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Hysteretic Fracturing Endochronic Theory for Concrete[a]

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Introduction

The endochronic theory, in which the inelastic strain increments are characterized by means of an intrinsic time—a nondecreasing scalar variable whose increments depend on strain increments—has appeared to be a powerful method for setting up the constitutive relation for nonlinear triaxial behavior of concrete. The original formulation (7) and its subsequent improved version (8,10) modeled realistically the basic features of triaxial response, such as the smooth continuation of the stress-strain diagrams through the peak stress points, the strain-softening, the hydrostatic pressure sensitivity, and the inelastic dilatancy due to shearing. The theory has been applied with success in several laboratories throughout the world for large finite element analyses of concrete structures (e.g. 33,39).

Certain shortcomings have, however, been detected. For example, the original endochronic model was found to give much too steep an unloading curve in the large strain region and too mild a degradation of elastic moduli at very large strain. When unloading was reversed to reloading the slope became smaller while it should have become steeper. Moreover, for an unload-reload cycle the net dissipated second-order work was negative while it should have been positive. Also, when a low stress cyclic loading was followed by a high stress load, the response was far too stiff.

The purpose of this work is to develop a more advanced endochronic model which is free of these shortcomings. Moreover, various conceptual improvements, partly based on the analysis of endochronic theory in Ref. 4, will be made.

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The behavior that we intend to model is characterized by representative data sets in Figs. 1-5 selected from the literature (1,2,12,20,21,23,24,25,26,27,30,32, 34,35,36) and our model will be shown to fit all these experimental data satisfactorily.

Nevertheless, one must not indulge in false hope that our result would be the "ultimate" model. In fact, in a simultaneous work an altogether different model, which fits the present test data (with the exception of a high number of cycles) equally well, has been found (9). It is a model of plasticity type, which is incrementally linear, whereas the endochronic theory is not (3). From the viewpoint of structural calculations, this is a disadvantage of the endochronic theory. However, lacking sufficient test data to decide which model is the more correct one, we must realize that the endochronic theory generally gives lower

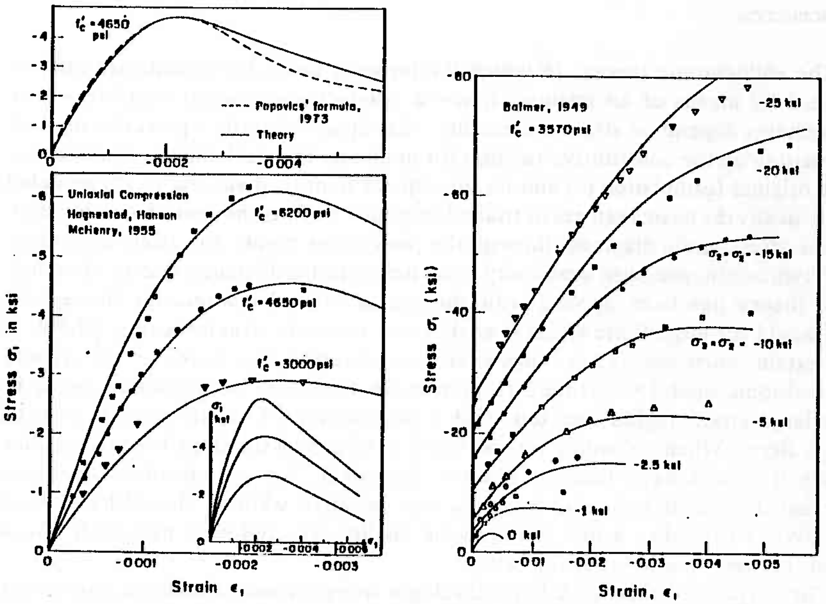

FIG. 1.—Fits of Uniaxial and Triaxial Experimental Data of Hognestad, Hanson, McHenry (14), Popovics (30), and Balmer (2)

predictions for the collapse load due to material instability (3) and is, therefore, on the safe side. In fact, with respect to this sort of collapse, classical incremental plasticity with normality is the least safe (most unconservative) model that one can assume (3).

Strain-Rate Dependent Time Shift in Viscoplasticity

The classical viscoplastic constitutive relation may be written as

$$\frac{d\epsilon_{ij}}{d\zeta} = D^{el}_{ijkm}\frac{d\sigma_{km}}{d\zeta} + \frac{\partial \Psi}{\partial \sigma_{ij}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

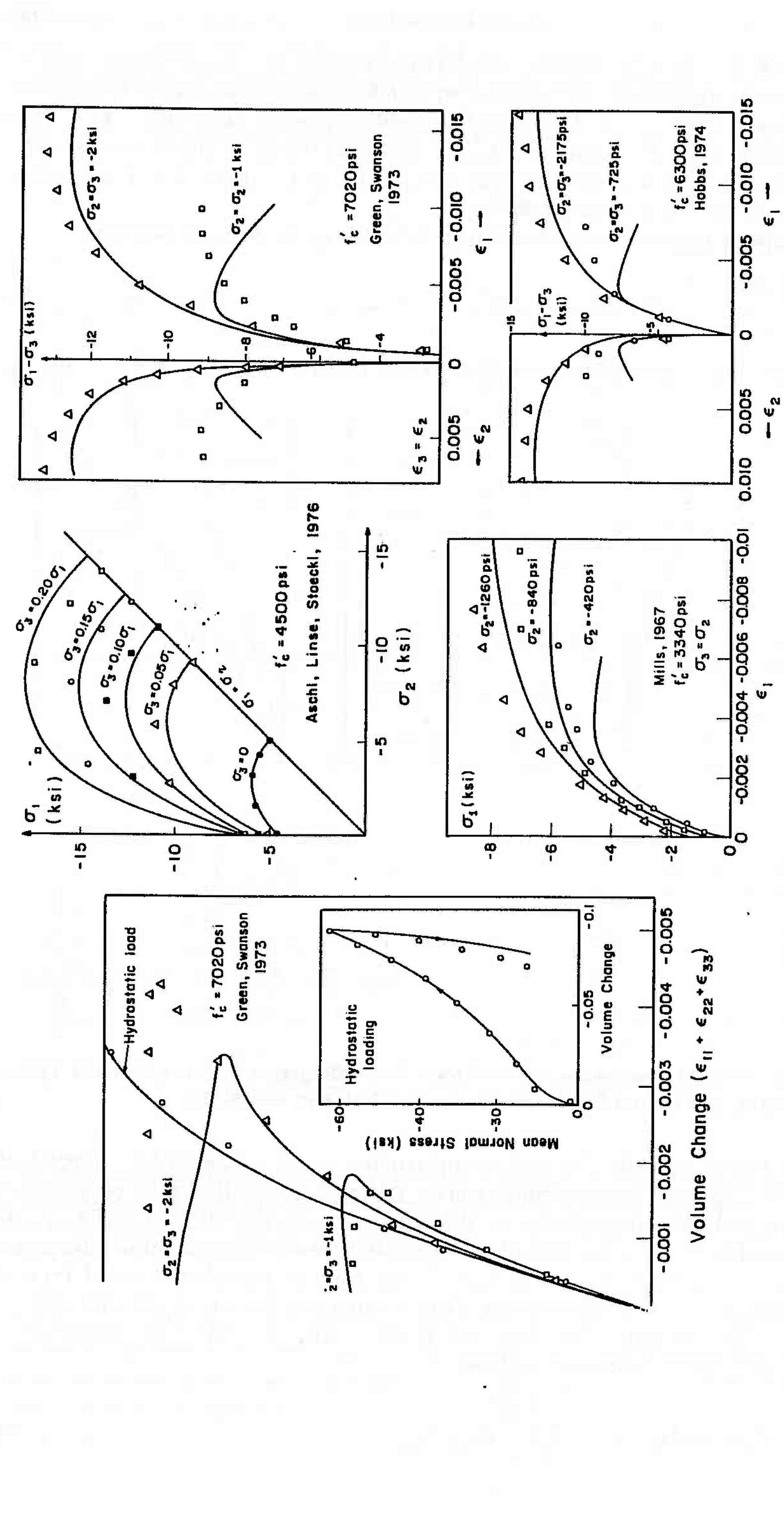

FIG. 2.—Fits of Experimental Triaxial Stress-Strain Diagrams, Volume Changes and Failure Envelopes of Mills (27), Hobbs (23), Green and Swanson (21), and Aschl, Linse, and Stoeckl (1)

in which ζ = time parameter called reduced time; ϵ_{ij}, σ_{ij} = components of small strain tensor ϵ and stress tensor σ; latin lower-case indices refer to cartesian coordinates x_i (i = 1, 2, 3) and index repetition implies summation; $\Psi = \Psi(\sigma)$ = current loading function; and D^{el}_{ijkm} = isotropic tensor of elastic compliances. Note that if Ψ is a quadratic form in σ_{ij} and $\zeta = t$ = time, Eq. 1 represents linearly viscoelastic Maxwell solid.

