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Stability and post-critical growth of a system of cooling or shrinkage cracks

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ABSTRACT

When a system of parallel equidistant cooling cracks propagates into an elastic halfspace, it reaches at a certain depth of the cracks a critical point of instability, and the equilibrium path of the system bifurcates. Further extension of equally long cracks is unstable and impossible. The stable post-critical path consists of extension of every other crack upon further cooling, initially with a crack jump at constant temperature, while the intermediate cracks stop growing and gradually diminish their stress intensity factor until it becomes zero. This represents a second critical state at which these intermediate cracks suddenly close over a finite length at no change in temperature and at constant length of the leading cracks. Subsequently, as the cooling front further advances, the leading cracks grow at equal length until they again reach a critical state, at which every other crack stops growing, and the process in which the crack spacing doubles is repeated. In this manner, the spacing of the opened cooling cracks fluctuates around roughly the one-half value of the cooling penetration depth.

The instability is determined by the sign of the second variation of the work needed to create the cracks, which leads to positive definiteness conditions for a matrix consisting of partial derivatives of the stress intensity factors with regard to crack lengths, subjected to admissibility conditions for the eigenvector of crack length increments. The first initial state of crack arrest is characterized by the vanishing of the diagonal element of the matrix, while the second critical state of crack closing is characterized by the vanishing of the determinant of this matrix.

The critical states and the postcritical crack growth are calculated numerically by finite elements. The solution is applied to the cooling of a hot granite mass, the cracking of which is important for one recently proposed geothermal heat extraction scheme. The solution is also of interest for drying shrinkage cracks, especially in concrete.

1. Introduction

When a brittle elastic solid is cooled from its surface, a system of cracks normal to the surface may be produced. The same type of cracking takes place in brittle porous materials, such as concretes, when they dry and shrink. Restricting attention to an elastic halfspace in plane strain, we obtain as one possible theoretical solution a system of equidistant parallel cracks of spacing $2h$. Seeking the solution according to the usual fracture mechanics criteria (satisfying the field equations, all boundary conditions, and criticality of the stress intensity factor), we find, however, that the solution is not unique. The length of the cracks, a_i ($i = 1, 2, 3, \dots$), may vary periodically from one crack to the next, and many such solutions exist for the same temperature or the same shrinkage stress profile.

What is the reason for the lack of uniqueness? Obviously, it must be some sort of instability of the crack system. The stability analysis was first carried out in [1] (based on [2]), in which the conditions of stability of a system of Mode I cracks propagating along given paths were determined by analyzing the second variation of the work needed to create the cracks, as well as by formulating conditions of adjacent equilibrium (a summary of this development was given in [3]. The later work of some other authors on this problem was commented upon in [1].

In this study, after briefly summarizing the stability conditions derived in [1] and [2], we will determine the critical states and bifurcations of equilibrium path for a system of parallel equidistant cracks in a halfspace, and we will also determine the postcritical behavior. Although the cases treated here can be solved (more accurately, for the same computer cost) by singular integral equations, we adopt the finite element approach because it is simple, general, and can be readily extended to more complicated surface geometries, arbitrary crack paths, as well as anisotropic, nonhomogeneous, and nonlinear solids.

2. Stability conditions for a crack system

Consider a brittle elastic body which contains a system of m cracks of length a_i ($i = 1, 2, \ldots, n$) propagating in Mode I along given paths. Stability of this system may be analyzed on the basis of the work, W, that has to be supplied to produce the cracks; $W = U(a_i, D) + \Sigma_i \int 2\gamma_i \, da_i$ where D is some loading parameter, which in our problem represents the cooling penetration depth; $2\gamma_i$ = given specific energy of extension of the i^{th} crack, a material property which is constant ($\gamma_i = \gamma$) if the solid is homogeneous; and U = strain energy of the elastic body. By analyzing the sign of the second variation $\delta^2 W$, it was shown [1] that the crack system is stable if, for any admissible variations δa_i,

$$2\delta^2 W = \sum_i \sum_j W_{ij} \delta a_i \delta a_j > 0; \quad W_{ij} = W_{ji} = \frac{\partial^2 U}{\partial a_i \, \partial a_j} + 2 \frac{\partial \gamma_i}{\partial a_i} \delta_{ij} H(\delta a_i) \tag{1}$$

in which H = Heaviside step function and $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ (Kronecker delta). The admissible variations [1] are:

$$\begin{aligned} &\text{for } K_i = K_c\text{: } \delta a_i \geq 0; \quad \text{for } K_i = 0\text{: } \delta a_i \leq 0; \\ &\text{for } 0 < K_i < K_c\text{: } \delta a_i = 0 \end{aligned} \tag{2}$$

where K_i is the stress intensity factor of the i^{th} crack and K_c = critical stress intensity factor (material property); K_i is here understood as $\lim(\sigma_y \sqrt{2\pi r})$ for $r \to 0$, not $\lim(\sigma_y \sqrt{r})$, where r is the distance from the crack tip and σ_y = normal stress on the crack extension line. It is well known [4] that $K_i^2 = -E' \partial U / \partial a_i$ and $K_c^2 = 2\gamma_i E'$; and for plane strain $E' = E/(1 - \nu^2)$ where E = Young's modulus, ν = Poisson ratio. Because $\partial^2 U / \partial a_i \partial a_j = \partial^2 U / \partial a_j \partial a_i$, we have the identity [1, 2]:

$$K_i \frac{\partial K_i}{\partial a_j} = K_j \frac{\partial K_j}{\partial a_i} \tag{3}$$

which must be always satisfied.

