Spac ing of Cracks in Reinforced Concrete

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ABSTRACT: The spacing and width of cracks in a parallel crack system is approximately analyzed using the energy criterion of fracture mechanics as well as the strength criterion. The energy criterion indicates that the crack spacing is a function of the axial strain of the bars and also depends on bar spacing, bar diameter, fracture energy of concrete, and its elastic modulus. Both the energy and strength criteria yield a minimum strain necessary to produce any cracks. The energy criterion and the bond slip conditions further yield a lower bound on possible spacing of continuous cracks. The rules for the formation of shorter, partial length cracks are also set up. Approximate expressions for the crack width at, and away, from the bars are derived. Numerical comparisons indicate satisfactory agreement with existing test data, and lend theoretical support to one aspect of the empirical Gerzegy-Lutz formula obtained by statistical regression analysis of test data. Finally, formation of skew cracks in a biaxially stressed and biaxially reinforced plate is analyzed and a crack spacing formula is derived for one typical case.

Introduction

The width and spacing of cracks in parallel crack systems in reinforced concrete structures have a major influence on structural performance, including shear, tensile and bending stiffnesses, energy absorption capacity, ductility, and corrosion resistance of reinforcement. Much has been learned already about crack width and spacing (1,2,5,14,16,18–20,22–25,29,30), but a completely rational and general method of prediction is still unavailable. We will attempt here an advance in this direction, paying particular attention to crack spacing since it is already approximately analyzed using the energy criterion of fracture mechanics, as well as the strength criterion, must be applied globally over regions that measure at least several aggregate sizes. As a consequence of this fact, the linear fracture mechanics cannot be used (except for very large structures such as dams) and the energy criterion of fracture mechanics, as well as the strength criterion, must be applied globally over regions that measure at least several aggregate sizes. This is clear even without experimental evidence, since structural analysis implies the hypothesis of smoothing of a heterogeneous material by an equivalent homogeneous continuum in which, if one uses the language of the statistical theory of randomly heterogeneous materials, the stresses and strains must be understood as the averages of the actual stresses and strains in the microstructure over the so-called representative volume whose size must be taken to be at least several times the size of the inhomogeneities. On a smaller scale, the continuum concepts are meaningless.

When the fracture process zone is not negligibly small, one must take into account the stress-strain relation of the material, including the decline of stress to zero at very large strains (strain-softening). Its two principal characteristics, which seem sufficient for most practical purposes, are the tensile strength, f, (determined as the peak stress in direct tension tests), and the fracture energy, Gf, which equals approximately the area under the stress-strain diagram times the effective width of the fracture process zone, thus depending on the mean slope of the strain-softening segment of stress-strain relation. The microcracking which must precede fracture formation begins to form approximately at the peak stress point, and fracture obviously cannot form if the stress (peak stress) is not reached. So the strength criterion (applied, of course, to the average stress in the representative volume for which the material can be treated as a continuum) is a necessary condition of fracture, but not a sufficient one. For the fracture to form, the stress must also reduce to zero, and for this to happen the energy characterizing the complete stress-strain curve must be supplied. Thus, the strength criterion indicates only whether the fracture formation can initiate, while the energy criterion indicates whether the fracture can actually form.

The standard approach to fracture analysis (21) is to calculate the rate at which the structure can release energy into the fracture front (the energy release rate) and equal it to the fracture energy (energy of fracture formation per unit crack extension). Equivalently to rates, one can equate the energies per infinitesimal crack length increment. This, however, has no direct physical meaning on a scale that is not large compared to material inhomogeneities. In this regard, it is more realistic to consider only crack length increments which exceed several aggregate sizes, i.e., to consider finite jumps in crack length. Concerning the problems of crack spacing which interest us here, the dimensions of the region in which we analyze fracture are normally less than 10–20 aggregate sizes. We therefore propose here, as an unconventional approach, to treat the formation of cracks in this region integrally, considering the entire crack to form simultaneously in one finite jump.
Thus, instead of considering the energy release rates, we will consider the total energies released by the structure and consumed by the formation of cracks. Note that this is not only more realistic but also more convenient, since the integral energy balance leads to algebraic equations, while the standard fracture mechanics analysis based on energy release rate leads to differential equations (21) [for a further explanation, see Appendix I of the Technical Note which follows (4)].

Note also that the strain energy released during the crack formation depends on both the states before and after the crack formation. This marks a crucial difference from the strength criterion which involves only the state before cracking.