Nonlinear behavior may be obtained by defining the reduced time as

$$\zeta = \int \frac{dt}{a_D}; \quad \frac{1}{a_D} = F_1(\sigma, \epsilon)\, R_1(\dot{\epsilon}) \quad \text{. . . . . (2)}$$

in which a_D is called the time-shift factor (because $\log \zeta = \log t - \log a_D$

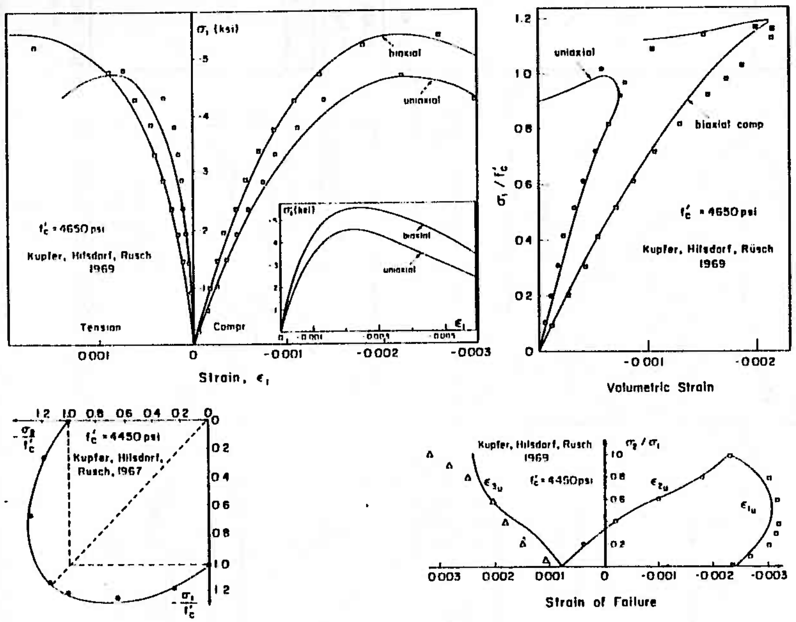

FIG. 3.—Fits of Experimental Biaxial Stress-Strain Diagrams, Volume Changes, Failure Envelopes, and Failure Strains of Kupfer, Hilsdorf, and Rüsch (26)

for constant a_D). In classical viscoplasticity, a_D is considered to depend on σ and ϵ (and of course temperature). It was a pioneering idea of Schapery (31) to make a_D depend also on the strain rate, $\dot{\epsilon}$, as indicated in Eq. 2. He proposed that a_D be a function of the octahedral shear component of the strain-rate tensor, i.e. a function of $(\dot{e}_{km}\dot{e}_{km})^{1/2}$, and thus he launched a novel type of constitutive relation representing what is currently known as the endochronic theory. Accordingly, we may set $R_1(\dot{\epsilon}) = (\dot{e}_{km}\dot{e}_{km}/2)^{1/2}$ or $R_1(\dot{\epsilon})\, dt = (de_{km}de_{km}/2)^{1/2}$, and this provides

$$d\zeta = F_1(\sigma, \epsilon)\, d\xi; \quad d\xi = \sqrt{\frac{1}{2}\, de_{km}\, de_{km}} \quad \text{. . . . . (3)}$$

in which e_{km} = deviator of ϵ_{km}. Note that with this form of $R_1(\dot{\epsilon})$ the time disappears from the definition of ζ and the constitutive relation is actually independent of the time rate of strain even though the dependence of a_D upon $\dot{\epsilon}$ was the starting assumption. By using, e.g., $R_1(\dot{\epsilon}) = (\dot{e}_{km}\dot{e}_{km} + \text{constant})^{1/2}$, we could generalize our formulation which follows to a strain-rate dependent

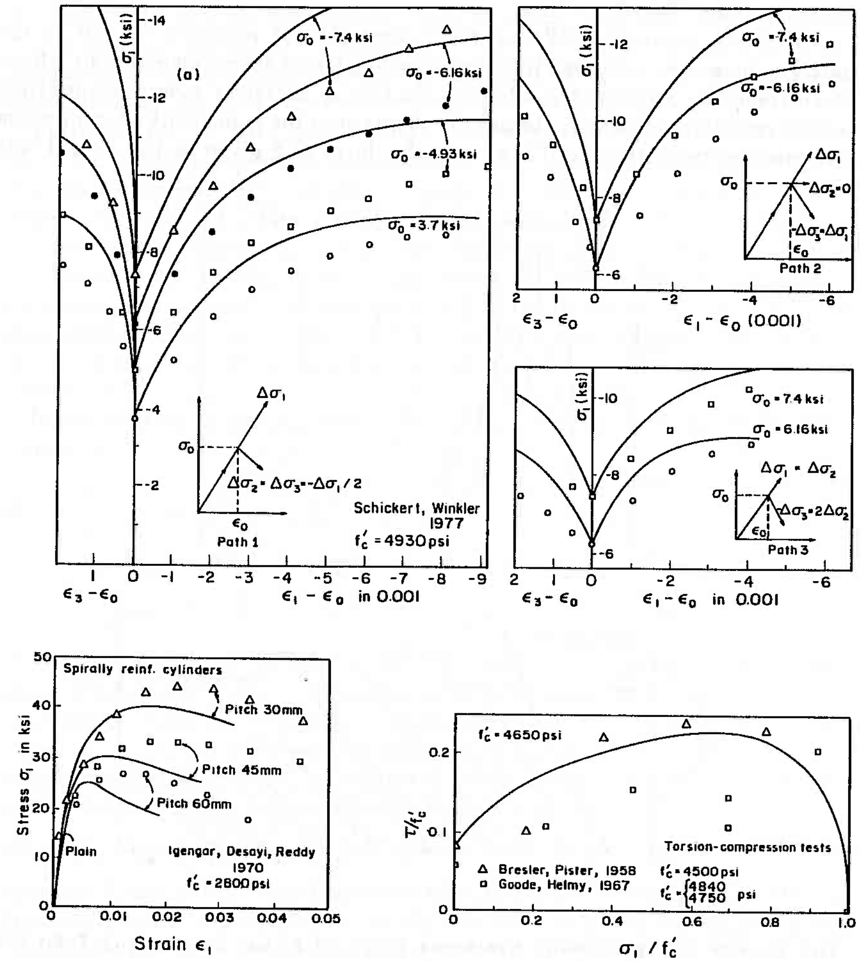

FIG. 4.—Fits of Experimental Nonproportional Triaxial Stress-Strain Diagrams, Compression Diagrams of Spirally Reinforced Cylinders, Torsion-Compression Failure Envelope of Schickert and Winkler (32), Iyengar, Desayi, and Reddy (25), and Bresler and Pister (12)

behavior of the type considered before (37,7). But we do not intend to seek improvement of the previously given endochronic treatment of strain-rate effects in concrete (7), and so we will content ourselves with a formulation that does not involve time-dependence.

Variable ξ may be called deviatoric path length as it represents the length of the path traced in a six-dimensional deviatoric strain space. More general forms of $R_1(\dot{\epsilon})$, leading to a more general expression for $d\xi$ are possible (3), but there are good reasons for Eq. 3 to grasp the essential behavior of concrete, as explained in Ref. 7.

