14933 INSTABILITY OF COOLING CRACKS

KEY WORDS: **Concrete; Cooling; Cracking;** Crack propagation; Cracks; Drying; **Fracture mechanics; Rock (material); Shrinkage cracking;** Thermal stresses

ABSTRACT: When a system of parallel equidistant cooling crack propagates into halfspace, it reaches at a certain depth a critical state at which the growth of every other crack is arrested. Later these cracks reach a second critical state at which they close. The intermediate cracks, at doubled spacing, open about twice as wide and advance further as cooling penetrates deeper. Determination of the critical states requires calculation of the derivatives of the stress intensity factors with regard to crack lengths, which is here accomplished by finite element method. It is found that the crack depth-to-spacing ratio at which the critical state is reached is extremely sensitive to the temperature profile. This ratio greatly increases as the cooling front becomes steeper. The effect of transverse isotropy of the material upon the location of critical states is found to be relatively mild. The results are of interest for one recently proposed geothermal energy scheme for hot dry rock, as well as for shrinkage cracks in concrete.

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Instability and Spacing of Cooling or Shrinkage Cracks

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Introduction

Extraction of geothermal heat from hot dry rock is of great interest because geothermal basins that are dry are far more numerous than those which have a natural water circulation. In one recently proposed scheme (12,15,16), a huge vertical crack of about 1 km in size is to be produced in the hot rock mass at the depth of several kilometers by the hydraulic fracturing process, in which water is forced under high pressure into a vertical bore. The crack must then be intersected by another bore, and by injecting cool water into one bore, hot water may be collected from the other bore to drive a power plant.

The withdrawal of heat by the circulating water gradually cools the rock near the crack walls, and it appears that after a few months of operation the rate of heat transport to the crack would become insufficient (6,12). Nevertheless, various mechanisms can be expected to increase the heat output. One of them is the thermal cracking, which would no doubt be induced in the rock. We may expect that, due to cooling, a system of secondary vertical cracks normal to the original (main) vertical crack would propagate into the hot rock mass, and part of the flux of circulating water would get diverted to these secondary cracks where the rock is still hot. The diverted flux is roughly proportional to the cube of secondary crack width, w, and to the inverse of crack spacing, b, while thermal strain calculation shows that w is proportional to b. The combined effect is, therefore, that the diverted flux of water is roughly proportional to b^2 (4,12). So, the heat extraction from the secondary cracks can become significant only if their spacing is sufficiently large. It is the purpose of this paper to investigate when this happens.

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The question of spacing is also important for drying shrinkage cracks in reinforced concrete. Here, we want the opposite to happen. To ensure sufficient transfer of shear stress across the rough crack surfaces to prevent corrosion of the embedded reinforcement, and to achieve good fatigue resistance and ductility, the cracks must remain hairline thin, and so we need the crack spacing to remain sufficiently small. Furthermore, crack spacing is of interest to geologists, e.g., to interpret the cooling cracks observed in solidified lava beds in the ocean (11), or the cracks in dried-up mud flats (9), or permafrost soils.

The situation may be idealized as a system of parallel Mode I cracks of lengths a_i (i = 1, 2, ... n), propagating into a brittle elastic halfspace $x \geq 0$ along planes y = constant (x, y, z are cartesian coordinates); see Fig. 1.

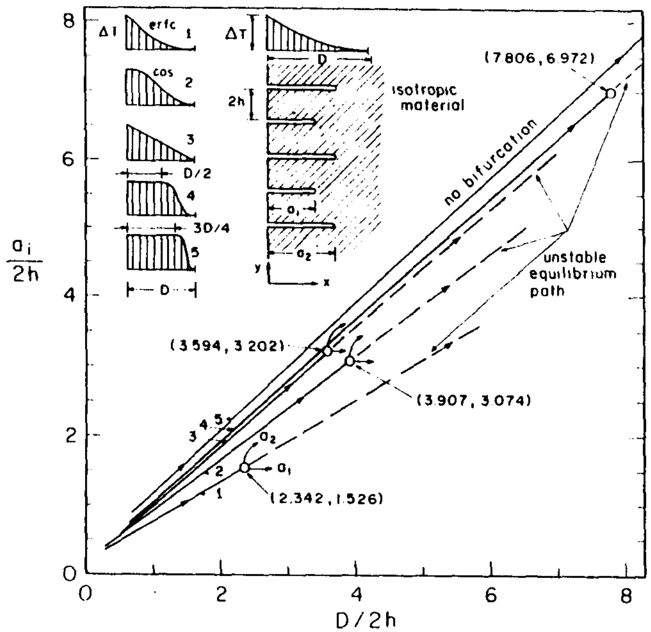

FIG. 1.—Critical States of Crack Arrest and Bifuration of Equilibrium Path, and Parallel Crack System

The loading consists of a prescribed thermal strain $\alpha\,\Delta T(x)$ (or shrinkage strain) as a function of depth x (α = thermal dilatation coefficient, and ΔT = temperature drop). We will consider fixed temperature profiles with the same value of the surface temperature drop $\Delta T(0)$, and the loading will then be fully characterized by the cooling penetration depth, D. Thus, temperature depends only on x/D and is assumed to be independent of y and z, which is certainly a simplification when water circulates through the secondary thermal cracks.

We shall assume that the cracks are planar. This is of course a simplification; the actual crack pattern must be expected to be three-dimensional, and cracks could also be curved.

Solving this problem of fracture mechanics, we notice an interesting feature—

the solution is not unique. We can find solutions of any chosen crack spacing s, as well as solutions with the same crack length or alternating crack lengths. What is causing the lack of uniqueness?

We must of course suspect instability. This was first investigated in Ref. 3 (and summarized earlier in Ref. 2), in which the second variation of the work needed to create the cracks and the conditions of adjacent equilibrium were analyzed and the stability conditions of a system of interacting cracks were determined. (In general terms these results were first given in a preceding unpublished work in 1976; see Acknowledgment.) The later work of some other authors on this problem was commented upon in Ref. 3.