For a system with two independent crack lengths a_1 and a_2, Eqn. (1) leads to the stability conditions [1, 2]:

$$W_{22} > 0 \text{ (or } W_{11} > 0) \tag{4}$$

and

$$\det(W_{ij}) > 0 \tag{5}$$

which, however, need to be satisfied only for *admissible* δa_i [1, 2]. This restriction marks a fundamental difference from the well-known stability problems in buckling. The crack system reaches a critical state when

$$W_{22} = 0 \text{ (or } W_{11} = 0) \quad \text{or} \quad \det(W_{ij}) = 0 \tag{6}$$

and the corresponding eigenvector δa_i is determined [1, 2] from the condition $\delta^2 W = 0$, i.e.,

$$\sum_i \sum_j W_{ij}\delta a_i \delta a_j = 0 \tag{7}$$

Let us now consider the case of a system of straight parallel equidistant cracks ($K_1 = K_2 = K_c$, $\gamma_i = \gamma = \text{const.}$) which are normal to the halfspace surface and are initially equally long, i.e., $a_1 = a_2$ and $K_1 = K_2$ (Fig. 1). It has been shown [1, 2] that Eqn. (5) can be violated at $a_1 = a_2$ only for eigenvectors $(\delta a_1, \delta a_2)$ such that $\delta a_2/\delta a_1 = -1$, which are inadmissible according to (2). So, the determinant condition (5) is of no concern if $a_2 = a_1$, and we have Eqn. (4) as a sufficient condition for stability. Because $\partial U/\partial a_2 = -K_2^2/E'$, the condition $W_{22} > 0$ or $\partial^2 U/\partial a_2^2 > 0$ is equivalent to [1, 2]:

$$[\partial K_2/\partial a_2]_{a_1=\text{const.}} < 0 \tag{8}$$

(at constant D). From $W_{21}\delta a_1 + W_{22}\delta a_2 = 0$ (7), the eigenvector which corresponds to the critical state $\partial K_2/\partial a_2 = 0$ is obtained in the form

$$\delta a_2 > 0, \quad \delta a_1 = 0 \quad \text{(at constant } D) \tag{9}$$

which is admissible according to Eqn. (2). Thus, as shown in [1], the critical state* $\partial K_2/\partial a_2 = 0$ is reached at a sufficiently large penetration depth of cooling, D.

At this critical state the equilibrium path of the crack lengths as a function of D bifurcates, and further equilibrium crack extension at $a_1 = a_2$ is unstable [1]. The stable postcritical crack growth consists of cracks a_2 advancing at constant D immediately after the bifurcation point ($\partial a_2/\partial D \to \infty$), while cracks a_1 cease growing ($a_1 = \text{const.}$) and gradually close [1] as their stress intensity factor gradually diminishes ($\partial K_1/\partial D < 0$) until it becomes zero.

When the value $K_1 = 0$ is reached, do the shorter cracks a_1 close only at their tip or do they close along a finite segment of the crack length? The condition $\partial K_2/\partial a_2$ is

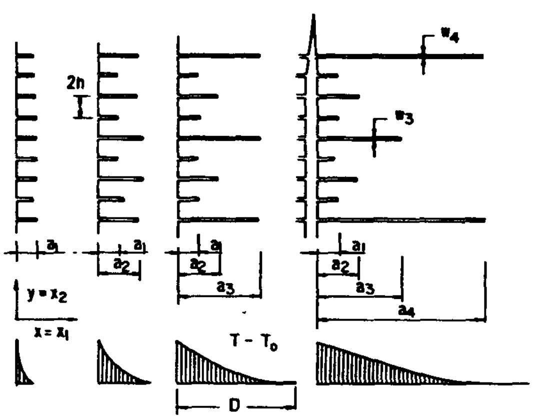

Figure 1. System of parallel cracks in a halfspace.

* Note that at critical state $\partial K_2/\partial a_2 = 0$ we have $\det(W_{ij}) = W_{11}W_{22} - W_{12}W_{21} = W_{22}^2 - W_{12}^2 < W_{22}^2 = 0$, i.e., $\det(W_{ij})$ is negative.

obviously not pertinent in this case because the corresponding eigenvector (δa_1, δa_2) is such that $\delta a_2 > 0$ and $\delta a_1 = 0$. Let us examine the determinant condition; $\det(W_{ij}) = W_{11}W_{22} - W_{12}^2$ and because $\partial U/\partial a_i = -K_i^2/E'$ we have $\det(W_{ij}) = 4K_1K_2(K_{1,1}K_{2,2} - K_{1,2}K_{2,1})/E'^2$ where $K_{i,j} = \partial K_i/\partial a_j$. Thus, at $K_1 = 0$ we must always have

$$\det(W_{ij}) = 0 \tag{10}$$

To determine the corresponding eigenvector, we first note that, according to identity (3), $K_2K_{2,1} = K_1K_{1,2}$, and because here $K_1 = 0$, we must have (at constant D):

$$[\partial K_2/\partial a_1]_{a_2=\text{const.}} = 0 \tag{11}$$

when crack a_1 closes. From Eqn. (7), $W_{22}\delta a_2 + W_{21}\delta a_1 = 0$, which yields the relation $K_2K_{2,2}\delta a_2 = -K_2K_{2,1}\delta a_1$, and assuming that $\partial K_2/\partial a_2 \neq 0$ (which is later confirmed by Fig. 7) and substituting (11) we conclude that this relation is satisfied by the eigenvector

$$\delta a_1 < 0, \quad \delta a_2 = 0 \text{ (at constant } D) \tag{12}$$

which at the same time meets the admissibility conditions (2). It is, of course, obvious directly from Eqn. (11) that cracks a_1 can close over some length δa_1 without any change in K_2 or D. This reveals the tendency of cracks a_1 to close not only at the tip but over a finite segment of the crack length.

This conclusion can be anticipated also from the fact that at $K_1 = 0$ the crack opening profile near the tip ceases to be elliptic and the crack surfaces become tangent to each other at the tip. (This is because the curvature radius at the crack tip is $4(1-\nu^2)K_i^2/E'^2$ [5].) Thus, an infinitely small further decrease of K_1 must cause the crack surfaces to contact each other over a finite length. It is interesting to note that just before reaching the second critical state (10), $\det(W_{ij})$ is negative. This is indicated by the fact that it is negative at the first critical state.