Variable ζ, originally called reduced time (31), is presently known as the intrinsic time. This term and the corresponding Greek term "endochronic" have been coined by Valanis (37), who was the first to present a general formulation of the endochronic theory. He put the theory into the framework of continuum thermodynamics, introduced the particular form of ξ given in Eq. 3, and was

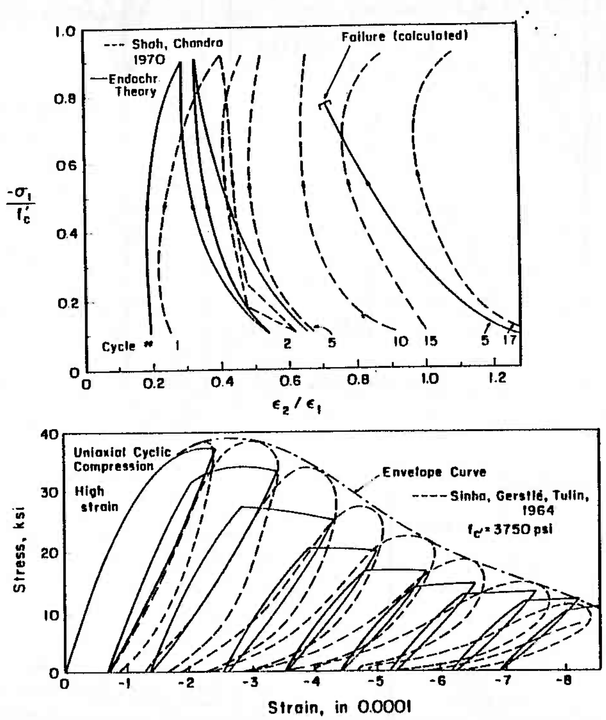

FIG. 5.—Fits of Experimental Hysteresis Loops of Sinha, Gerstle, and Tulin (35), and Shah and Chandra (34)

first to demonstrate the great potential of the endochronic approach by modeling the inelastic behavior of work-hardening metals, especially the cross-hardening in axial loading due to preceding twist (37).

Endochronic Loading Functions

The decrease of stress at increasing strain, called strain-softening, must be regarded as loading rather than unloading. Indeed, if it were not, then not only the decreasing stress at decreasing strain but also the strain-softening would represent unloading, and no loading regime would exist, which is conceptually

unacceptable. Then, however, our starting equation (Eq. 1) is stained by a fundamental fault: Function Ψ cannot actually distinguish between loading and unloading if strain-softening takes place, since Ψ is a function of stress. How can we remedy this situation?

Obviously, the loading-unloading criterion for the strain-softening regime must be defined in terms of strains. Therefore, as a novel step in endochronic theory we propose to express the inelastic strain increments as $(\partial\Phi/\partial\epsilon_{ij})\,d\zeta$, where Φ is a loading function which depends on strain. It should be noted, however, that in incrementally linear theories, particularly plasticity, loading functions that depend on the strain tensor have been used before (e.g., 14–17).

Furthermore, we must realize that, according to the physical mechanism, there are basically two types of inelastic strains—plastic and fracturing (5,4,9). The plastic strains, $d\epsilon_{ij}^{pl}$, are due to plastic slip, which takes place at constant stress and does not affect elastic stiffness, and the fracturing strains, $d\epsilon_{ij}^{fr}$, are due to microfracturing or microcracking, which is accompanied by a stress drop and results in a degradation of material stiffness. Obviously each of these components of inelastic strains must be expected to be associated with a different time shift function, i.e., with a different intrinsic time, although both of them should depend on $d\xi$ because they both stem from deviatoric rather than volumetric straining.

In consequence of these arguments, Eq. 1 may be replaced, in case of deviatoric strains, by:

$$de_{ij} = \frac{ds_{ij}}{2G} + de''_{ij};\quad de''_{ij} = de_{ij}^{pl} + de_{ij}^{fr} \quad \ldots \ldots \ldots \ldots \ldots \ldots (4)$$

$$de_{ij}^{pl} = \frac{1}{2G}\frac{\partial\Psi}{\partial e_{ij}}\,d\zeta;\quad de_{ij}^{fr} = \frac{\partial\Phi}{\partial e_{ij}}\,d\kappa \quad \ldots \ldots \ldots \ldots \ldots \ldots (5)$$

in which Ψ and Φ are the plastic and fracturing loading functions; G = elastic shear modulus, introduced for dimensionality convenience; $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma$ = stress deviator; $\sigma = \sigma_{kk}/3$ = volumetric stress; $e_{ij} = \epsilon_{ij} - \delta_{ij}\epsilon$ = strain deviator; $\epsilon = \epsilon_{kk}/3$ = volumetric strain; δ_{ij} = Kronecker delta; de_{ij}^{pl} = plastic strain increments; and de_{ij}^{fr} = fracturing strain increments. The corresponding intrinsic times may be expressed similarly to Eq. 3:

$$d\zeta = c\,F_1\,d\xi;\quad d\kappa = c\,F_2\,d\xi \quad \ldots \ldots \ldots \ldots \ldots \ldots (6)$$

in which F_1 and F_2 = invariant functions of σ, ϵ, and ξ, called hardening-softening functions; and c = a coefficient which will later be useful for cyclic loading. We choose $c = 1$ for monotonic loading.

The loading functions Ψ and Φ must be defined in such a manner that they increase along both the rising and falling segments of uniaxial (or biaxial) loading diagrams. Noting that the source of inelastic strain is essentially the deviatoric straining, we may then assume

$$\Psi = \int s_{km}^{*}\,de_{km}^{*} + h_1(\sigma)\int \sigma\,d\epsilon - H_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots (7)$$

$$\Phi = \frac{1}{2}\,e_{km}^{*}\,e_{km}^{*} + h_2(\epsilon) - H_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots (8)$$

in which $\quad s_{ij}^{*} = s_{ij} - \alpha_{ij};\quad e_{ij}^{*} = e_{ij} - \alpha'_{ij} \quad \ldots \ldots \ldots \ldots \ldots \ldots (9)$

Parameters α_{ij} and α'_{ij} represent the centers of the deviatoric loading surfaces; these parameters will be needed later for cyclic loading, whereas for monotonic loading we will use $\alpha_{ij} = \alpha'_{ij} = 0$.

The deviatoric terms in Eqs. 7 and 8 reflect the view that inelastic strains are due chiefly to deviatoric straining. The deviator part of Ψ represents the distortional work, which is the simplest choice to make and is related to von Mises' yield function. The deviator part of Φ is, likewise, of a von Mises type. The precise form is no doubt more complicated, and insertion of cubic terms, such as $\int s_{ik} s_{km} de_{mj}$ and $e_{ik} e_{km} e_{mj}$ (third invariant) into Eqs. 7 and 8 has, in fact, been explored. However, because of the scatter of available test data, no appreciable improvement of data fits could be detected. Thus, the question of the precise form of the loading surface appears to be a moot point, anyhow less important than the constitutive equation structure itself.

Furthermore, in Eqs. 7 and 8 we have included volumetric terms $\int \sigma d\epsilon$ and $h_2(\epsilon)$. Their purpose is to reflect the inelastic dilatancy and the pressure sensitivity of deviatoric inelastic strains (which is similar to Drucker-Prager yield function).

Substituting Eqs. 7 and 8 into Eq. 5, we obtain:

$$de^{pl}_{ij} = \frac{s^*_{ij}}{2G} d\zeta; \quad de^{fr}_{ij} = e^*_{ij}\, d\kappa \quad \text{(10)}$$

The fact that de^{pl}_{ij} is proportional to s^*_{ij} could alternatively be also obtained from a plastic loading function that depends on the stress and contains a von Mises type deviator part, $s_{km} s_{km}$ (3). This, however, would require differentiation of Φ with respect to σ_{ij}, which would be questionable in case of strain-softening, as already explained.

In classical incremental plasticity, a more fundamental approach, based on Drucker's postulate, is used for deriving the inelastic strains. Shouldn't we have used a similar approach here? This would not make much sense, though, because Drucker's postulate is not always satisfied by endochronic formulations (3) and, besides, materials which exhibit internal friction and microcracking do not have to follow this postulate either (6). The purpose of the loading functions is mainly to aid us in choosing the expressions for inelastic strains.

Hardening-Softening Functions

To illustrate the practical effect of de^{pl}_{ij} and de^{fr}_{ij}, consider the uniaxial stress-strain diagram with analogous inelastic strains: $d\epsilon^{pl}_x = \sigma_x F_1\, d\xi/E$; $d\epsilon^{fr}_x = \epsilon_x F_2\, d\xi$; and the stress-strain relation $d\epsilon_x = d\sigma_x/E + d\epsilon^{pl}_x + d\epsilon^{fr}_x$, in which subscript x refers to x_1-direction ($i = 1$) and E is Young's modulus. To be able to integrate explicitly, assume that E, F_1 and F_2 are constant. As a uniaxial analog of Eq. 3 we may also set $d\xi = |d\epsilon_x|$. For $d\epsilon_x > 0$, we thus get the differential equation $d\sigma_x/d\epsilon_x = E - F_1\sigma_x - EF_2\epsilon_x$, whose solution for the initial condition $\sigma_x = 0$ at $\epsilon_x = 0$ is

$$\sigma_x = \frac{E}{F_1}\left[\left(1 + \frac{F_2}{F_1}\right)(1 - e^{-F_1\epsilon_x}) - F_2\epsilon_x\right], \quad \text{for} \quad F_1 > 0$$

$$\sigma_x = E\epsilon_x\left(1 - \frac{F_2}{2}\epsilon_x\right), \quad \text{for} \quad F_1 = 0 \quad \text{(11)}$$

This solution is sketched in Fig. 6(*a*), where we see that for $F_2 = 0$ there is no strain-softening. The previously developed endochronic theory (7,10) corresponded to $F_2 = 0$, and to obtain a peak and the falling segment on the stress-strain curve it was then necessary to make F_1 variable, increasing it greatly for large strain. Thus the response curve was rather sensitive to the choice of F_1 which made it more difficult than here to model strain-softening.