In this work, after summarizing the stability conditions from Ref. 3, we will determine the critical states of crack arrest for various temperature or shrinkage stress profiles in a halfspace. Although for linear homogeneous materials and simple geometries these problems can be solved more accurately by a Green's function approach leading to singular integral equations, the finite element approach has been adopted because it has required the least human effort and cost (since a suitable finite element program was already available), and because it is simple, general, and can be easily extended to various surface geometries, arbitrary crack paths, as well as nonhomogeneous, nonlinear, and anisotropic materials. The latter extension will actually be also included in this study because anisotropy is an important characteristic of rock.

Griffith Cracks

Consider the work, W, that needs to be supplied to create cracks of length a_i:

$$W = U(a_1, a_2, \ldots, a_n;\quad D) + \sum_{i=1}^{N} \int_0^{a_i} 2\gamma_i\, da_i' \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Here U represents the strain energy of the elastic body; $2\gamma_i$ = specific energy of crack extension of the ith crack, considered to be a material property; and D = loading parameter (in our problem, the penetration on depth of cooling or drying). The equilibrium state of the cracks is determined by vanishing of the first variation of work:

$$\delta W = \sum_{i=1}^{m} \left(\frac{\partial U}{\partial a_i} + 2\gamma_i\right)\delta a_i + \sum_{j=m+1}^{k} \frac{\partial U}{\partial a_j}\,\delta a_j = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which i = 1, ... m are the cracks that extend ($\delta a_i > 0$); $j = m + 1$, ..., k, are those that close ($\delta a_i < 0$); and $i = k + 1$, ... N are further cracks that remain stationary ($\delta a_i = 0$). Because Eq. 2 must be satisfied for any admissible δa_i, the equilibrium crack extensions are characterized by (3):

$$\text{for}\quad \delta a_i > 0:\quad -\frac{\partial U}{\partial a_i} = 2\gamma_i;\quad \text{for}\quad \delta a_i < 0:\quad \frac{\partial U}{\partial a_i} = 0 \quad \ldots\ldots\ldots\ldots (3)$$

The first of these conditions is the well-known Griffith fracture criterion, and $-\partial U/\partial a_i = G_i$ is the energy release rate of the ith crack (9). Eq. 2 can be equivalently stated in terms of the Mode I stress intensity factor, which is here defined as $K_i = \lim \sigma_y\,[2\pi(x - a_i)]^{1/2}$ where σ_y = normal stress on the crack extension line (8), and not as $K_i = \lim \sigma_y (x - a_i)^{1/2}$. For plane strain, we have $\partial U/\partial a_i = -K_i^2/E'$ where $E' = E/(1 - \nu^2)$, E = Young's modulus, and ν = Poisson ratio (9). We may, therefore, rewrite Eq. 3 in the form (3):

$$\text{for} \quad \delta a_i > 0: \quad K_i = K_{c_i}; \quad \text{for} \quad \delta a_i < 0: \quad K_i = 0 \qquad (4)$$

in which $K_{c_i} = (2\gamma_i E')^{1/2}$ = critical value of the stress intensity factor for the given material.

The case $K_i > K_{c_i}$ or $-\partial U/\partial a_i > 2\gamma_i$ is impossible because Eq. 2 yields $\delta W < 0$ for $\delta a_i > 0$, which is an unstable condition (energy is released rather than consumed by crack extension). The case $K_i < 0$ or $-\partial U/\partial a_i < 0$ yields also $\delta W < 0$ for $\delta a_i < 0$, which is again unstable. Therefore, it is necessary that $0 \leq \partial U/\partial a_i \leq 2\gamma_i$ or $0 \leq K_i \leq K_{c_i}$ at any time. Consequently, only the following variations of cracks lengths are admissible (3):

$$\text{for} \quad K_i = K_{c_i}: \quad \delta a_i \geq 0; \quad \text{for} \quad 0 < K_i < K_{c_i}: \quad \delta a_i = 0;$$

$$\text{for} \quad K_i = 0; \quad \delta a_i \leq 0 \qquad (5)$$

Instability of System of Griffith Cracks

Let us now explore the possibility of a critical state (bifurcation point of equilibrium path) in which there exists an adjacent equilibrium state characterized by different crack lengths $a_i + \delta a_i$ but the same loading parameter D. The equilibrium state is characterized by the condition $W_i = \partial W/\partial a_i = 0$, and this condition would have to hold both for a_j and $a_j + \delta a_j$ at the same D. Thus, there must be no change in W_i between these two states, i.e., $\delta W_i = \Sigma_j (\partial W_i/\partial a_j)\, \delta a_j = 0$ at constant D, and because $\partial W_i/\partial a_j = \partial^2 W_i/\partial a_i \partial a_j$, we thus obtain the critical state condition:

$$\sum_{j=1}^{n} W_{ij} \delta a_j = 0 \qquad (6)$$

$$\text{in which} \quad W_{ij} = \frac{\partial^2 W}{\partial a_i \partial a_j} = \frac{\partial^2 U}{\partial a_i \partial a_j} + 2\frac{\partial \gamma_i}{\partial a_i} \delta a_i H(\delta a_i) \qquad (7)$$

according to Eq. 1; $j = 1, \ldots n$ runs through the given set of crack lengths which are assumed to vary independently; H is the Heaviside step function; and for a homogeneous material, in which $\gamma_i = \gamma$ = constant, we have $W_{ij} = \partial^2 U/\partial a_i \partial a_j = -\partial G_i/\partial a_j$. Noting that $\partial U/\partial a_i = -K_i^2/E'$, Eq. 6 yields for a homogeneous material ($\partial \gamma_i/\partial a_i = 0$) the expression $W_{ij} = K_i\, \partial K_i/\partial a_j$, and because $W_{ij} = W_{ji}$ we have the following useful relation (3):

$$K_i \frac{\partial K_i}{\partial a_j} = K_j \frac{\partial K_j}{\partial a_i} \qquad (8)$$