The following picture now emerges. First, the parallel cracks of equal length reach a critical state of crack arrest, which corresponds to vanishing of the diagonal terms in matrix W_{ij} (Eqns. (8), (9)). Every other crack (of length a_2) suddently jumps ahead at constant temperature, while the intermediate cracks (of length a_1) stop growing. With the increase of D the stress intensity factor (as well as the opening width) of the shorter cracks a_1 gradually decreases as the intermediate cracks a_2 extend. When K_1 vanishes a second critical state, corresponding to vanishing of the determinant of W_{ij}, is reached (Eqns. (10)–(12)). Cracks a_1 then close over a finite length during an infinitesimal extension of leading cracks a_2 at constant temperature. Upon further increase of D cracks a_2 grow gradually, and later a critical state of crack arrest is again reached for each other of these cracks, and the process is repeated. The precise behavior will be quantified by the numerical analysis which follows.

3. Material parameters, thermal load, and minimum spacing

For all numerical calculations, the following values of numerical parameters, typical of granite [5], have been considered: linear thermal dilatation coefficient $\alpha = 8 \times 10^{-6}$ per °C, Young's modulus $E = 37\,600$ MN/m^2, Poisson ratio $\nu = 0.305$, mass density 2650 kg/m^3, heat capacity $C_p = 954\ J(\text{kg °C})^{-1}$, heat conductivity $k = 2.72\ J(\text{m s °C})^{-1}$, and surface energy density $\gamma = 104$ J/m^2, which yields $K_c = 2.94$ MN m$^{-3/2}$. The half-space is assumed to be initially (at $t = 0$) at temperature $T = T_0$, and for $t > 0$ the surface is cooled to constant temperature T_s.

As non-dimensional geometric parameters, $a_1/2h$, $a_2/2h$ and $D/2h$ are a logical choice (Fig. 1). To determine the nondimensional stress intensity factor, we assume that the halfspace is first cooled in absence of cracks. Since the halfspace can freely contract in the x-direction we have $\sigma_x = 0$ (and not $\epsilon_x = 0$), while $\epsilon_y = \epsilon_z = 0$ (σ_x, σ_y, σ_z, ϵ_x, ϵ_y, ϵ_z denote the normal stresses and strains in respective directions, Fig. 1). From Hooke's law, $\epsilon_y = \alpha\Delta T + (\sigma_y - \nu\sigma_z)/E = 0$, $\epsilon_z = \alpha\Delta T + (\sigma_z - \nu\sigma_y)/E = 0$, from which we solve $\sigma_y = \sigma_z = -\alpha\Delta TE/(1-\nu)$ where $\Delta T(x) = T(x) - T_0$ = temperature drop. So the effect of cooling may be replaced by the effect of surface tractions $\sigma_y^0 = \sigma_z^0 = \alpha\Delta T(x)E/(1-\nu)$ applied on the halfspace at $y = \pm\infty$ and $z = \pm\infty$.

Note that before cracking we have plane stress conditions on the planes x = const., and the thermal stress in each of them is independent of $\Delta T(x)$ in other parallel planes. Also note that the use of $1-2\nu$ instead of $1-\nu$ would be incorrect and would give 1.78 times smaller values. This would correspond to $\epsilon_x = 0$, and non-zero thermal stresses σ_x^0 would be induced. To balance them, loads of direction x would have to be applied on the halfspace surface ($x = 0$), but such loads do not exist.

Subsequently the body is allowed to crack at constant $\Delta T(x)$, the crack system being large but not extending to very remote locations ($y = \pm\infty$, $z = \pm\infty$). Because the solution of a plane elasticity problem for given surface loads and a singly connected domain is independent of Poisson ratio ν, the stress intensity factors must be proportional to σ_y^0 and σ_z^0, i.e., to $\alpha(T - T_s)E/(1-\nu)$. Furthermore, from linear fracture mechanics we know that the stress intensity factor is proportional to $\sqrt{L}$ where L = size parameter [4]. Here L may be identified with $2h$, and so K_i is proportional to $\sqrt{2h}\,\alpha(T - T_0)E/(1-\nu)$. Hence, the nondimensional form of the stress intensity factor for our problem is

$$\kappa_i = \frac{K_i(1-\nu)}{E\alpha(T_0 - T_s)\sqrt{2h}}. \tag{13}$$

Assuming that the heat is transferred solely by heat conduction through the solid, the temperature profile at any time (Fig. 1) is given by the relation

$$\frac{T - T_0}{T_s - T_0} = \operatorname{erfc}(\xi) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-s^2}\,ds; \quad \text{with } \xi = \frac{x}{D}\sqrt{3}. \tag{14}$$

Here $c = k/\rho C_p$ = heat diffusivity $= 1.08 \times 10^{-6}\,\mathrm{m^2/s}$, erfc = complementary error function. Although, according to the exact solution, the temperature nowhere remains equal T_0 exactly, the values of $T - T_0$ become negligible at a certain depth $X = D$, called penetration depth of cooling. D may be defined as the depth of an equivalent approximate parabolic temperature profile, given by $T - T_0 = (1 - x/D)^2(T_s - T_0)$ for $0 \leq x \leq D$, with $T = T_0$ for $x \geq D$. To achieve overall heat balance for this profile, the heat flux at $x = 0$ must equal the rate of increase of the heat stored in the entire halfspace, i.e., $[k\partial T/\partial x]$ at $x = 0$ must equal $(d/dt)\int_0^D (T - T_0)\rho C_p\,dx$ or $2k/D = (\rho C_p/3)\,dD/dt$ where t = time. Integration yields the well-known expression $D = \sqrt{12ct}$ and because ξ in Eqn. (14) is known to equal $x/\sqrt{4ct}$ we obtain $\xi = x\sqrt{3}/D$, which substantiates ξ in (14).