Instead of constants, we will actually use variable F_1 and F_2. The effect of this variation may be regarded as a change in horizontal scale in Fig. 6(*a*). By this, e.g., the slope of the downward asymptote in Fig. 6(*a*) may be reduced and the strain-softening segment may be made to exhibit an inflexion point.

Functions F_1 and F_2, called hardening-softening functions, have mainly the following threefold purpose: (1) To model the hardening of response during cyclic loading at small strain; (2) to offset this cyclic hardening effect at large

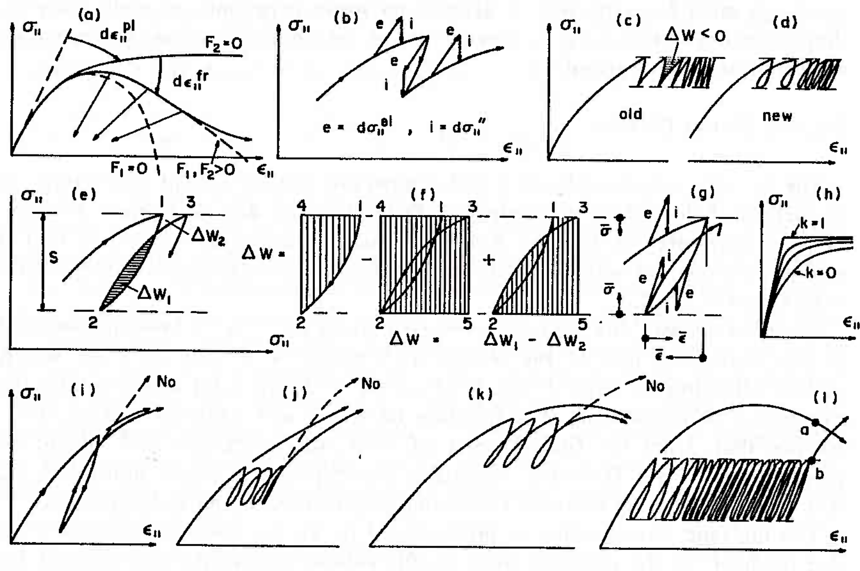

FIG. 6.—Characteristic Stress-Strain Diagrams

strains; (3) to model the effect of hydrostatic pressure and triaxial loading. They may depend only on invariants and may be assumed basically in the form:

$$F_1 = \frac{g_1(\bar{\gamma}, \sigma)}{f_1(\xi)}; \quad F_2 = \frac{g_2(\bar{\gamma}, \sigma)}{f_2(\xi)} \quad \text{(12)}$$

in which $\bar{\gamma} = (e_{km} e_{km}/2)^{1/2}$ = strain intensity. The first effect is modeled by increasing functions $f_1(\xi)$ and $f_2(\xi)$ called cyclic hardening functions. Since ξ represents the path length, it grows during unloading as well as loading and is proportional to the number of loading cycles. (A function of this type was first introduced by Valanis in Ref. 37.) In the previous endochronic theory for concrete (7,10), f_1 was slightly different, being a function of ζ rather than ξ.

Functions $f_1(\xi)$ and $f_2(\xi)$ cause the tail of the uniaxial stress-strain diagram from Fig. 1(*a*) to be raised upward to the extent that no strain-softening (downward slope) might be obtained at constant g_1 and g_2. This must be offset by increasing g_1 and g_2, not of course as a function of ξ but as a function of strain magnitude. This magnitude is mainly characterized by the second deviator invariant J_2^ϵ or by $\bar{\gamma} = (J_2^\epsilon)^{1/2}$ because microcracking, the source of strain-softening, is produced chiefly by deviatoric distortion.

The microcracking is greatly reduced by hydrostatic pressure p ($p = -\sigma$), and so g_2 must greatly decrease with p. The plastic response is also stiffened by hydrostatic pressure, and so g_1 should also decrease with p. Furthermore, the same hydrostatic pressures applied together with uniaxial, biaxial, and triaxial loadings have generally somewhat different effects (7,10), and so functions g_1, g_2, f_1, f_2 must be expected to depend on other invariants as well, such as the third stress invariant, I_3. To tune up the fits, relatively complicated expressions are necessary; see Appendix I.

Inelastic Volume Changes

The inelastic volume dilatancy and compaction are the salient characteristics of inelastic behavior of concrete, yet little attention has been given to them in most literature. In fact, it was first with the endochronic theory that a constitutive relation which models the inelastic volume changes adequately has been found (7,8,10).

In the formal way, the inelastic volumetric strain $d\epsilon'' = d\epsilon''_{kk}/3$ may be obtained as the volumetric part of the tensor $(F_1\, \partial\Psi/\partial\epsilon_{ij} + F_2\, \partial\Phi/\partial\epsilon_{ij})\, d\xi$, which yields, according to Eqs. 7 and 8, $(F_1\, \sigma\, h_1 + F_2\, dh_2/d\epsilon)\, d\xi$, in which the expression in parenthesis is a function of stress and strain invariants. Thus we see that, from the formal point of view, the expression $d\epsilon''$ should be proportional to $d\xi$. However, regarding the expression which multiplies $d\xi$, it is better to set it up directly, combining physical reasoning and experience.

The inelastic strain which is proportional to $d\xi$ describes the dilatancy $d\lambda$ due to shear. In the previous work (7,10), various arguments were offered for $d\lambda$ to be of the form

$$d\lambda = F_3(\bar{\gamma}, \sigma, \lambda)\, d\xi \quad \ldots \ldots \ldots \ldots \ldots (13)$$

in which function F_3 is characterized basically by the following properties: (1) It is negligible at small strain and increases sharply at large deviatoric strain (large $\bar{\gamma}$); (2) it decreases with increasing hydrostatic pressure p; and (3) it decreases with the accumulated dilatancy λ and cannot exceed a certain maximum dilatancy (λ_0 in Appendix I). The detailed expression for F_3, derived by data fitting, is given by Appendix I. It is almost the same as in the previous work.

Although the dilatancy (Eq. 13) is the most important inelastic volumetric strain, there are two more components to be considered (10). One is also due to shear strain and consists of the inelastic decrease of volume which occurs at the start of shear straining in presence of triaxial compression. It was called shear compaction (10) and may be described by

$$d\lambda'' = -F_4(\bar{\gamma}, \sigma, \lambda'')\, d\xi \quad \ldots \ldots \ldots \ldots \ldots (14)$$

in which function F_4, also given in Appendix I, increases with $\bar{\gamma}$ more rapidly

than $d\lambda$ but reaches a maximum value (λ''_0 in Appendix I) much earlier, upon which its growth is terminated to let λ prevail afterwards.

Finally, there is a third component of volume change, which was called hydrostatic compaction (4), λ'. It is observed under pure hydrostatic loading (Fig. 2) and is significant only at very high hydrostatic pressure. Unlike $d\lambda$ and $d\lambda''$, it is not a cross-effect due to shear deformation, and does not result from microcracking. Rather it is caused by a collapse of pore walls due to hydrostatic pressure or volumetric compression. Consequently, the intrinsic time to describe this behavior must involve the first invariant of $d\epsilon_{ij}$, i.e., $d\epsilon_{kk}$, and its increment must be always positive, i.e., $(d\epsilon_{kk})^2$ must be used. One possibility would be to use $d\xi^2 = de_{km}\, de_{km}/2 + c(d\epsilon_{kk})^2$ with c = constant. This would be, however, questionable because entirely different physical mechanisms are at work, as just mentioned (accumulations of microcracking or plastic slip and of pore collapse would be lumped).