Alternatively, we may obtain Eqs. 6–8 directly from the condition that a crack system that is in equilibrium ($\delta W = 0$) is stable if the second variation, $\delta^2 W$, is always positive, and unstable if it can be negative (3), i.e.

$$\delta^2 W = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} W_{ij} \delta a_i \delta a_j \left. \begin{array}{ll} > 0 & \text{stable (all admissible } \delta a_i) \\ = 0 & \text{critical (some admissible } \delta a_i) \\ < 0 & \text{unstable (some admissible } \delta a_i) \end{array} \right\} \qquad (9)$$

Obviously $2\delta^2 W = \Sigma\Sigma \delta a_i (W_{ij} \delta a_j)$, and this equals zero if Eq. 6 holds; so Eq. 6 provides the critical state according to Eq. 9. If matrix $\mathbf{W}_{ij}$ is positive definite, stability is assured. Note, however, that if it is not positive definite, the crack system may still be stable if those δa_i that make $\delta^2 W$ negative are not admissible according to Eq. 5. This last restriction marks an important difference from buckling problems. There indeed exist cases where W_{ij} is not positive definite yet the crack system is stable (3).

In our problem of cracked halfspace (Fig. 1), we have infinitely many cracks, but because we assume a periodic pattern of crack lengths we may restrict ourselves to only a few independent crack lengths, whose number we denote as N ($N = 2$ in Fig. 1). We may then consider the body to be limited by

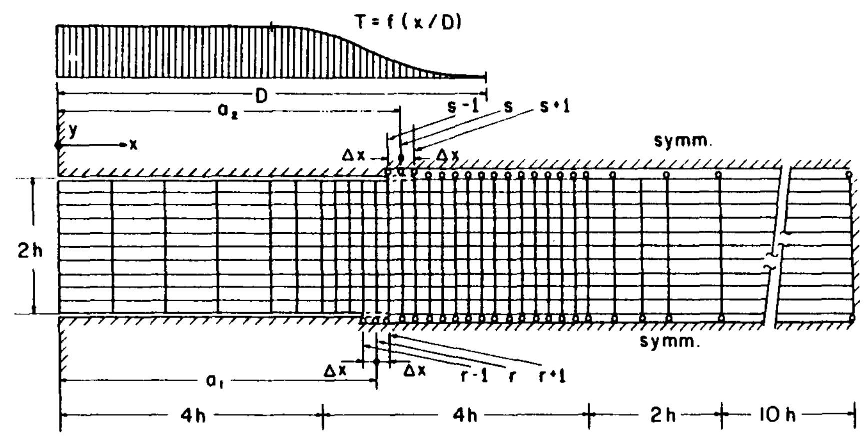

FIG. 2.—Finite Element Grid Used (Each Rectangle Consists of Four Triangles)

the symmetry lines (with sliding supports, as pictured in Fig. 2).

If Eq. 6 is satisfied for some nonzero δa_i that are admissible by Eq. 5, we have a critical state in which the crack lengths vary at no change of loading. Since Eq. 6 is a system of homogeneous linear equations, the critical state can be obtained only if some principal minor, $\det_n(W_{ij})$, of ($N \times N$) matrix $\mathbf{W}_{ij}$ vanishes ($n \leq N$). Positive definiteness of the quadratic form in Eq. 9 is assured if each principal minor $\det_n(W_{ij})$ is positive. Thus

$$\det_n(W_{ij}) \left. \begin{array}{ll} > 0 & \text{stable (all } n\text{, any admissible } \delta a_i) \\ = 0 & \text{critical (some } n\text{, admissible } \delta a_i) \\ < 0 & \text{unstable (some } n\text{, admissible } \delta a_i) \end{array} \right\} \qquad (10)$$

An important point is that we must check the principal minors of all sizes $n \leq N$. The eigenvector that corresponds to the vanishing of $\det_n(W_{ij})$ may be determined from Eq. 6 and must satisfy Eq. 5 to obtain instability.

When we consider, e.g., two independent crack lengths a_1 and a_2 (Fig. 1), one possibility is that only a_2 extends while a_1 remains constant. Then, we need to check $\det_n(W_{ij}) = W_{22} = \partial^2 W/\partial a_2^2$, and Eq. 9 yields $\delta^2 W = \frac{1}{2} W_{22}(\delta a_2)^2$. Noting that for a homogeneous material $W_{22} = \partial^2 U/\partial a_2^2 = -\partial G_2/\partial a_2$ where $G_2 = K_2^2/E'$, and assuming that $K_2 \neq 0$, we thus have the condition (3):

$$\frac{\partial K_2}{\partial a_2} \left.\begin{matrix} < 0 \text{ stable} \\ = 0 \text{ critical} \\ > 0 \text{ unstable} \end{matrix}\right\} \quad \text{(11)}$$

at D = constant, $\delta a_2 < 0$ ($K_2 = K_c$), and $\delta a_1 = 0$. This condition is indeed found to govern in our problem (3).

Furthermore, we must check the 2×2 determinant condition. The critical state is now characterized by $\det_2(W_{ij}) = W_{11}W_{22} - W_{12}W_{21} = 0$, and the eigenvector, $\delta a_1, \delta a_2$ is given by $W_{11}\delta a_1 + W_{12}\delta a_2 = 0$ and $W_{21}\delta a_1 + W_{22}\delta a_2 = 0$. Thus, if $K_1 > 0$ and $K_2 > 0$, we have $\delta a_2/\delta a_1 = -W_{11}/W_{21} = -W_{12}/W_{22}$, and because $W_{12} = W_{21}$ (Eq. 8) we get $W_{12} = W_{21} = \pm\sqrt{W_{11}W_{22}}$. For the system of parallel cracks in a halfspace, numerical calculations indicate that $\partial K_1/\partial K_2$ is always negative, i.e., $W_{12} > 0$. So, $W_{12} = W_{21} = \sqrt{W_{11}W_{22}}$, and we have for the critical state

$$W_{11}W_{22} - W_{12}^2 = 0 \quad \text{at} \quad \frac{\delta a_2}{\delta a_1} = \sqrt{\frac{W_{11}}{W_{22}}} \quad \text{(12)}$$

provided that $\delta a_1 \neq 0$ and $\delta a_2 \neq 0$ are admissible increments according to Eq. 5. Obviously, if $K_1 = K_2 = K_c$, the signs of W_{11} and W_{22} are both positive (Eq. 11), and so $\delta a_2/\delta a_1$ is obtained from Eq. 12 as negative (3). This, however, violates the admissibility condition in Eq. 5 because $K_1 = K_2 = K_c$. Therefore, violation of the 2×2 determinant condition does not indicate instability of our system of parallel cooling cracks when the cracks are equally long (see Ref. 3; this fact and Eqs. 6–12 were obtained in the unpublished work mentioned in the Acknowledgment).

