for \( \beta' \) (Eq. 36) may be divided by an increasing function, \( \beta_s(\sigma) \). A lower friction coefficient means a more plastic response, and the response is most plastic when \(-\sigma\) is high and shear \( \tau \) is low (triaxial test). So, \( \beta' \) should increase with \( \tau \), which is introduced by an increasing function, \( \beta_s(\tau) \).

The plastic dilatancy factor, \( \beta \), must be always non-negative to give dilatancy rather than densification. The dilatancy is high at high shear stresses or shear strains and low at high confining pressures. Thus, \( \beta \) must increase with \( \tau \) and \( \gamma \) [Fig. 1(l)], and it must decrease with \(-\sigma\), which is modeled by increasing functions \( \beta_s(\tau) \), \( \beta_s(\gamma) \), and \( \beta_s(\sigma) \).

Compared to the basic form of the expressions for \( h, \phi, \alpha', \beta', \) and \( \beta \) deduced before, various further terms were introduced to "tune up" the fits empirically. For this purpose the desired trends of various functions were sketched as in Figs. 1(g), 1(h), 1(l), 1(j), and 1(k). The interrelation of the fits of uniaxial, biaxial, and triaxial tests was controlled by noting that \( I_1 \) is nonzero for all three, \( I_2 \) is nonzero only for biaxial and triaxial tests, and \( I_3 \) is nonzero only for triaxial tests. The ratio \( I_2/I_1 \) appeared to be useful to control the inelastic dilatancy since it grows with triaxial compression and decreases with shear stress. The invariant \( I_3 \) may also be used to model the deviation from the hydrostatic stress influence.

Simulation of Test Conditions.—The incremental equation, Eq. 21, may be rewritten as \( \Delta \sigma = C \Delta \epsilon \), in which \( \sigma, \epsilon = (6 \times 1) \) column matrices of stress and strain components, i.e., \( \sigma = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})^T \), \( \epsilon = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{23}, \epsilon_{31})^T \), in which \( T \) denotes a transpose; and \( C = (6 \times 6) \) matrix of tangential moduli (nonsymmetric). To integrate this stress-strain relation for a particular test, some stress components and some other strain components are prescribed. To exemplify this process of stress, for which \( \Delta \epsilon_{11} \) and \( \Delta \epsilon_{22} \) are specified for each load step and \( \Delta \sigma_{33} = \Delta \sigma_{11} = \Delta \sigma_{22} = 0 \). Denoting the column submatrices of the known stress and increments known as \( \Delta \sigma'' = (\Delta \sigma_{11}, \Delta \sigma_{22}, 0) \) and \( \Delta \sigma' = (\Delta \sigma_{33}, \Delta \sigma_{11}, \Delta \sigma_{22}, \Delta \sigma_{33}) \), and the unknown ones as \( \Delta \epsilon'' = (\Delta \epsilon_{11}, \Delta \epsilon_{22}, \Delta \epsilon_{33}, \Delta \epsilon_{12}, \Delta \epsilon_{23}, \Delta \epsilon_{31}) \) and \( \Delta \sigma'' = (\Delta \sigma_{11}, \Delta \sigma_{22}, 0) \), we may write the incremental relations in the partitioned form

\[
\begin{bmatrix}
\Delta \sigma'' \\
\Delta \sigma'
\end{bmatrix} =
\begin{bmatrix}
P & Q \\
R & S
\end{bmatrix}
\begin{bmatrix}
\Delta \epsilon'' \\
\Delta \epsilon'
\end{bmatrix}
\] (37)

The solution of \( \Delta \sigma'' \) and \( \Delta \epsilon'' \) is then obtained as

\[
\begin{bmatrix}
\Delta \sigma'' \\
\Delta \epsilon''
\end{bmatrix} =
\begin{bmatrix}
P - QS^{-1}R & QS^{-1} \\
-S^{-1}R & S^{-1}
\end{bmatrix}
\begin{bmatrix}
\Delta \sigma' \\
\Delta \epsilon'
\end{bmatrix}
\] (38)

An equation of this type is automatically set up in the program (in component form) for various specified types of tests to be simulated and fitted.

Basic Information on Test Data Used.—This has been summarized in Ref. 9, except for the following.

Fig. 4(a,b).—2.7-in. \times 6-in. cylinders; age: over 200 days; strain-controlled test; strain rate: 10\(^{-7}\)/sec; water-cement-aggregate ratio: 0.6:1:6.3 by weight; cured for 4 days at constant environment (curing room); covered with wet burlap, then exposed to drying lab environment for several days until testing.

Fig. 4(c).—10-cm cubes; cured for 1 week in wet room, then sealed and stored at 18\(^\circ\)C and 65% relative humidity for 20 days until testing; age: 3 months-4 months at test; water-cement-sand-gravel: ratio 0.81:1.36:3.19; stress-controlled test: specimens glued on brush bearing platens.