Consequently, we must keep $d\xi^2$ defined as $de_{km}de_{km}/2$ and must introduce another, volumetric, path length $d\xi' = |d\epsilon_{kk}|$ to account for pore collapse. In analogy to Eqs. 6 and 10 we may then set:

$$d\lambda' = \frac{\sigma - \alpha'}{3K}\, d\zeta'; \quad d\zeta' = c'\frac{g_0}{f_0}\, d\xi'; \quad d\xi' = |d\epsilon_{kk}| \quad \ldots \ldots \ldots \ldots (15)$$

Here α' = center of the loading surface, which is considered as $\alpha' = 0$ during monotonic loading and is further defined in the sequel, and c' = coefficient needed later for cyclic loading; for monotonic loading $c' = 1$. Functions f_0 and g_0 describe the hardening and softening properties, which are rather different from those given by functions f_1, g_1, f_2, g_2. The main difference is that after the initial inelastic volume decrease a stiffening is subsequently observed, which may be explained by closing of pores, voids, and microcracks. Thus, function g_0 must first increase with pressure p but later the increase must stop, while function f_0 must steadily increase with ξ'. By using the endochronic approach for this inelastic volume change, we automatically provide the irreversibility upon unloading (Fig. 2). For the detailed form of f_0 and g_0, identified by data fitting, see Appendix I.

Combining the three different inelastic volume changes, we finally have:

$$d\epsilon = \frac{d\sigma}{3K} + d\lambda + d\lambda'' + d\lambda' \quad \ldots \ldots \ldots \ldots \ldots (16)$$

Degradation of Elastic Moduli

Whereas plastic strain does not significantly reduce the elastic moduli, microcracking causes both inelastic strain and a reduction of elastic moduli. Let us imagine a material which has a purely elastic matrix but undergoes microcracking; if it is isotropic, $s_{ij} = 2G\, e_{ij}$, and by differentiation $ds_{ij} = 2G\, de_{ij} + 2e_{ij}\, dG$ or $de_{ij} = (ds_{ij}/2G) - e_{ij}\, dG/G$. The last term is readily recognized to be the fracturing inelastic strain increment de^{fr}_{ij} because it is proportional to e_{ij} rather than s_{ij}, i.e., $de^{fr}_{ij} = -e_{ij}\, dG/G$. According to Eq. 10, $de^{fr}_{ij} = e_{ij}\, d\kappa$, and so we conclude that the physical interpretation of $d\kappa$ is $d\kappa = -dG/G$. Since $d\kappa$ has already been expressed, we may use this relation to calculate:

$$dG = -G\, d\kappa \qquad (17)$$

To determine the degradation of bulk modulus K, we may benefit from a recent work of Budianski and O'Connell (13) who used the self-consistent method for composites to determine the macroscopic elastic moduli of the elastic solid containing randomly oriented identical elliptical cracks. They expressed both K and G as a function of crack concentration, from which a relation between K and G results. This relation may be approximately fitted by the equation:

$$\frac{K}{K_0} = f_K\left(\frac{G}{G_0}\right) = 0.46\left(\frac{G}{G_0}\right) + 0.54\left(\frac{G}{G_0}\right)^2 \qquad (18)$$

in which G_0, K_0 = the initial shear and bulk moduli of concrete.

The degradation of G and K according to Eqs. 17 and 18 is different and physically more reasonable than that in Refs. 7, 8, 10 in which G and K were made to decrease as inelastic dilatancy λ increases. This did not provide sufficient decrease of the mean slope of unload-reload loops as the strain at the start of unloading, ϵ_1, becomes higher. This decrease is particularly pronounced when ϵ_1 goes into the strain-softening range (Fig. 5), and it is just in this range that κ and the fracturing strains begin to increase strongly. So it is even empirically apparent that κ, introduced in this study, is the most significant parameter for the degradation of elastic moduli.

Our treatment of degradation neglects the fact that microcracks are actually not randomly oriented but exhibit a prevalent orientation, thus giving rise to stress-induced anisotropy of incremental elastic moduli. However, except for tensile cracking not covered by our model, the prevalence of one orientation is probably not too pronounced.

Jump-Kinematic Hardening for Unloading and Reloading

The basic appeal of the endochronic theory lies in the fact that quite a reasonable prediction of unloading and cyclic loading responses can be obtained without using any inequality conditions. It has even been thought by some that such continuity and smoothness of the formulation is the necessary attribute of a "correct" formulation, and that the inelastic theories with inequality conditions are nothing but "patchwork theories." A more sober view is, however, more reasonable—namely, that the intrinsic time is not a physically fundamental properly but just a convenient variable the use of which reduces the need for inequality restrictions but does not remove it completely. It automatically models only the loading-unloading irreversibility, but predicts incorrectly the unloading-reloading irreversibility. When unloading is reversed to reloading, the observed slope of the stress-strain curve becomes steeper [Fig. 6(*b*)] while in the original endochronic theory the slope becomes less steep [Fig. 6(*b*)]. Besides, the second order work ΔW dissipated during an unload-reload cycle [Fig. 6(*c*)] is negative, which violates Drucker's stability postulate (3). This is not strictly impossible but is rather questionable if no particular phenomenon, such as internal friction, is known to be the culprit (6).

How can we modify the endochronic theory so that the slope becomes steeper when unloading is reversed to reloading, and simultaneously get a positive second-order work ΔW as well as a net increase (rather than a decrease) of inelastic strain upon the completion of an unload-reload cycle? A general constitutive relation which achieves this has been proposed in Ref. 3, and we will now outline it in our context and at the same time generalize it for fracturing strains. We call this constitutive relation hysteretic since it satisfies the aforementioned conditions for the proper form of hysteresis loops.

To make the slope become steeper as unloading reverts to reloading, the inelastic stress increments $d\sigma''_{ij}$ for unloading must point upwards rather than downwards, as they do during loading; see Fig. 6(*g*). This may be achieved by kinematic hardening, such that the centers of the loading surfaces, i.e. α_{ij}, α'_{ij} and α', are made to jump into the maximum stress point at the start of unloading, and again into the minimum stress point (or below it) at the start of reloading. [This approach, which was called jump-kinematic hardening (3), bears some resemblance to Phillips' use of two loading functions (22,29).]

Furthermore, we must use different values of coefficients c and c' (Eqs. 6, 15) for unloading and reloading. To deduce them, let us consider, similarly to Ref. 3, the uniaxial analog of the deviatoric stress-strain relation defined by Eqs. 4, 5, 10. Noting that $d\xi = -d\epsilon$ for reloading, we have

$$d\bar{\epsilon} = \frac{d\bar{\sigma}}{E} + \frac{\bar{\sigma}}{E} F_1\, c\, d\bar{\epsilon} + \bar{\epsilon}\, F_2\, c\, d\bar{\epsilon} \qquad (19)$$

in which $\bar{\sigma} = |\sigma - \alpha_1|$; $\bar{\epsilon} = |\epsilon - \alpha'_1|$; α_1, α'_1 = stress and strain at the preceding maximum (or minimum) point [Fig. 6(*g*)]. The initial condition is $\bar{\epsilon} = 0$ at $\bar{\sigma} = 0$. Consider now an unload-reload cycle [Fig. 6(*e*)] in which a small stress S is removed and reapplied. Noting that $\bar{\epsilon} \approx \bar{\sigma}/E$ up to the first order terms in $\bar{\sigma}$, we see that the integral of Eq. 19 will remain correct up to the second order terms in $\bar{\sigma}$ or S if $\bar{\epsilon}$ is replaced by $\bar{\sigma}/E$. Thus, Eq. 19 may be written as $d\bar{\sigma} = E(1 - a\bar{\sigma})d\bar{\epsilon}$ where $a = c(F_1 + F_2)/E$. Up to second-order small terms in $\bar{\sigma}$ and $\bar{\epsilon}$, this is equivalent to the equation $d\bar{\epsilon} = (1 + a\bar{\sigma})d\bar{\sigma}/E$. The work $\bar{W}$ of stress $\bar{\sigma}$ is defined by $d\bar{W} = \bar{\sigma}\, d\bar{\epsilon}$, which yields $d\bar{W} = \bar{\sigma}(1 + a\bar{\sigma})\, d\bar{\sigma}/E$. Realizing that for a small enough cycle S the coefficient a may be considered constant, we can easily integrate the expressions for $d\bar{\epsilon}$ and $d\bar{W}$:

$$\bar{\epsilon} = \frac{\bar{\sigma}}{E}\left(1 + \frac{a}{2}\bar{\sigma}\right), \quad \bar{W} = \frac{\bar{\sigma}^2}{2E}\left(1 + \frac{2a}{3}\bar{\sigma}\right) \qquad (20)$$

Substituting $\bar{\sigma} = S$ we get the integral from 0 to S.