The foregoing conclusion is contingent upon the negativeness of $\partial K_1/\partial a_2$. There exist other problems where $\partial K_1/\partial a_2$ can be positive (3), yielding $\delta a_1/\delta a_2 > 0$, and then the 2×2 determinant condition may indicate unstable crack extension.

In case that $K_1 = 0$ and $K_2 = K_c$, it is interesting to examine the 2×2 determinant condition again. Because $\partial U/\partial a_i = -K_i^2/E'$, we get $\det_2(W_{ij}) = 4K_1K_2(K_{1,1}K_{2,2} - K_{1,2}K_{2,1})/E'^2$ where $K_{i,j} = \partial K_i/\partial a_j$. Thus, at $K_1 = 0$ we must always have $\det_2(W_{ij}) = 0$ (5). To determine the corresponding eigenvector, we first note that, according to Eq. 8, $K_2K_{2,1} = K_1K_{1,2} = 0$ if $K_1 = 0$. Thus, we see that (5):

$$\det_2(W_{ij}) = 0; \quad \frac{\partial K_2}{\partial a_1} = 0 \quad (\text{at } K_1 = 0) \quad \text{(13)}$$

The eigenvector corresponding to the 2×2 determinant condition satisfies Eq. 6, i.e., $W_{22}\delta a_2 + W_{21}\delta a_1 = 0$. Therefore, $K_2K_{2,2}\delta a_2 = -K_2K_{2,1}\delta a_1$, and so, assuming that $\partial K_2/\partial a_2 \neq 0$ and substituting Eq. 13, we conclude that the eigenvector, $\delta a_1, \delta a_2$, which corresponds to Eq. 13, is (5):

$$\delta a_1 < 0, \ \delta a_2 = 0 \quad (\text{at } K_1 = 0 \text{ and constant } D) \quad \text{(14)}$$

which is admissible according to Eq. 5. So, here we have found a critical state of parallel cracks in a halfspace that is characterized by vanishing of the 2×2 determinant.

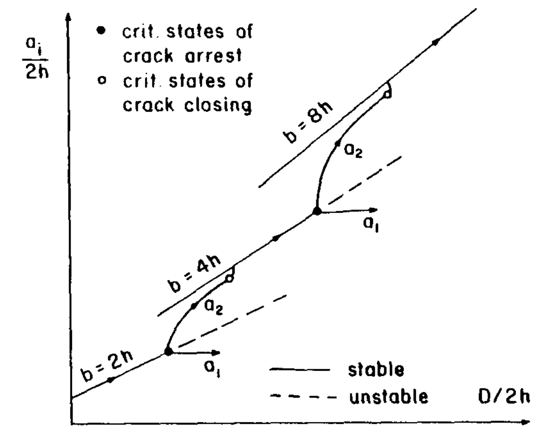

FIG. 3.—Bifurcations, Post Critical Paths, and Critical States of Crack Arrest

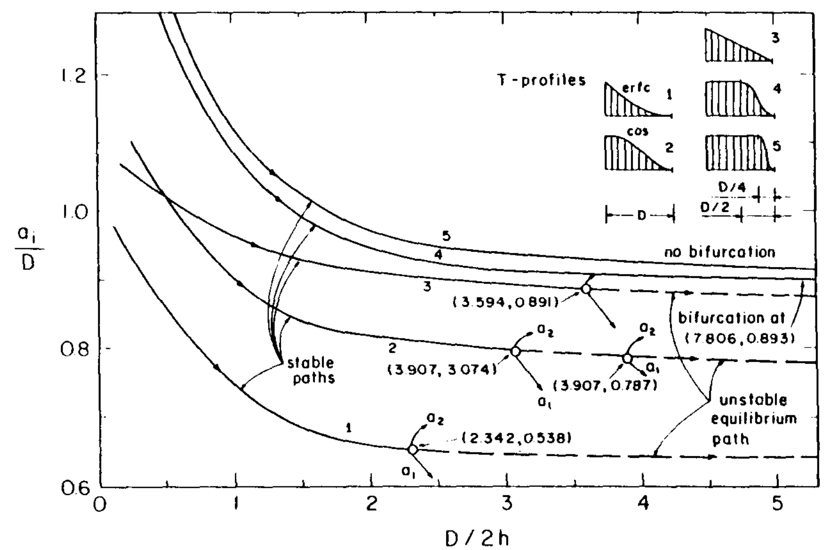

FIG. 4.—Results from Fig. 1 Replotted in Relation to Cooling Penetration Depth D

Note that only two independent crack lengths have been assumed in this analysis. Whether there could be three independent crack lengths before the state $K_1 = 0$ is reached has not been checked because this question seems unimportant.