To simulate the growth of the crack system, we must start with some realistic initial crack spacing, b ($b = 2h$). This spacing cannot be less than a certain minimum, which ensues from the restriction that the surface energy of cracks cannot exceed the drop in strain energy caused by cracking, ΔU. Approximating the temperature profile as parabolic, the strain energy in absence of cracks is $U_0 = \int_0^D \frac{1}{2}\alpha\Delta T(x)(\sigma_y^0 + \sigma_z^0)\,dz$ where $\Delta T(x) = (1 - x/D)^2(T_s - T_0)$, $\sigma_y^0 = \sigma_z^0 = \alpha\Delta T(x)E/(1-\nu)$, as derived before Eqn. (13). By integration, $U_0 = 0.2D\alpha^2(T_s - T_0)^2E/(1-\nu)$ per unit area of halfspace surface. We must now subtract the strain energy U_1 that is left after the cracks are introduced, and doing this we must not forget that a significant part of U_1 is due to the normal stress σ_z^1 in the direction parallel to crack planes as well as halfspace surface. If all stress in the (xy)-planes were relieved by cracking, the remaining strain energy would be due solely to uniaxial thermal stress $\sigma_z^1 = \alpha \Delta T(x) E$, and would equal $U_1 = \int_0^D \frac{1}{2}\alpha \Delta T(x) \sigma_z^1 \, dx = 0.1 D\alpha^2(T_s - T_0)^2 E$. Consequently, the relief of strain energy cannot be larger than $\Delta U = U_0 - U_1 = 0.1 D\alpha^2(T_s - T_0)^2 E(1+\nu)/(1-\nu)$. Then, if $a_1 = a_2$, we must have $2\gamma a_1 \leqslant b\Delta U$, which yields:

$$b = 2h \geqslant \frac{20(1-\nu)\gamma}{(1+\nu)\alpha^2(T_s - T_0)^2 E} \frac{a_1}{D} = 10\left(\frac{(1-\nu)K_c}{\alpha(T_s - T_0)E}\right)^2 \frac{a_1}{D}. \tag{15}$$

From the diagrams of crack length in [1] (Fig. 1f), we may estimate that for $D \ll 2h$ the ratio a_1/D equals at least 1.5. Numerical evaluation of Eqn. (15) for granite then yields the minimum spacing $2h \approx 6.9$ cm, and we may note that this result is independent of D. Because only a certain fraction of ΔU is consumed by creation of the cracks, the actual spacing must be higher; if 50% is consumed, it would be 13.8 cm.

4. Finite element solution of critical states

The stress intensity factors for various crack lengths have been calculated by finite elements, applying Yamamoto's superposition method [6]. Meshes that are highly refined near the crack tip have been used (Fig. 2), which allowed rather accurate value of K_i to be obtained. This was thought to be important because for stability checks we need the derivatives of K_i, and we know that the derivatives of a numerical approximation have a much larger relative error than the approximation itself. No comparative study of the effectiveness of various highly accurate singularity elements has been undertaken and the method was chosen because of availability of a program and the need to minimize the human programming effort.

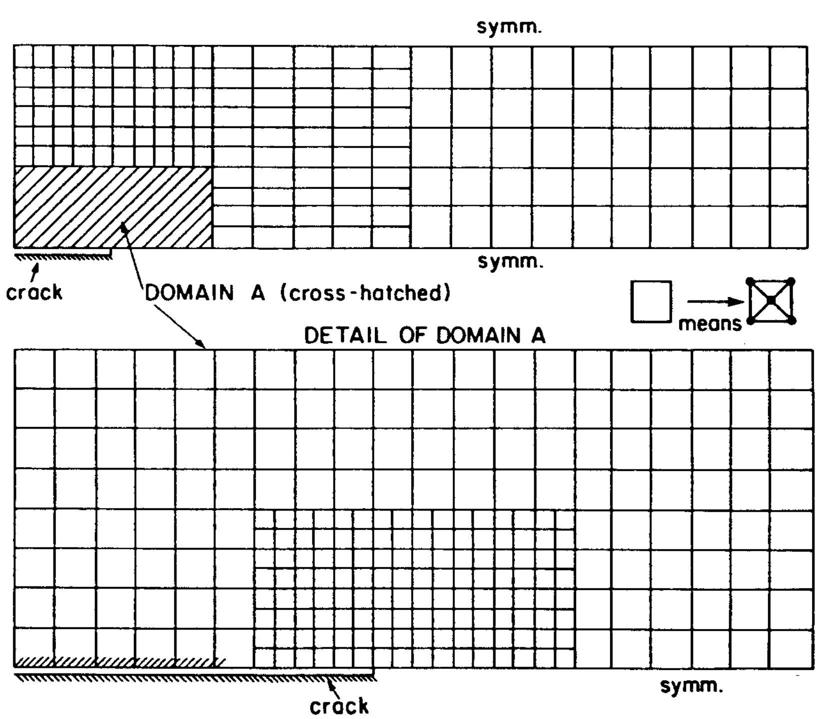

Figure 2. Example of a finite element grid used to calculate stress intensity factors and their derivatives.

To determine the equilibrium states for equally long cracks ($a_1 = a_2$), the finite element program was run for various values of a_2 and various penetration depths of cooling, D. The results of these calculations are plotted in Fig. 3. For a given K_c (and fixed $2h$, E, ν, $T_s - T_0$, α), the equlibrium states are represented in Fig. 3 by intersections with a horizontal line $\kappa = \text{const.}$ ($K_2 = K_c$), and the values for the intersection points may be used to construct the equilibrium path of the system in the plot of a_2 versus D; see Fig. 4.