Assume now that for unloading $c = c_u$ and for reloading $c = c_r$. The net second-order work ΔW dissipated during the cycle equals $\Delta W_1 - \Delta W_2$ where the ΔW_1 and ΔW_2 are the cross-hatched areas in Fig. 6(*e*); ΔW may be calculated as area 1241, minus area 24352, plus area 2352 [Fig. 6(*f*)]; areas 1241 and 2352 are equal to $\bar{W}$ for $\bar{\sigma} = S$ and $c = c_u$ or $c = c_r$, respectively, and area 24352 is equal to $\sigma\bar{\epsilon}$ where $\bar{\epsilon}$ is evaluated from Eq. 20 by substituting $\bar{\sigma} = S$ and $c = c_r$. Summing up these areas, and also subtracting $\bar{\epsilon}$ for $c = c_u$ from $\bar{\epsilon}$ for $c = c_r$ (both for $\bar{\sigma} = S$), we obtain for a small unload-reload cycle

$$\Delta\epsilon = \frac{F_1 + F_2}{2E}(c_r - c_u)\, S^2; \quad \frac{\Delta W}{W_1} = \frac{2}{3E}(F_1 + F_2)\left(c_u - \frac{c_r}{2}\right) S \qquad (21)$$

in which $\Delta\epsilon$ = the strain increase after the cycle; ΔW = the net dissipated second-order work; and $W_1 = S^2/2E$.

It is now apparent that a net strain increase during the cycle will occur if $c_r > c_u$. Furthermore, ΔW will be non-negative if $c_r/2 \leq c_u$. Thus, we must require that

$$c_u < c_r \leq 2c_u \quad (22)$$

which is the result we sought. (In Ref. 3, this result was given for an endochronic theory that was more general in one respect but lacked the fracturing terms.)

To decide when c must switch from 1 to c_u and then to c_r, we need three-way loading-unloading-reloading criteria. These may be defined only in terms of invariants, which may be most reasonably chosen (3) as the deviatoric and volumetric work, W and W':

$$dW = s_{km}\,de_{km}; \quad dW' = 3\,\sigma\,d\epsilon \quad (23)$$

The criteria may be formulated as follows:

1) for $dW \leq 0$ and $W = W_0$: $c = 1$;

for $dW' \leq 0$ and $W' = W_0'$: $c' = 1$ (virgin loading) .... (24*a*)

2) for $dW < 0$: $c = c_u$; for $dW' < 0$: $c' = c_u$ (unloading) (24*b*)

3) for $dW \geq 0$ and $W < W_0$: $c = c_r$;

for $dW' \geq 0$ and $W' < W_0$: $c' = c_r$ (reloading) ...... (24*c*)

Here W_0 and W_0' denote the maximum values of W and W' attained up to the current time. As already mentioned, coefficients α_{ij}, α_{ij}' and α' are set equal to the values that s_{ij}, e_{ij} and σ, respectively, have at the maximum or minimum point. In practical calculations, the following coefficients, satisfying Eq. 22, worked well:

$$c_u = 0.6; \quad c_r = 0.8 \quad (25)$$

To signal the extreme points in the computer program, criteria other than the sign of Eq. 23 could also be used; e.g., the sign of $\Delta e_{ij}\,\Delta e_{ij}^{pl}$, $\Delta e_{ij}\,\Delta e_{ij}^{fr}$ and $\Delta\epsilon\,\Delta\lambda'$ turning negative. Available test data do not let us decide whether this would be better or worse.

The previously published endochronic theory for concrete gives correct strain values at the extreme points of the small cycles superimposed on a static load, but the cycle shape is not correct and gives $\Delta W < 0$. A similar jump-kinematic hardening has been used with success in a parallel work on plastic-fracturing theory (9).

Another loading for which the ordinary endochronic theory is clearly in error is when an unloading-reloading cycle is followed by a new virgin loading in which the stress increases well beyond the previous maximum stress [Fig. 6(*j*–*k*)]. During the cyclic loading ξ becomes large, and the hardening functions $f_1(\xi)$, $f_2(\xi)$ (Eq. 12) then give too stiff a response during the subsequent virgin loading. How can we remedy it? Simply, we consider f_1 and f_2 as functions of ξ_c, which is the same as ξ, $d\xi_c = d\xi$, except that at the reloading-virgin loading transition point we reset ξ_c to the value

$$\xi_c = \xi_c^0 \quad (26)$$

where ξ_c^0 is the value that ξ_c had when the preceding virgin loading changed to unloading. With this device, the response during the renewed virgin loading comes always very close to the response for monotonic loading (without the unloading episode), as it should according to tests; see Fig. 6(*i*–*k*).

The theory expounded so far would not properly distinguish the different stiffnesses of concrete at large strain when this strain is reached through monotonic loading (point a, Fig. 6*l*) or through low-stress cyclic loading (point b, Fig. 6*l*). Under the former loading ξ is of the same order of magnitude as $\bar{\gamma}$, whereas under the latter loading ξ becomes much larger than $\bar{\gamma}$. Thus, a parameter which may distinguish between these two types of loading is

$$C = \left| \frac{\xi}{\bar{\gamma}} - 1 \right| \quad (27)$$

We call it cycling indicator. It is nearly zero for monotonic loading but it may become very large after cyclic loading. (Moreover, moderately large values of C are obtained for nonproportional monotonic tests, and C is also helpful to provide proper distinction there.)

The cyclic response is greatly influenced by hardening-softening functions F_1 and F_2. The expressions in Appendix I give good results only up to about 1,000 cycles. (It must be pointed out that the formulations in Refs. 7, 8, 10 work well for fewer cycles.)

At this point our model is complete. The constitutive relation is defined by Eqs. 4, 6, 10, 13, 14, 15, 16, 17, 18, 24, 26.

Identification of Material Parameters from Test Data

For data fitting, a computer program, similar to that described previously (listed in Ref. 8), has been developed. Small loading steps are used to integrate the constitutive relation numerically for specified forms of material functions and given material parameters. The numerical algorithm is analogous to that in Refs. 7 and 8 and the simulation of various types of tests (uniaxial, biaxial, triaxial, shear-compression, etc.) is done in the same manner. The response curves are automatically generated by the computer on Calcomp plotter (and a sum of the square deviations from the given characteristic data points, representing the objective function to be minimized, may be also evaluated). The data fitting is carried out collectively for all data sets considered, searching simultaneously for the dependence of the material parameters on the strength of concrete. (For details, see Ref. 8).

The data fits achieved are exhibited by the solid lines in Figs. 1–5, and the corresponding numerical parameters are given in Appendix I. It must be stressed that all fits correspond to the same set of material parameters given as functions of the 28-day standard cylindrical strength, f_c'. The theory also fits reasonably well some further test data on cyclic loading [of the type in Fig. 6(*i*–*k*)] which are exhibited in Ref. 6.

The fits in Figs. 1–5 represent a distinct improvement and broadening of scope over the previous endochronic formulations (7,8,10).

The basic information on test procedures and concrete types is summarized in Ref. 9.

Further Considerations

Strain-Rate Effect.—The fact that the effect of strain rate has been neglected in fitting the data in Figs. 1–5 is obviously a simplification. It is nevertheless an admissible simplification because the strain-rates in all these tests were of the same order of magnitude. The present formulation would require further extension when rates of different orders of magnitude are considered.

Number of Parameters in the Theory.—It is certainly disappointing that such a large number of material parameters is needed. Much fewer parameters would, however, suffice for a qualitatively correct model. The effort to closely fit the shape of experimental curves greatly increases the number of parameters. We need to describe a certain curve, such as the dependence of $y = g$, on $x = \bar{\gamma}$ (Appendix I), and if we are lucky we may, for example, discover that it is well-described by some "odd" expression such as $y = c_1 \arctan \sqrt{\exp x^3}$ involving a single parameter c_1. By crude force approach we would represent this same curve by some rational expression, e.g., $y = (c_1 + c_2 x + c_3 x^3)/(c_4 + c_5 x + c_6 x^3)$ involving six parameters. From a fundamental viewpoint this source of difference in the number of parameters causes no difficulty.