We now arrive at the following picture of the system of parallel cooling cracks in a halfspace (3). The cracks, which may be assumed to start at equal spacing $b (b = 2h)$ that must not be smaller than the minimum possible spacing (see Appendix I), grow at first at equal length as the penetration depth of cooling D increases. To examine possible transition to a case of alternating crack lengths, two crack lengths a_1, a_2 and the halfspace strip within the symmetry lines (Figs. 1 and 2) may be considered. The 2×2 determinant vanishes first but this does not signify instability (3) as explained before. Subsequently, $\partial K_2/\partial a_2$ (at constant a_1 and D) vanishes, and this represents a critical state of crack arrest (bifurcation of equilibrium path). Further crack extension at $a_1 = a_2$ represents an unstable, impossible path because $\partial K_2/\partial a_2 > 0$. As shown qualitatively in Ref. 3 and quantitatively in Ref. 5, cracks a_2 at critical state jump ahead at constant a_1 and constant temperature (constant D) (Fig. 3). Then cracks a_2 gradually extend in a stable manner with increasing D while cracks a_1 stop growing and K_1 gradually diminishes at increasing a_1 and D until K_1 becomes zero; this represents the second critical state of crack closing (Fig. 3). In this state, which corresponds to vanishing 2×2 determinant, cracks a_1 suddenly close over a finite length at constant a_2 and constant temperature (constant D) (Fig. 3), and subsequently cracks a_2 grow in a stable manner at increasing D. Because cracks a_1 are closed, this is equivalent to the initial situation of a single crack length, but with a doubled spacing. The entire process is then repeated at doubled spacing, then at quadrupled spacing, etc.

Numerical Analysis

The finite element method, with grids exemplified in Fig. 2 was used to solve the problem. All rectangular finite elements consist of four constant strain triangles and the displacements of the central node are eliminated in advance by static condensation. The stress intensity factors and their derivatives are calculated from the first and second derivatives of the total strain energy of the system (7,8,13,14) which was evaluated as

$$U = \int_V \left(\frac{1}{2} C_{ijkm} \epsilon_{ij} \epsilon_{km} - \sigma^o_{ij} \epsilon_{ij} \right) dV$$

$$= \int_V \left[G \epsilon_{ij} \epsilon_{ij} + \frac{G\nu}{1 - 2\nu} (\epsilon_{kk})^2 - \frac{E\alpha}{1 - 2\nu} (T - T_o) \epsilon_{kk} \right] dV \qquad (15)$$

in which the first expression applies for anisotropic materials and the second one for isotropic materials; G = elastic shear modulus; E = Young's modulus; ν = Poisson's ratio; T_o = initial uniform temperature; T = current temperature; C_{ijkm} = anisotropic elastic moduli; σ^o_{ij} = thermal stresses produced by $T - T_o$ at $\epsilon_{ij} = 0$; indices i, j, k, m refer to Cartesian axes $x_1 = x$, $x_2 = y$, $x_3 = z$; and summation over $i = 1, 2, 3$ is implied by repeated indices. The derivatives $\partial U/\partial a_i$ and $\partial^2 U/\partial a_i \partial a_j$ are calculated from their central finite difference approximations:

$$\left(\frac{\partial U}{\partial a_2} \right)_{r,s} = \frac{U_{r,s+1} - U_{r,s-1}}{2\Delta x} \quad \left(\frac{\partial^2 U}{\partial a_2^2} \right) \approx \frac{U_{r,s+1} - 2U_{r,s} + U_{r,s-1}}{\Delta x^2}$$

$$\left(\frac{\partial^2 U}{\partial a_1 \partial a_2} \right)_{r,s} \approx \frac{U_{r+1,s+1} - U_{r-1,s+1} - U_{r+1,s-1} + U_{r-1,s-1}}{4\Delta x^2} \qquad (16)$$

in which the subscripts refer to cracks terminating at adjacent grid nodes $r - 1$, r, $r + 1$, and $s - 1$, s, $s + 1$ shown in Fig. 1; Δx = spacing of these nodes.

To get an idea of accuracy, several cases of known K value have been solved by finite elements. For example for the case of equally long cracks (Fig. 1) with $a_1 = a_2 = 2h = 10$ m and $\Delta x = 2$ m, grid depth of 40 m and a uniform stress $\sigma_y = 1$ at infinity, the solution is known to be (16) $K_1/\sqrt{h} = 0.999$ while the finite element calculation provided 1.01. For the case of a strip of length $2h$ and width $2h$, which is subject to a uniform uniaxial tension $\sigma_y = 1$ at ends and contains in the midlength cross section two symmetrical cracks penetrating from the boundary to the depth of $a_1 = h/2$, the finite element method gave $K_1\sqrt{2/h\pi} = 1.311$ while more accurate methods previously provided 1.311 and 1.330 (16).

In calculations for the hot dry rock geothermal energy scheme, the material properties typical of granite have been used: linear thermal dilatation coefficient $\alpha = 8 \times 10^{-6}/°C$; Young's modulus $E = 37{,}600$ MN/m^2; Poisson ratio $\nu = 0.305$; and surface energy density $\gamma = 104$ J/m^2, which yields $K_c = 2.94$ MN$m^{-3/2}$. The halfspace surface was assumed to be cooled by 100° C, i.e., $T_s - T_o = 100°$ C, and the initial crack spacing was $b = 2h = 1$ m. The results can be, however, transformed to other spacings and other K_c values because the expression

$$\kappa_i = \frac{K_i(1 - \nu)}{E\alpha(T_o - T_s)\sqrt{2h}} \qquad (17)$$

is independent of E, ν, $2h$, α, and $T_o - T_s$ and may be regarded as the nondimensional stress intensity factor; for derivation see Ref. 5. (Note that the use of $1 - 2\nu$ instead of $1 - \nu$, later made by some researchers, would be incorrect and would result in a 78% error; see Ref. 5.)

The diagrams of crack length a_1 versus cooling penetration depth D (equilibrium path of the crack system) are calculated as follows. We fix the crack lengths a_2 and a_1, choose various values of D and run the finite element program to obtain K_2. Then we use Newton iteration to find the D value which yields $K_2 = K_c$ (given critical value). We repeat this for various lengths a_2 (plotting the points in Figs. 1, 3, 4, 5), and for each a_2 we also evaluate $\partial K_2/\partial a_2$ using Eq. 16 (by running the program also for $a_2 + \Delta x$ and $a_2 - \Delta x$). Then we use Newton iteration with respect to a_2 so as to finally locate the a_2 value that gives $\partial K_2/\partial a_2 = 0$. For calculation of the post-critical path and the second critical state, see Ref. 1.