Fig. 4(d).—10-cm cubes; demolded after 1 day; cured for 1 week in 100% relative humidity, then sealed in plastic bags; age: 150 days-250 days at test; water-cement-sand-gravel ratio: 0.84:1:16.32 by weight; stress-controlled test with flexible steel platens; loading rate 0.075 N/mm\(^2\)/min.

Fig. 5(e).—Specimens cured in a fog room for at least 7 days, then sealed in plastic bags for 7 days until testing; maximum aggregate size: 4.76 mm.

Fig. 2(a,b).—250-mm \times 75-mm \times 75-mm\(^3\) prisms; strain-controlled test; strain rate: 2 \times 10^{-4}/min; epoxy between specimen face and platen.

Fig. 2(e-g).—10-cm\(^3\) \times 30-cm\(^3\) \times 50-cm\(^3\)\(^3\) prisms; water-cement-aggregate ratios: (1) 0.71:5.7:7, (2) 0.99:1:7, and (3) 0.58:1.5:1; demolded after 1 day, in water at 18\(^\circ\)C-20\(^\circ\)C for 7 days, then stored at 55%-70% relative humidity and 15\(^\circ\)C-19\(^\circ\)C until test; age: 56 days at test.

Appendix II—References


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41. Willam, K. J., and Warnke, E. P., “Constitutive Model for the Triaxial Behavior of Concrete,” presented at the 1974 International Association for Bridge and Structural Engineering Seminar on Concrete Structures Subjected to Triaxial Stresses, held at Bergamo, Italy.

Errata

Page 413, 1-st line after Eq. 16: Should read "\( \dot{\gamma} \)" instead of "\( \gamma \)" and "\( \dot{\epsilon} \)" instead of "\( \epsilon \)".

Page 417, Eq. 23: Should read "\( dW = 3d\xi \), \( d\dot{e} = d\dot{e}/3K \), \( dW = \xi \dot{e} \)".

Page 417, Line 4 and Eq. 23: Symbols "W" and "\( \dot{W} \)" should be mutually interchanged.

Page 417, 1-st line after Eq. 24f: Should read "\( W_0 \)" instead of "\( W \)".

Page 417, 1-st and 2-nd line after Eq. 26: Symbols "\( \dot{W}_0 \)" and "\( \dot{W}'_0 \)" should be interchanged.

Page 423, Eq. 29: Should read "\( \nu_\gamma - \nu \)" instead of "\( \psi - \nu \)".

Page 423, Eq. 33b: Should read "\( \nu_\gamma = (45,000 \text{ psi} / f'_{c})^{2.5} / \text{psi} \)" and "\( \nu_\gamma = (f'_{c}/860 \text{ psi})^{2.5} / \text{psi} \)" instead of "\( \nu_\gamma = (45,000 \text{ psi} / f'_{c})^{2.5} \) psi and "\( \nu_\gamma = (f'_{c}/860 \text{ psi})^{2.5} \) psi."

Page 423, Line 4 and 5 after Eq. 32: Should read "\( J_3 = J_3 + \sigma J_2 - \sigma^2 \)" instead of "\( J_3 = J_3 - \sigma J_2 + \sigma^2 \)."

Page 423, Eq. 33a: Should read "\( \alpha = \left[ 15 \times 10^3 (\text{psi})^2 / f'_{c} \right]^{2} \)" instead of "\( \alpha = (15 \times 10^3 (\text{psi})^{2} / f'_{c} \)" and "\( f'_{c} = \alpha \)" instead of "\( f'_{c} = \alpha \)".

Page 423, line after Eq. 32: Should read "\( J_3 = J_3 \)" instead of "\( J_3 = \left| J_3 \right| \)".

Page 424, Eq. 33d: Should read "\( \alpha = (f'_{c} - 5,500 \text{ psi})^{2} \)" instead of "\( \alpha = (f'_{c} - 5,500 \text{ psi})^{2} \)".

14653 PLASTIC-FRACTURING THEORY FOR CONCRETE

KEY WORDS: Concrete; Concrete structures; Constitutive equations; Cracking; Ductility; Failure; Fracturing; Inelastic action; Mathematical models; Plasticity; Tests

ABSTRACT: Incremental plasticity and fracturing (microcracking) material theory are combined to obtain a nonlinear triaxial constitutive relation that is incrementally linear. The theory combines the plastic stress decrements with the fracturing stress decrements, which reflect microcracking, and accounts for internal friction, pressure sensitivity, inelastic dilatancy due to microcracking, strain softening, degradation of elastic moduli due to microcracking, and the hydrostatic nonlinearity due to pore collapse. Failure envelopes are obtained from the constitutive law as a collection of the peak points of the stress strain response curves. Six scalar material functions are needed to fully define the monotonic response. One function, the dilatancy due to microcracking, is determined theoretically based on Budianski-O'Connell's calculation of the effective elastic contents of a randomly microcracked elastic material by the self consistent method for composites.