Adjusting the Theory to Own Data.—To modify the theory by changing many of the material parameters is a task of major difficulty. However, a need to change many parameters would seldom arise for a structural analyst. He could hardly acquire for his concrete sufficient experimental information for changing but a few material parameters.

Consider, for example, that the complete uniaxial stress-strain curve has been measured. It may then happen that when we plot σ_x versus ϵ_x the shape of the curve differs from the response of the present model for this particular strength. Using Fig. 1 we may, however, construct for the present formulation the normalized plots of σ_x/f'_c versus ϵ_x/ϵ_p where ϵ_p is the strain at peak stress. Among these plots we may pick the one which is closest to our test data regardless of f'_c. Let this plot correspond to some other strength f'_{c_2} and some other strain ϵ_{p_2} at peak stress. We may then use strength f'_{c_2} in our expressions (Eq. 35) for material parameters and carry out in all constitutive relations the transformations in which σ_{ij} is replaced by $k_a \sigma_{ij}$ and ϵ_{ij} is replaced by $k_b \epsilon_{ij}$ in which $k_a = f'_c/f'_{c_2}$ and $k_b = \epsilon_p/\epsilon_{p_2}$. (The coefficients k_a and k_b may then be worked into the material parameters.) These transformations (geometrically representing affinity transformations) may significantly improve data fits and can be easily carried out by hand.

When the analyst's test data for his concrete are more extensive, including for example both uniaxial and triaxial tests, and when our theory indicates a somewhat different relationship of triaxial and uniaxial responses, we need a computer to adjust the parameters of the theory. However, the analyst's data for his concrete would normally be far less extensive than those in Figs. 1–5, and then the task is easy since the information is insufficient anyway for adjusting more than a few parameters, which can be done by a trial-and-error procedure.

The various tests exemplified in Figs. 1–5 are of various degrees of usefulness.

For example, if we first fit both the uniaxial tests and the triaxial tests, the fit of biaxial tests is unlikely to be very bad. Thus, next to the uniaxial test, the most useful test is the triaxial test at a relatively high confining pressure (say $\sigma_2 = \sigma_3 = f'_c$). The axial-torsional test and the uniaxial cyclic test would be the next choice.

Other Endochronic Theories.—Valanis (38) recently proposed a rather interesting generalization of endochronic theory in which $d\xi$ is defined not only in terms of de_{ij} but also $d\sigma_{ij}$, i.e., $d\xi^2 = d\theta_{mn} d\theta_{mn}$ in which $d\theta_{mn} = de_{mn} - k_{mnpq} d\sigma_{pq}$; and k_{mnpq} = given constants. The uniaxial counterpart of this may be written as

$$d\epsilon = \frac{d\sigma}{E} + \frac{\sigma}{E} F\, d\xi; \quad d\xi = \left| d\epsilon - k \frac{d\sigma}{E} \right| \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (28)$$

in which $0 \leq k \leq 1$. For $k = 1$ the differential equation has two solutions: $\sigma = E\epsilon$ and $\sigma = \pm E/F$, which obviously represents an elastic perfectly plastic material [Fig. 6(*h*)]. For $k < 1$ and for $d\epsilon > k\, d\sigma/E$ (loading) we have $(1 - F\sigma/E)\, d\epsilon = (1 - k\, F\sigma/E)\, d\sigma/E$, which gives the slope $E_t = E(E - F\sigma)/(E - kF\sigma)$. As k grows from 0 to 1, the stress-strain curve gradually assumes a more distinct plastic plateau and a sharper transition to this plateau [Fig. 6(*h*)]. With regard to concrete and other geological materials, this property is not useful, and the best fit of test data is obtained for $k \approx 0$ (as has been checked for Fig. 3).

The intention behind the generalized definition of $d\xi$ (Eq. 28) was to satisfy Drucker's postulate. However, this goal is achieved only for $k = 1$, which is of no interest to us. (Moreover, although for $k = 1$ the Drucker's postulate is satisfied, no inelastic strain accumulation is obtained under cyclic loading.) When $0 < k < 1$, the positiveness of the second-order hysteretic work required by Drucker's postulate is achieved only for unload-reload cycles of sufficiently large stress amplitude, greater than a certain limiting amplitude (which is getting smaller as k increases); but for a small enough amplitude this work is always negative.

Criticisms of Endochronic Theory.—Useful criticisms of some elementary forms of the endochronic theory have been raised by I. Sandler and by R. J. Rivlin. They were concerned chiefly with stability and continuity in a uniaxial cyclic loading, and in a multiaxial loading that proceeds in the space $(\epsilon_{11}, \epsilon_{12})$ along a staircase path for which the number of stairs approaches infinity and their size zero. The former criticism has been dealt with in detail in Ref. 3, and the latter one is analyzed in Ref. 6 and a simple remedy is shown. These criticisms do not detract from the present form of the endochronic theory (3,6).

Summary and Conclusions

An improved endochronic constitutive relation is developed to model nonlinear triaxial short-time deformations of concrete. Good fits of representative test data from the literature are achieved and material parameters are identified as functions of concrete strength. The conclusions are:

1. To model strain-softening, the inelastic strain increments which depend on the strain tensor and are called fracturing strain increments are more appropriate

than the previously used ones which depend on the stress tensor.

2. The fracturing and plastic strain increments may be derived from a loading surface that depends on strains rather than stresses and also depends on work. The theory is equivalent to viscoplasticity with a time shift factor that depends on strain rate.

3. Different intrinsic times, both depending on the deviatoric path length, should be used in the plastic and fracturing strain increments. Still another intrinsic time which depends on the volumetric path length is needed to describe the inelastic strain which is due to hydrostatic pressure and results from pore collapse.

4. Degradation of the elastic shear modulus is determined by the intrinsic time for the fracturing strain increments.

5. Degradation of the elastic bulk modulus may be related to that of the shear modulus if one uses Budianski-O'Connell's calculations of elastic moduli of randomly microcracked material by the self-consistent method.

6. To achieve that the energy dissipation under cyclic loading be non-negative and the slope would increase when unloading reverses to reloading, we may use jump-kinematic hardening, in which the center of the loading surface jumps into the current stress point (or strain point) whenever loading reverses to unloading or vice versa. For unloading and reloading the intrinsic time increments must be multiplied by coefficients c_u and c_r (less than 1) such that $c_u < c_r \leq 2\,c_u$.

7. The sign of the work increment and the previous maximum work may be used to define criteria for virgin loading, reloading, and unloading.

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Appendix I.—Material Functions

By optimization of the fits of test data in Figs. 1–5 the following material functions have been identified:

$$g_1 = \frac{1}{g_3}\left[a_0\sqrt{\bar{\gamma}} + a_1 C\bar{\gamma}\left(\frac{a_2\bar{\gamma}}{a_3+\bar{\gamma}}\right)^{1/2} + a_4 g_4\right];$$

$$g_4 = |I_3|^{1/3}\left(\frac{a_2\bar{\gamma}}{a_5(1+a_6|I_1|)+\bar{\gamma}}\right)^3 \quad \ldots\ldots\ldots\ldots \quad (29)$$

$$g_3 = 1 + a_7|I_1|^{1/2} + |a_8 I_3|^{0.4}\,\bar{\gamma}^{2/3}\frac{1}{g_5};$$

$$f_1 = Z_1\left(1 + \frac{\beta_1\xi_c + \beta_2\xi_c^2 + a_{11}(C\bar{\gamma})^{3/4}}{1+|a_{12}I_3|^{1/4}g_1}\right) \quad \ldots\ldots\ldots\ldots \quad (30)$$

$$g_2 = \frac{1}{g_3}\left\{\left[\frac{a_2\bar{\gamma}}{a_{13}(1+a_{14}I_1^2)+\bar{\gamma}}\right]^2 + a_{15}|I_2|^{1/2}\bar{\gamma} + a_{16}g_4\right\} \quad \ldots\ldots \quad (31)$$

$$g_5 = 1 - a_9 I_1\frac{\sigma_{\min}-\sigma_{\mathrm{med}}}{a_{10}-\sigma_{\mathrm{med}}};\quad f_2 = Z_2\left(1 + \frac{\beta_5\xi_c + \beta_6\xi_c^2 + a_{11}(C\bar{\gamma})^{3/4}}{1+|a_{17}I_3|^{1/4}g_2}\right) \quad (32)$$

$$F_3 = \frac{c_0}{1-c_1 I_1}\left(1-\frac{\lambda}{\lambda_0}\right)\left\{\left(\frac{\lambda}{\lambda_0}\right)^2 + \left[\frac{c_2\bar{\gamma}^2}{c_3(1+c_4|I_1|+\bar{\gamma}^2)}\right]^2\right\} \quad \ldots\ldots \quad (33)$$

$$g_0 = \left(\frac{I_1}{b_1-I_1}\right)^2\frac{b_0}{1+b_2\bar{\gamma}^2};\quad f_0 = Z_2(1+\beta_3\xi''+\beta_4\xi''^2);\quad \xi'' = \int g_0\,d\xi \quad (34)$$

$$F_4 = \left[1-\left(\frac{\lambda''}{\lambda_0''}\right)^2\right]\frac{c_6 I_1 - c_5|I_3|^{1/3}}{1+c_7 g_6^2}g_6;\quad g_6 = c_9\bar{\gamma} - c_8|I_1| \quad \ldots\ldots \quad (35)$$