Effect of Temperature Profile (Loading)

The previous conclusion on the doubling of crack spacing appears to be relatively favorable for heat withdrawal from the secondary thermal cracks in the hot dry rock mass. The spacing of the opened cracks indeed increases, roughly in proportion to the penetration depth D. For the error function profile of temperature, it has been calculated (5) that the opening width, w, of the leading cracks fluctuates roughly around the values $w = (a - x/D)^2 \alpha (T_s - T_o) D/2$ in which α = thermal dilatation coefficient; T_o = initial uniform temperature; and T_s = temperature to which the halfspace surface is cooled) and the total vertical flux of water along the secondary thermal cracks is roughly between 9% and 12% of the vertical flux in the main crack and does not depend on D (1). However, these results have been obtained assuming the error function profile of temperature, which is correct only if water circulation in the secondary cracks has a negligible effect on temperatures and all heat is transported by conduction in rock. This assumption of course cannot be true because the vertical water flux in the secondary cracks is not negligible (1).

The water circulating in the thermal cracks provides additional cooling of

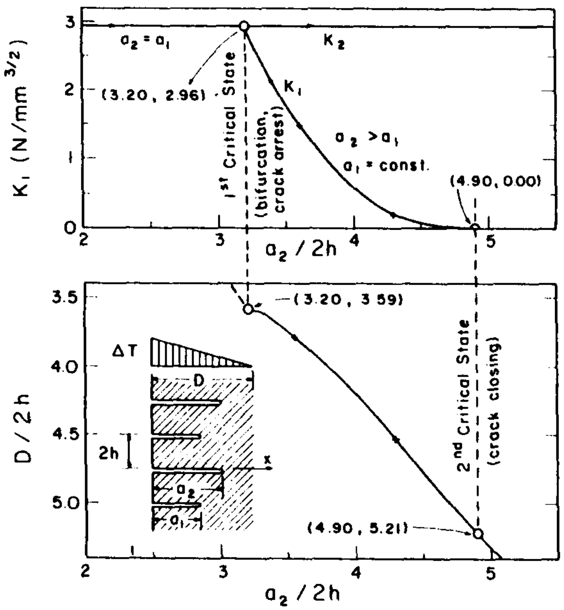

FIG. 5.—Post-Critical Behavior and Second Critical State for Linear Temperature Profile

the rock below the halfspace surface. We neglect here the fact that the temperature distribution becomes two-dimensional and consider that the same (averaged) temperature profile, depending only on x/D, may be considered to exist along all normals y = constant. This profile must be at small depth x more uniform that the error function profile whereas near the cooling front it must have a steeper gradient. Therefore, we have arbitrarily selected several temperature profiles shown in Fig. 4. Profile 1 is $\Delta T(x) = \text{erfc}\,(x\sqrt{3}/D)$, where erfc is the complementary error function, the case already mentioned. Profile 3 is linear, and profiles 2, 4, and 5 have a temperature drop in the form of a halfwave of cosine function extending over lengths D, $D/2$, and $D/4$ from the cooling front, respectively.

The equilibrium crack lengths and the critical states of crack arrest (bifurcation points of equilibrium paths) obtained for various temperature profiles by finite element calculations are plotted in Figs. 1 and 4. It is seen that generally the steeper the cooling front, the longer are the cracks for the same cooling penetration depth, and the deeper is the location of the critical point at which every other

TABLE 1.—Critical States of Crack Arrest for Various Temperature Profiles

Profile (1)	x_c/D (2)	a_2/D (3)	$a_2/2h$ (4)	Eq. 19 (5)	Eq. 18 (6)
1	≈0.250	0.538	1.53	0.539	1.59
2	0.297	0.787	3.07	0.783	2.51
3	0.333	0.891	3.20	0.868	3.89
4	0.383	0.893	6.97	0.899	8.94
5	0.439	—	?(>30)	0.900	50

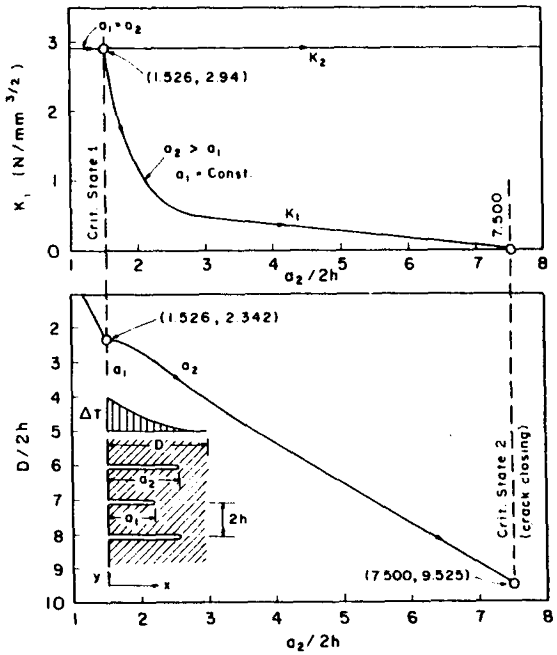

FIG. 6.—Post-Critical Behavior and Second Critical State for Error Function Profile of Temperature

crack would stop growing. For profile 5, which has a very steep cooling front, no critical state has been found, i.e., the cracks continuously extend at equal length, equal spacing, and equal width.