There are now basically two possibilities for the states to be considered before and after cracking. Either we consider the formation of the first cracks, in which case there are no cracks in the initial state and the stress in concrete may be considered uniform, or we consider the formation of intermediate (secondary) cracks during further loading, by which the spacing, s, of the initially existing cracks is reduced to s/2. In the latter case, the stress states before and after cracking are similar and, consequently, the expressions for the energy change of the region come out simpler. At the same time, we should note that the immediate formation of a complete regular system of parallel cracks from an uncracked state is unlikely for reasons of random scatter in the values of fracture energy and other material properties. The material is weaker at some points than others, and so the very first cracks will always be few and very far apart. Accordingly, we now restrict attention to the question when the crack spacing, 2s, is halved, i.e., becomes s. (Some of the formulas will, nevertheless, also apply to the formation of the first cracks.)

**METHOD OF APPROXIMATE ANALYSIS**

We consider a prismatic region of concrete (Fig. 1) which contains at its axis one reinforcing bar and has a rectangular cross section of sides $b_1$ and $b_2$, denoted so that $b_1 \leq b_2$. This can serve as a model not only for a concrete member with a single reinforcing bar, but also for the situation around one bar in a row of parallel bars of spacing, $b_1$, in a wall [plate, shell, slab, panel (Fig. 1)], or in a rectangular array of bars. In treating all these situations in the same manner, we must abandon the conditions for compatibility of strains in the transverse direction, and we must limit consideration to only uniaxial stress in the bar direction, distributed uniformly over the region, for the sake of simplicity.

To simplify our analysis, we will further treat the prismatic region as a cylindrical region with diameter $b$ such that the cross-sectional area, $A$, of the region remains the same, i.e., $A = b_1 b_2 = \pi b^2 / 4$ or

$$b = 2 \left( \frac{A}{\pi} \right)^{1/2}; \quad A = b_1 b_2 \quad \text{.................................................. (1)}$$

To simplify the stress analysis, we will exploit the old concept of inclined "stress lines" along which a concentrated force—in our case the force transmitted into concrete by bond—spreads into an elastic solid. Referring to Fig. 1(g), we thus imagine that the formation of an intermediate crack relieves the stress within the shaded triangular regions, 012, and that the preceding cracking has already relieved the stress in the triangular regions, 456 [Fig. 1(g)]. The slope of the stress lines limiting these regions is roughly 0.7, the precise value to be determined empirically since this concept is approximate (Appendix I, Ref. 4).

The method of stress lines works reasonably only when the geometry of the stress relief region is simple. This includes the cases of very sparse cracks $[s >> b/2$, Fig. 1(g)] or very dense cracks $[s << b/2$, Fig. 1(i)], but not the case in Fig. 1(h) for which $s/b \approx 1$. For this situation, we will therefore have to devise a certain approximate interpolation technique.

It must be admitted that the method of stress lines is often correct only in the order of magnitude, and a two or three-fold error is possible. However, the relative error remains about the same for similar geometries, and the method gives correct structure of the expression for the energy release, except for a numerical factor. Since we will calibrate the numerical factor by test data, the method seems acceptable.

It is well known that the bond shear stresses may cause the steel bars to slip. To make explicit formulas attainable, we will assume that either there is no bond slip or the bond slip occurs over the entire length of the bar and the bond shear force per unit length, $F_b$, is constant and equal to its ultimate value, $F'_b$. In reality, $F_b$ is of course variable; it equals...
zero at intersections with the cracks and at the midlength between the cracks, causing the length $L_b$ of bond slip to become in reality less than the crack spacing. Furthermore, the value of $F_b$ need not be the same as that from pullout tests. Because we deal here with a much shorter bond slip length and smaller slip displacements, $F_b$ can be higher than in these tests.

Do we need to consider that a transition from a no-slip situation to bond slip can occur during crack formation? Within our assumptions we do not, as long as we assume the bond slip to occur over the entire length of the bar between the cracks. We consider the crack to form at a constant overall deformation, and so the crack formation generally relieves the stresses. After the crack spacing is halved, the situation is geometrically similar and stress distribution is then also similar for our assumptions. Thus, if there is no bond slip before cracking, there is none after cracking in our model. An exception is the formation of first cracks during which the bond slip can occur. Although it would be possible to analyze this case with our method, we omit it for the reasons already explained.

**Conditions for Initiation of Microcrack Bands.**—If there is no bond slip, the strength criterion requires that

$$\epsilon_s \geq \frac{f'_t}{E_t} \quad \text{(no slip)}$$

in which $f'_t =$ tensile strength of concrete; and $E_t =$ its Young's modulus. (This condition also applies to the formation of first cracks.)