Alternatively, if we are not interested in the behavior at various κ values, it is better to proceed differently. We choose the lengths a_2 ($= a_1$) and then make computer runs for different D till we find (by iterative Newton method) the value of D for which $K_1 = K_2 = K_c$. Repeating this for various a_2, we obtain one of the paths shown in Fig. 4.

The derivatives $\partial K_2/\partial a_2$ at constant a_1 have been calculated from the central difference approximation $\partial K_2/\partial a_2 \approx (K_2^+ - K_2^-)/2\Delta x$ where K_2^+ and K_2^- are the values of K_2 for crack lengths $a_2 = a_1 + \Delta x$ and $a_2 = a_1 - \Delta x$, with a_1 being such that K_1 and K_2 for $a_2 = a_1$ equal K_c. Some of the calculated derivatives $\partial K_2/\partial a_2$ are indicated by dashed arrows in Fig. 3, whose slope represents $\partial K_2/\partial a_2$. We see that the slope increases with a_2 and eventually changes its sign. By interpolating with regard to a_2 we can then locate the value of a_2 for which $\partial K_2/\partial a_2 = 0$. This is the critical state in which bifurcation of the path of equilibrium states occurs.

The calculated bifurcation points are marked in Fig. 4 for three values of κ_c corresponding to three different crack spacings $2h$. Bifurcation is found to occur when roughly $a_2/2h = 1.8$ and $D/2h = 2.5$. In Fig. 3 we see that for a_2 longer than the critical value the derivative $\partial K_2/\partial a_2$ is positive, which indicates that the cracks whose length

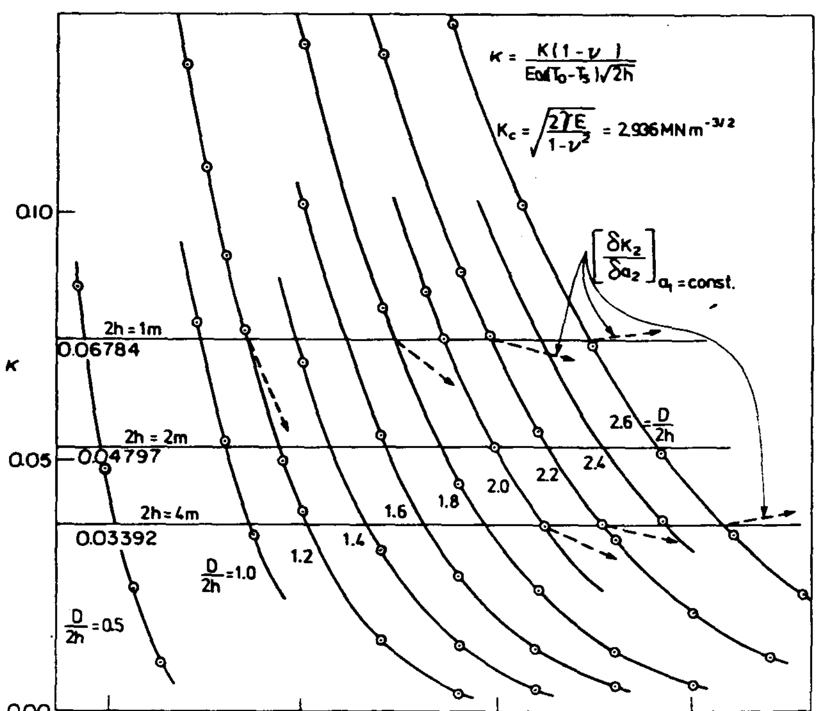

Figure 3. Stress intensity factors for equally long cracks at various cooling penetration depths.

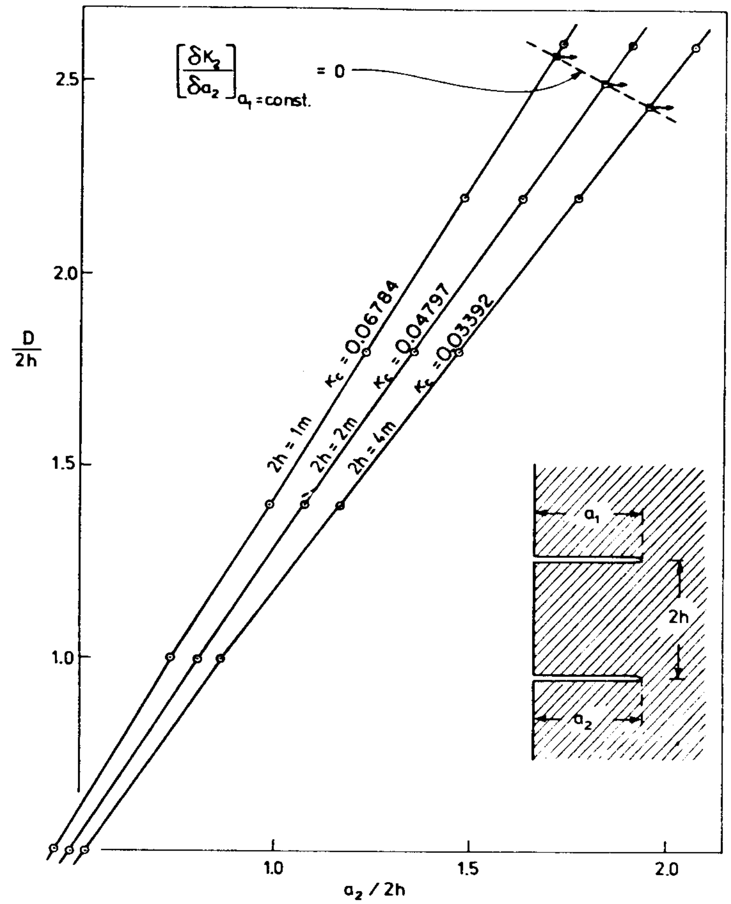

Figure 4. Crack paths in terms of cooling penetration depth and crack length, with critical states of crack arrest (bifurcation of equilibrium path).

is greater than critical are unstable. Such crack states cannot practically occur even though they represent proper solutions according to the standard approach of linear fracture mechanics.