To organize the results, can we introduce some simple approximate characteristic of the steepness of the temperature profile? The parameter x_c/D where x_c is the distance of the centroid of the profile (i.e., of the cross-hatched area in Fig. 1) from the halfspace surface is suitable for this purpose. This parameter can vary between 0 (for a temperature drop that is concentrated at the surface) and 0.5 (for a temperature drop that is concentrated at cooling penetration depth). The results are tabulated as a function of x_c/D in Table 1. It is seen that the critical state values of $a_2/2h$ and a_2/D vary monotonically as a function of x_c/D, which confirms the suitability of using x_c/D as the profile characteristic. The following formulas approximate the results tabulated in Table 1:

$$\left(\frac{a_2}{2h}\right)_{\mathrm{crit}} = \left(\frac{0.29}{0.48 - x_c/D}\right)^2 \quad \text{if} \quad \frac{x_c}{D} < 0.48; \quad \text{else} \quad \infty \qquad (18)$$

$$\left(\frac{a_2}{D}\right)_{\mathrm{crit}} = 0.9 - 107\left(0.4 - \frac{x_c}{D}\right)^3 \quad \text{if} \quad \frac{x_c}{D} \leq 0.40; \quad \text{else} \quad 0.9 \qquad (19)$$

More accurate formulas could no doubt be obtained by using two variables: x_c/D, and the second moment of the temperature profile divided by its area

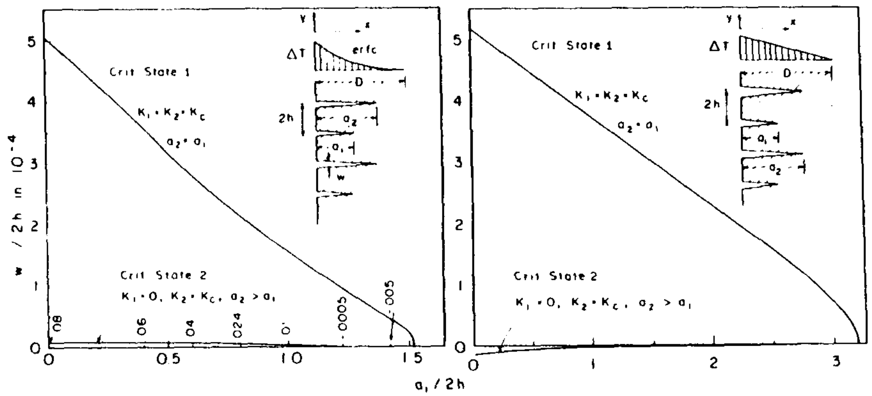

FIG. 7.—Profiles of Crack Width at the Critical States

and by D^2. By including profile moments of high enough order, one could of course obtain formulas as accurate as desired.

The critical stages of crack closing (second critical states) were calculated in Ref. 5 for the error function profile. How are these critical states influenced by the temperature profile? This is apparent from Figs. 5 and 6 where some finite element results are plotted for the linear profile of temperature and for the error function profile. For the former profile, the critical state of crack closing is seen to occur at $a_2/2h = 4.90$, whereas for the latter profile (5) it is found to occur at $a_2/2h = 2.56$. So, we see that the effects of temperature profile on the critical states of crack closing and of crack arrest are similar. The deeper the centroid of the profile, the deeper are located both critical states. The profiles of crack width at critical states are plotted in Fig. 7. We see that, approximately, the crack becomes closed over its entire length when $K_1 = 0$ is attained.

Now that we have seen the numerical results we consider their practical

consequences. We see that the results are not particularly favorable for the heat transfer effectiveness of the secondary thermal cracks in the hot dry rock geothermal scheme.

We have already noted that, because of cooling by water circulating in the secondary cracks, the temperature profile must have a steeper front than the error function profile. However, we see that for a steeper front things get worse; i.e., the point where the crack spacing is doubled (which means roughly a doubling in the opening width of the leading cracks), is pushed deeper into the halfspace. This tends to decrease the water flux and the heat withdrawal rate in the secondary cracks. We must keep in mind, though, that these conclusions are reached under certain simplifying assumptions, stated before.

The temperature profile that gets established is obviously somewhere between

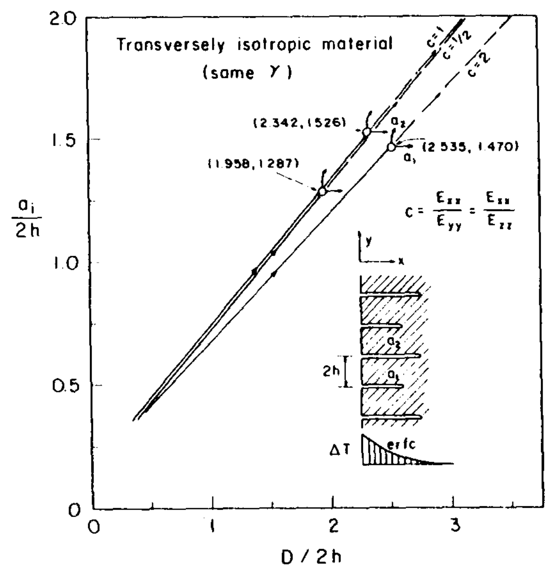

FIG. 8.—Critical States of Crack Arrest and Bifurcation of Equilibrium Path for Error Function Profile of Temperature and Various Orthotropic Materials

profile 1 which implies no heat transport through the secondary cracks yet gives the highest water flux in these cracks, and profile 5, which implies the largest heat transfer from rock to the secondary cracks yet gives the smallest crack spacing and, therefore, the smallest crack width and the smallest water flux in the secondary cracks. The present results can be used in the analysis of the water flow and heat convection to determine the actual temperature profile, but this is beyond the scope of this work.

Similar phenomena are observed for shrinkage cracks in concrete. Here, the thermal strain $\alpha\Delta T$ must be replaced by linear shrinkage strain ϵ_{sh}, and the problem is perfectly analogous, except that the error function distribution of ϵ_{sh} does not satisfy the diffusion equation precisely because the dependence of ϵ_{sh} on pore relative humidity is nonlinear and the diffusivity is dependent on pore humidity (1). By contrast to the cases of interest for hot dry rock geothermal energy, the profile is usually steeper near the surface than the error function profile and then rather flat at greater depth below the surface. Extrapolating the results from Figs. 1 and 4 and Table 1, we may, therefore, expect that the critical state of shrinkage crack arrest is reached at a smaller crack depth than it is for the error function profile.