Here I_1, I_2, and I_3 are the first, second, and third invariants of stress σ_{ij}, and $\sigma_{\min}$, σ_{med} are the minimum and medium principal stresses (in the sense of compression). The material parameters are:

$$a_0 = 31.5;\quad a_1 = 400;\quad a_2 = 2;\quad a_3 = 0.003;\quad a_4 = \frac{15}{f_c'};\quad a_5 = 0.025;$$

$$a_6 = \frac{0.03}{f_c'};\quad a_7 = \frac{0.1}{f_c'};\quad a_8 = \left(\frac{25.2}{f_c'}\right)^3;\quad a_9 = \frac{0.14}{f_c'};\quad a_{10} = 0.2f_c';\quad a_{11} = 100;$$

$$a_{12} = \left(\frac{0.0339}{f_c'}\right)^3;\quad a_{13} = 0.0011;\quad a_{14} = \frac{1}{f_c'^2};\quad a_{15} = \frac{5{,}000}{f_c'};\quad a_{16} = \frac{10}{f_c'};$$

$$a_{17} = \left(\frac{0.0137}{f_c'}\right)^3;\quad b_0 = \frac{f_c'}{771};\quad b_1 = f_c';\quad b_2 = 10^5;\quad c_0 = 0.3;\quad c_1 = \frac{2}{f_c'};$$

$$c_2 = 2;\quad c_3 = 1.8\times10^{-6};\quad c_4 = \frac{1}{f_c'};\quad c_5 = \frac{1.8}{f_c'};\quad c_6 = \frac{0.12}{f_c'};\quad c_7 = 5;$$

$$c_8 = \frac{1.8}{f_c'};\quad c_9 = 1{,}500;\quad Z_1 = 0.001;\quad Z_2 = 0.0125;\quad Z_3 = 0.01;\quad \beta_1 = 50;$$

$$\beta_2 = 50{,}000;\quad \beta_3 = 12.5;\quad \beta_4 = 18.9;\quad \beta_5 = 50;\quad \beta_6 = 20{,}000;$$

$$\lambda_0 = 0.001;\quad \lambda_0'' = 0.001 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (36)$$

in which f_c' = 28-day cylinder strength of concrete. A computer program for producing the diagrams in Figs. 2–6 from given material parameters is listed in Ref. 11.

The term with $\sigma_{\min} - \sigma_{\mathrm{med}}$ in g_5 is used mainly for controlling the differences between proportional and standard triaxial tests, and partly also the Schickert's nonproportional triaxial tests. The term $1 + a_6|I_1|$ in g_4 causes the peak stress to occur at a larger strain as the hydrostatic pressure increases. The terms with I_2 and I_3 are used for empirical adjustment of the differences between uniaxial, biaxial and triaxial tests; $I_2 = I_3 = 0$ applies for a uniaxial test; $I_2 \neq 0$, $I_3 = 0$ for a biaxial test; and $I_2 \neq 0$, $I_3 \neq 0$ for a triaxial test. Thus, for example, function g_4 controls response in traixial tests, and has no effect on uniaxial and biaxial tests. So does the term $|a_8 I_3|^{0.4}\,\bar{\gamma}^{2/3}$ in g_3; here I_1 mainly adjusts the initial slopes in triaxial tests for various σ_3 (Figs. 1, 2).

and $\bar{\gamma}$ prevents the strain-softening branch to get too steep. The terms $a_{12}I_3$ and $a_{17}I_3$ prevent excessively strong hardening in triaxial tests. The term $|I_2|^{1/2}$ in g_2 is used to control mainly the initial slope of biaxial tests. The term with C in g_1 controls strictly cyclic tests because $C = 0$ for monotonic loading. The terms $a_{14}I_1^2$ in g_2 and $c_4|I_1|$ in F_3 delay the onset of softening in biaxial tests and especially in triaxial tests, compared to uniaxial tests. The term $a_{15}|I_2|^{1/2}$ $\bar{\gamma}$ in g_2 affects mainly the intensity of strain softening in biaxial tests compared to uniaxial ones. The terms $c_5|I_3|^{1/3}$ causes the shear compaction to be most significant in triaxial tests.

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15781 FRACTURING ENDOCHRONIC THEORY FOR CONCRETE

KEY WORDS: **Concrete; Constitutive equations;** Cyclic loads; **Fracturing; Hysteresis;** Inelastic action; Nonlinear systems; Plasticity; **Triaxial tests;** Viscoplasticity

ABSTRACT: An improved endochronic nonlinear triaxial constitutive relation for concrete is developed: (1)In addition to plastic strain increments proportional to stress tensor, fracturing strain increments proportional to strain tensor are used to model strain-softening; (2)the plastic and fracturing inelastic strain increments are characterized by different intrinsic times; (3)degradation of the elastic shear modulus is determined by the intrinsic time for the fracturing strain increments, and degradation of the bulk modulus is related to that of the shear modulus using Budianski-O'Connell's expressions for the elastic moduli of randomly microcracked material; (4)jump-kinematic hardening, in which the center of the loading surface jumps into the current stress and strain point whenever loading reverses to unloading or vice versa, is introduced. The theory also exhibits hydrostatic pressure sensitivity, inelastic dilatancy, initial shear compaction in triaxial loading, and inelastic strain due to hydrostatic pressure.

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HYSTERETIC FRACTURING ENDOCHRONIC THEORY FOR CONCRETE*

Errata

The following corrections should be made to the original paper:

Page 937, Fig. 6(*c*): The wedge under 13 should be cross-hatched. Fig. 6(*f*): The wedge under 13 should not be cross-hatched.
Page 946, Eq. 30: Should read $a_7|I_2|^{1/2}$ sign (I_2) instead of $a_7|I_1|^{1/2}$
Page 946, Eq. 31: Should read $a_{15}|I_2|^{1/2}\bar{\gamma}^2$ instead of $a_{15}|I_2|^{1/2}\bar{\gamma}$
Page 947, Eq. 33, denominator on right hand side: Should read $c_3(1 + c_4|I_1|) + \bar{\gamma}^2$ instead of $c_3(1 + c_4|I_1| + \bar{\gamma}^2)$
Page 947, Eq. 34: Should read $Z_0(1 + \beta_3\xi'' + \beta_4\xi''^2)$; $\xi'' = \int g_0 d\xi'$ instead of $Z_2(1 + \beta_3\xi'' + \beta_4\xi''^2)$; $\xi'' = \int g_0\, d\xi$
Page 947, Eq. 36, line 4: Should read $b_0 = \frac{f'_c}{771\text{ psi}}$; instead of $b_0 = \frac{f'_c}{771}$;

Page 947, Eq. 36, line 6: Should read $c_8 = \frac{1.8}{f'_c}$; $c_9 = 1{,}500$; $Z_0 = 0.0125$; $Z_1 = 0.001$; $Z_2 = 0.01$; $\beta_1 = 50$; instead of $c_8 = \frac{1.8}{f'_c}$; $c_9 = 1{,}500$; $Z_1 = 0.001$; $Z_2 = 0.0125$; $Z_3 = 0.01$; $\beta_1 = 50$;
Page 947, Eq. 36, line 8: Should read $\lambda_0 = 0.001$; $\lambda''_0 = 0.001$; $E_0 = 4.0 \times 10^6$ psi $+ 10^3(f'_c - 4{,}650$ psi) instead of $\lambda_0 = 0.001$; $\lambda''_0 = 0.001$
Page 947, line 6: Should read "Ref. 8" instead of "Ref. 11"

*October, 1980, by Zdeněk P. Pažant and Ching-Long Shieh (Proc. Paper 15781)