5. Post-critical behavior

From the theory in [1] we know that as D grows beyond the critical value cracks a_1 remain at constant length. Cracks a_2, the leading cracks, suddenly extend their length at constant D and later grow gradually as D increases.

To calculate the post-critical path, we fix the length of shorter cracks a_1 at the critical value, $a_1 = a_{crit}$, at which bifurcation occurred. Then we select the length a_2 of the leading cracks ($a_2 > a_1$) and run the finite element program for various values of D (such that $D > D_{crit}$), and iterate by Newton method until we find the value of D for which $K_2 = K_c$. We also evaluate K_1, record it (and check that indeed $K_1 < K_c$). We then repeat this entire calculation for several values of a_2 and obtain the corresponding values of D and K_1. We may now interpolate (or iterate by Newton method) to find the value of a_2 for which $K_1 = 0$. Thus we obtain the case in which cracks a_1 become fully closed at their tips. The post-critical path of the crack system obtained by connecting the points calculated in this manner is shown in Fig. 5, and the values

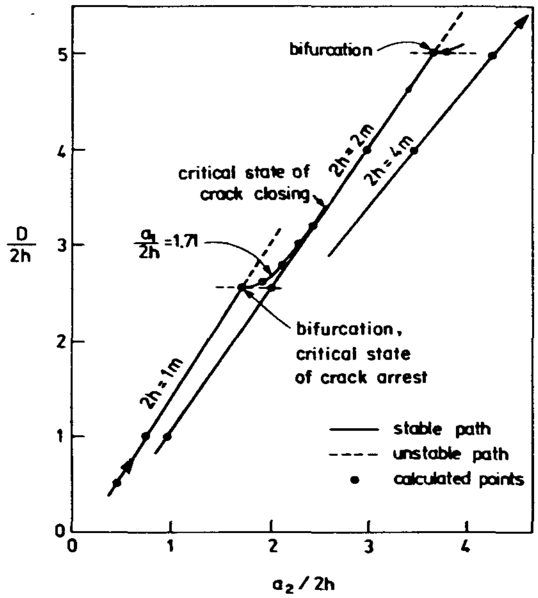

Figure 5. Postcritical crack path after bifurcation, with critical state of crack closing, and repetition for larger spacing.

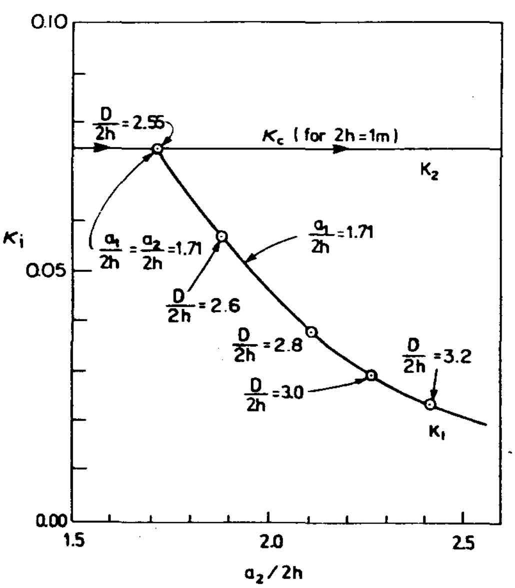

Figure 6. Postcritical stress intensity factors as function of leading crack length, with critical state of crack closing ($K_1 = D$).

of K_1 obtained for various a_2 are plotted in Fig. 6. The second critical state of crack closing is also marked in Fig. 5.

To get a more complete picture, we may, alternatively, begin by constructing the lines of K_2 as a function of a_2 at various fixed values of D; see Fig. 7. The post-critical path in Fig. 5 can then be constructed from this diagram, by plotting in Fig. 5 the intersection points of the lines of constant D with the line $K_2 = K_c$ (Fig. 7).

It is interesting to note that at the critical point the curve of constant D is tangent to the line $K_2 = K_c$ (Fig. 7). This must of course be so because at the critical state there exists an adjacent equilibrium state [1, 2] at which the crack length a_2 is different but K_2 is the same (as shown in [1] and [2]).

The critical state at $a_1 = a_2$ is characterized by the condition $\partial K_2/\partial a_2 = 0$. Therefore, each of the peak points of the curves at constant D in Fig. 7 represents a critical state for that K_c-value which equals the K_2-value at this peak point. The post-critical path for any of these critical states can be obtained by passing at the appropriate peak point in Fig. 7 a horizontal tangent (e.g., dash-dot line in Fig. 7) and determining the slopes of other curves at their intersections with this tangent. The interaction points in which the curve of constant D in Fig. 7 has a downward slope represent stable states, and those where it has an upward slope represent unstable (and statically unattainable) states.

It is noteworthy that if K_2 $(= K_c)$ increases the peak points in Fig. 5 shift to the right, albeit only slightly. This means that for a higher K_c, the bifurcation occurs at deeper cracks a_2 and a deeper cooling penetration D. Moreover, noting the non-dimensional form of the stress intensity factor, Eqn. (13), we see that the smaller the value of $T_s - T_0$, α, h, E or $1/\nu$, the larger the value of $a_2/2h$ and $D/2h$ at bifurcation. Thus, in case of very high K_c, very small E and very small h we could presumably obtain stable thermal cracks of a very high depth-to-spacing ratio.

As determined by (9)–(11), at the moment when $K_1 = 0$ (Fig. 6), the post-critical path reaches a second critical state at which cracks a_1 suddenly close over a finite length (at $D = $ const.) before further growth of cracks a_2 occurs. Do cracks a_1 close suddenly over their entire length? Although very accurate calculations have not been made for this case, based on crude finite element results it seems that this is just about what happens.