Thus, the spacing of drying shrinkage cracks in plain concrete would increase more rapidly than the spacing of our cooling cracks in rock. In reinforced concrete, however, the instabilities are completely altered by the effect of reinforcement (6).

Effect of Material Anisotropy

The preceding calculations are based on isotropic material properties. However, the granitic rocks usually exhibit significant orthotrophy. Therefore, the critical

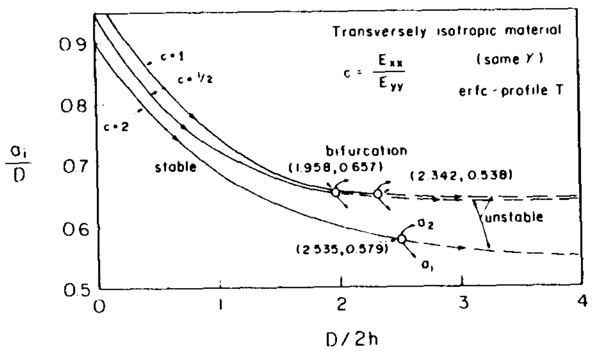

FIG. 9.—Results from Fig. 6 Replotted in Relation to Cooling Penetration Depth D

states have been calculated by finite elements also for several cases of tranverse isotropy whose axis is normal to halfspace surface. The material compliance matrix was the same as before except that the uniaxial compliances $1/E_{yy}$ and $1/E_{zz}$ in the y and z directions were altered. The uniaxial compliance $1/E_{xx}$ was kept equal $1/E$, all three shear compliances were kept equal $1/G$, and the off-diagonal terms were kept equal $-\nu/E$. Cases in which the transverse stiffnesses, E_{yy}, E_{zz}, are either higher or smaller than E_{xx} have been considered. The crack extension criterion was defined in terms of 2γ, the specific energy of crack extension, which was kept the same as before.

Some results for transversely isotropic rock are plotted in Figs. 8 and 9. Although the transverse stiffnesses differ by a factor of 2, the effect of anisotropy is seen to be relatively small. A likely and practical situation is that of a halfspace whose weaker direction is along the normal x (because the main hydraulic fracture, whose wall we treat as the halfspace surface, is likely to form in a plane normal to the weak direction). In this case the difference from the solution for isotropic properties is particularly small (Figs. 8 and 9).

Summary and Conclusions

When a system of parallel equidistant cooling cracks propagates into a halfspace, it reaches at a certain depth a critical state at which the growth of every other crack gets arrested. Later these cracks reach a second critical state at which they close. The intermediate cracks, at doubled spacing, open about twice as wide and advance further as cooling penetrates deeper. Determination of the critical states requires calculation of the derivatives of the stress intensity factors with regard to crack lengths, which is here accomplished by finite element method. It is found that the crack depth-to-spacing ratio at which the critical state is reached is extremely sensitive to the temperature profile. This ratio greatly increases as the cooling front becomes steeper. The effect of transverse isotropy of the material upon the location of critical states is found to be relatively mild. The results are of interest for one recently proposed geothermal energy scheme for hot dry rock, as well as for shrinkage cracks in concrete.

Acknowledgment

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Appendix I.—Minimum Crack Spacing

Cracks may be expected to initiate with the minimum possible spacing, which may be determined as the spacing for which the surface energy of the cracks equals the drop in strain energy, ΔU, caused by a complete relief of the stress on all planes z = constant (1) (as proposed in the work mentioned in the Acknowledgment). Approximating the initial temperature profile as parabolic, the strain energy in absence of cracks is $U_0 = \int_0^D \frac{1}{2}\, \alpha\, \Delta T(x)\, (\sigma_y^0 + \sigma_z^0)\, dx$ in which $\Delta T(x) = (1 - x/D)^2\, (T_s - T_0)$, $\sigma_y^0 = \sigma_z^0 = \alpha\, \Delta T(x)\, E/(1-\nu)$ = thermal stress which is created in absence of cracks (1). By integration, $U_0 = 0.2\, D\, \alpha^2\, (T_s - T_0)^2\, E/(1-\nu)$ per unit area of halfspace surface. The strain energy, U_1, that is left after the cracks are introduced must now be subtracted. Determining it we must not overlook that a significant part of U_1 is due to normal stress σ_z' in direction z parallel to halfspace surface as well as to crack planes. If all stress in the (x, y) planes were relieved by cracking, the remaining energy would be due solely to uniaxial thermal stress $\sigma_z' = \alpha\, \Delta T(x) E$ and would equal $U_1 = \int_0^D \frac{1}{2}\, \alpha\, \Delta T(x)\, \sigma_z'\, dx = 0.1 D\, \alpha^2\, (T_s - T_0)^2\, E$. Consequently, the strain energy relief cannot be larger than $\Delta U = U_0 -$

$U_1 = 0.1D\,\alpha^2\,(T_s - T_o)^2\,E(1+\nu)/(1-\nu)$. Then, if $a_1 = a_2$, we must have $2\gamma\,a_1 \leq b\,\Delta U$, which yields (1):

$$b = 2h \geq \frac{20(1-\nu)\gamma}{(1+\nu)\,\alpha^2\,(T_s - T_o)^2\,E}\,\frac{a_1}{D} = 10\left(\frac{(1-\nu)\,K_c}{\alpha\,(T_s - T_o)\,E}\right)^2 \frac{a_1}{D} \quad \ldots \quad (20)$$

According to Fig. 1 and Ref. 5, we may estimate that initially the ratio a_1/D is not less than about 1.5. For this value and properties of granite, Eq. 20 yields minimum spacing of about 7 cm. Note that this result is essentially independent of D. This means that later, as the crack spacing doubles, smaller and smaller fraction of the strain energy is available for creating cracks.

Note again that the use of $1 - 2\nu$ rather than $1 - \nu$ in Eq. 20, later made by some, would be incorrect.

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