It should be observed that the critical state, corresponding to $\det(W_{ij}) = 0$, does not represent bifurcation of the equilibrium path because a smooth continuation of the regime through the point $K_1 = 0$ does not give an equilibrium state acceptable according to linear fracture mechanics, since it would lead, according to Fig. 6, to

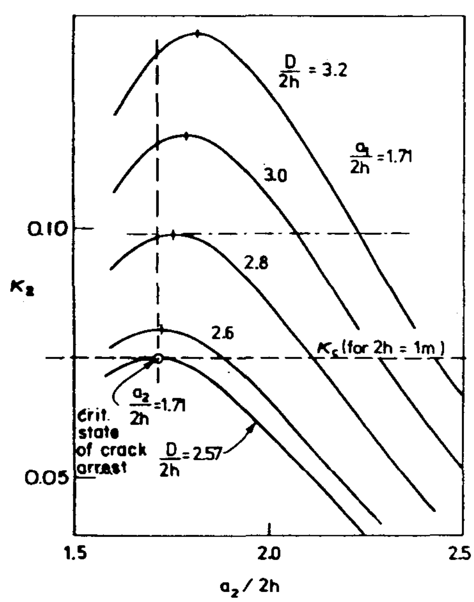

Figure 7. Stress intensity factor of cracks a_2 at constant crack length a_1, with critical states of crack arrest.

negative K_I. By transferring the point $K_I = 0$ from Fig. 6 to Fig. 5, the critical state of crack closing (on the extension of the plot in Fig. 5) can be obtained.

The folowing behavior for the error function profile of temperature now comes to light. At the beginning of cooling, the cracks are no doubt closely spaced, possibly with the minimum spacing estimated in (15). Later every other crack closes and the spacing of the opened cracks, the leading cracks, becomes effectively doubled, i.e., $4h$. For the purpose of calculation, closed cracks are the same as no cracks at all, as long as there is no slip, which is in our case guaranteed for the next to the longest cracks by the symmetry conditions. Therefore, the same behavior must repeat itself for the leading cracks. Every other one among them will close when their length-to-spacing ratio reaches the critical value determined before (Fig. 4). The spacing of opened cracks then doubles again, becoming $8h$, and so forth.

The existence of critical states is no doubt characteristic of "perfect" crack systems, just like in column buckling. If the crack system is "imperfect", e.g., if the spacing of cracks is not precisely equal, or if the halfspace exhibits inhomogeneities, surface unevenness, etc., the equilibrium path of the system may be expected to have no critical points and trace a smooth path in close vicinity of the solid line path in Fig. 5. Note, however, that the calculation of such a path would be more difficult; it is easier, and conceptually more useful, to determine the critical states of the perfect crack system.

6. Practical consequences for the geothermal energy scheme

If one cubic mile of granite of 300°C temperature should be cooled to 200°C, the heat released could drive a power plant of 1000 MW (thermal) for 33 years. In the United States there exist numerous geothermal basins where hot rock is close enough to the surface, but most of them are dry, i.e., have no natural water circulation. A study of the methods of tapping the hot dry rock energy is, therefore of considerable importance.

The preceding results are of interest for one recently proposed geothermal energy scheme for heat extraction from hot dry rock [7, 8, 9]. This scheme relies on hydraulic fracture. By forcing water under very high pressure into a vertical bore, a huge circular vertical crack, of about 1 km diameter, is to be produced in a hot rock mass at the depth of several kilometers beneath the surface. The top region of the crack is then intersected by another bore. Water is injected into the first bore, gets heated by circulating through the crack and exits to the surface by the second bore [7, 8, 9]. However, the withdrawal of heat by the circulating water gradually cools the rock adjacent to the crack walls, and the rate of heat transfer by heat conduction through the rock rapidly declines as the cooling penetration depth increases. Detailed numerical calculations of the water circulation and heat distribution have been made [9, 10] and it appeared that within a few months the heat output would become insufficient to drive a power plant of any reasonable size.

Are there any further means of making the scheme viable? One hope is that further thermal cracking, which would certainly be induced by cooling of the rock, might be sufficient to bring enough circulating water close to the cooling front. Basically two mechanisms can be envisaged. One consists in a downward extension of the large crack, and calculations of water circulation in a crack whose thickness varies as a function of rock cooling indicate that this mechanism bears some promise [9, 10]. Another possible mechanism is a horizontal growth of a system of parallel secondary vertical cracks normal to the main crack (Fig. 1). In the present case, a cracked halfspace serves as a simplified model for this second mechanism.

As the secondary cracks advance into the rock mass, part of the water flux will divert into the secondary cracks. The water flux in a single crack is roughly proportional to the cube of crack width, w, and to the inverse of the spacing b ($b = 2h$) of the opened cracks. The opening width of the cracks farther away from their tips is roughly $\alpha(T - T_0)b$. Therefore, the combined diverted flux in all secondary cracks is proportional to w^3/b or b^3/b, i.e., to b^2 [9]. Clearly, for this flux to become significant, spacing b must become large.

For the error function profile of temperature we have found the behavior to be favorable. According to Figs. 5–7, the spacing of the opened cracks fluctuates between $D/2.57$ just before crack arrest and $2(D/3.3)$ just after closing, and so, on the average, the spacing b of the opened, leading cracks is about $b \approx (D/2.57 + 2D/3.3)/2 \approx D/2$. Noting that at the surface of the cracked halfspace the stress in negligible, we conclude that the opening width w_0 at the entrance to the leading cracks is about

$$w_0 \approx \alpha(T_s - T_0)D/2 \tag{16}$$

The thermal opening of the main crack, whose wall was treated as the half-space surface (Fig. 1e), is about $w_1^T \approx 2\int_0^D \alpha\Delta T(x)\,dx$ where $\Delta T(x) = (1 - x/D)^2(T_s - T_0)$ or $w_1^T = 2\alpha(T_s - T_0)D/3$, and so

$$w_0/w_1^T \approx \tfrac{3}{4}. \tag{17}$$

Thus, based on the thermal opening alone one could expect a water flux of the same order of magnitude in the main crack and the secondary cracks. The length a_2 of the opened cracks fluctuates, according to Figs. 5–7, between $(1.71/2.57)D$ and $(2.55/3.3)D$, and is on the average $(1.71/2.57 + 2.55/3.3)D/2$ or $a_2 \approx 0.72D$. If the crack depth a_2 equaled D, the opening width would be $w(x) = b\alpha(T_s - T_0)(1 - x/D)^2$; on the other hand, if D equaled a_2, it would be $w(x) = b\alpha(T_s - T_0)(1 - x/a_2)^2$. The actual $w(x)$ should lie between these two estimates (and must have the form $(1 - x/a_2)^{1/2}$ near the crack tip, $x = a_2$). For a laminar flow, the water flux in the opened cooling cracks would be $q_0 \approx k\int_0^{a_2} w(x)^3\,dx$ where $k = p_{,z}\rho/12\mu$, $p_{,z}$ = hydraulic pressure gradient and μ = viscosity of water. Integration for the two extreme estimates of $w(x)$ yields the approximate bounds $kw_0^3a_2/7 < q_0 < kw_0^3D/7$ where $a_2 \approx 0.72D$. The water flux in the main crack is $q_1 = kbw_1^3$, and because $b \approx D/2$ and $w_1 \approx 4w_0/3$ (17), $q_1 = 1.19hDw_0^3$. Thus, the ratio of the water flux in the secondary cracks to that in the main crack is approximately bounded as

$$0.09 < q_1/q_0 < 0.12 \tag{18}$$

provided the opening of the main crack is due mainly to temperature. If this opening is further increased by pressure, then q_1/q_0 is less. It is interesting to note that q_1/q_0 is independent of D, the cooling penetration depth.

The foregoing crude calculations do not indicate that a significant water flow would get diverted toward the tips of the secondary cooling cracks, and thus the prospects of this postulated mechanism to work in practice do not seem to be especially promising, even though the phenomenon of increasing spacing of the opened cracks is favorable. It remains to be seen, however, what could be the effect of the water flux in the secondary cracks on the temperature profile $\Delta T(x)$ and on crack closing due to instability. This will be explored separately [13].

There exist other important applications. The spacing and opening of drying shrinkage cracks in concrete is a similar problem [14] influencing corrosion of embedded steel and shear transfer across the crack surfaces by aggregate interlock. The problem of crack spacing has been studied with regard to drying of mud flats [11] and cooling of lava flows in the ocean [12], and it was in these works that the phenomenon of crack closing and multiplication of the leading crack spacing has been first suggested, on the basis of observation in nature. (Everybody knows how the cracks gradually open at the bottom of a dried lake.) Shrinkage cracks in wood or in reactor fuel elements, and spacing of radial cracks in a star pattern around a pressurized borehole, are other possible applications.

7. Conclusion

The system of parallel equidistant thermal cracks in a halfspace becomes unstable when cracks extend too deep compared to spacing. In the first critical state, characterized by vanishing of $\partial K_2/\partial a_2$ at a_1 = constant, each other crack is arrested, and in the second critical state, characterized by $\det(W_{ij}) = 0$, these cracks close, and the process is at a greater depth repeated for the remaining cracks of doubled spacing. The finite element method provides a simple and general approach to determining these critical states as well as the post-critical path.

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RÉSUMÉ

Lorsque un système de fissures équidistantes parallèles survenant lors d'un refroidissement se propage dans un demi espace élastique, il atteint à une certaine profondeur de fissuration un état critique d'instabilité et le chemin d'équilibre du système bifurque. Une extension ultérieure de fissures de longueurs égales se présente comme instable est impossible. Le chemin post-critique stable correspondra à une extension de toute autre fissure correspondant à un refroidissement ultérieur, et ce à l'origine avec un resaut de la fissure à temperature constante, tandis que les fissures intermédiaires cessent leur croissance et voient leur facteur d'intensité de contrainte graduellement diminuer jusqu'à la valeur nulle. Ceci correspond à un deuxième état critique pour lequel les fissures intermédiaires se ferment brusquement sur une longueur finie sans que ne se produise un changement de température, et à longueur constante des fissures principales. Par après lorsque le front de refroidissement continue d'avancer, les fissures principales croissent d'une longueur égale jusqu'à ce qu'elles atteignent à nouveau un état critique, état pour lequel les autres fissures cessent de croître et pour lequel le processus d'espacement des fissures est répété. De cette manière, l'espacement de fissures s'ouvrant lors du refroidissement fluctue autour d'une valeur correspondant sensiblement à la moitié de la valeur de la pénétration de refroidissement.

L'instabilité est déterminée par le signe de la deuxième variation du travail nécessaire pour créer les fissures, ce qui conduit à des conditions positives non définies pour une matrice comportant les dérivées partielles des facteurs d'intensité d'entaille par rapport aux longueurs de la fissure, l'ensemble étant sujet aux conditions d'admissibilité de l'eigenvector pour des accroissement de longueurs de la fissures. Le premier état initial de l'arrêt de la fissure est caractérisé par la disparition de l'élément diagonal de la matrice tandis que le deuxième état critique de fermeture de la fissure est caractérisé par la disparition du déterminant de cette matrice.

Les états critiques et la croissance post-critique de la fissure sont calculés de façon numérique par des éléments finis. La solution est appliquée au refroidissement d'une masse granitique à haute température dont la fissuration est importante, dans le cadre d'un schéma d'extraction de la chaleur géothermique proposé récemment. La solution est également intéressante à appliquer dans le cas de fissures procédant d'une contraction due au séchage, et ce en particulier dans le cas du béton.