If there is bond slip, the cracks initiate only if the stress produced in segment 01 [Fig. 1(g,i)] by the bond force accumulated over the bond slip length, $L_b$, equals $f'_t$. We now distinguish the case of very sparse cracks ($s \gg b/2$) and very dense cracks ($s \ll b/2$). For $s \gg b/2$ [Fig. 1(g)], the equilibrium condition is $F_b L_b = \pi(b^2 - D^2)f'_t/4$, and since $L_b \leq s$, we may write for $s >> b/2$:

$$s \geq \frac{\pi f'_t}{4 F_b h_1}; \quad h_1 = \frac{4}{b^2 - D^2}$$

For $s << b/2$ [Fig. 1(i)], the equilibrium condition is $F_b L_b = \pi[(2ks + D)^2 - D^2]f'_t/4$, and since $L_b \leq s$ we get for $s << b/2$:

$$s \geq \frac{\pi f'_t}{4 F_b g_1(s)}; \quad g_1(s) = \frac{1}{ks(b^2 - D^2)}$$

For any $s/b$ we may now expect, upon comparing Eqs. 3–4, that $s \geq \frac{\pi f'_t}{F_b \phi_1(s)}$, in which $\phi_1(s) =$ a smooth function which approaches $h_1$ when $s >> b/2$, and $g_1(s)$ when $s << b/2$ [Fig. 2(a)]. A simple function which has these two asymptotic properties is $\phi_1(s) = h_1 + g_1(s)$, and with it we acquire the general approximate relation

$$\frac{4s}{b^2 - D^2} + \frac{1}{k(ks + D)} = \frac{\pi f'_t}{F_b} \quad \text{(bond slip)}$$

which we now need to solve for $s$. We may transform Eq. 5 into the quadratic inequality $s^2 + 2p_1 s + p_2 \geq 0$, in which

$$p_1 = \frac{D}{2k} - \frac{\pi f'_t}{8 F_b (b^2 - D^2)}; \quad p_2 = \frac{b^2 - D^2}{4k} \left(1 - \frac{\pi f'_t k D}{F_b}ight)$$

Denoting $s_{1,2} = -p_1 \pm (p_1^2 - p_2)^{1/2}$ (the roots), we may write this inequality as $(s - s_1)(s - s_2) \geq 0$. The solution then is

$$s \geq s_1 \quad \text{or} \quad \text{(if } s_2 > 0\text{)} \quad s \leq s_2 \quad \text{(bond slip)}$$

We see that the strength criterion provides a lower bound, $s_1$, on the crack spacing at which cracks can initiate if there is bond slip. However, there may also exist an upper bound, $s_2$, above which no regularly spaced cracks can exist.

Eqs. 2 and 7 indicate whether the microcracking can start but not whether it can lead to the formation of a distinct crack. This question is decided by the energy criterion, to which we turn our attention now.

**Energy Approximations and Conditions for Continuous Crack Formation.**—Consider first that there is no bond slip. If the cracks are sparse ($s >> b/2$), the stress relief region is a body of revolution whose cross

![FIG. 2.—(a) Auxiliary Functions; (b,c,e,f) Spacing-Strain Diagrams; (d) Stress-Strain Diagram; (g) Cracks of Partial Length; (h) Interacting Concrete Area for Bending; (i-n) Parallel Crack Systems Skew to Bars](image)
A simple function which exhibits these two asymptotic properties is \( c_1(S) \) (no slip) ................................ (8)

\[
\Delta U = \frac{E_c \varepsilon_t^2}{2h_2} \quad h_2 = \frac{24k}{\pi(b-D)^2(2b+D)}
\]

If the cracks are dense (\( s << b/2 \)), the stress relief region is a body of revolution whose cross section is crosshatched as in Fig. 1(g); its volume is \( V = \pi(b-D)^2(2b+D)/24k \). Since there is no bond slip, the strain energy density in this volume before cracking is \( E_c \varepsilon_t^2/2 \). So, the energy release due to the formation of the intermediate cracks is \( \Delta U = VE_c \varepsilon_t^2/2 \) for \( s >> b/2 \):

\[
\Delta U = \frac{E_c \varepsilon_t^2}{2h_2} \quad g_2(s) = \frac{4}{\pi ks^2(ks + D)}
\]

Comparing Eqs. 8–9, we may now expect that for any \( s/b \), \( \Delta U = E_c \varepsilon_t^2/2g_2(s) \), in which \( g_2(s) \) is a smooth function which asymptotically approaches \( h_2 \) when \( s >> b/2 \), and \( g_2(s) \) when \( s << b/2 \) [Fig. 2(a)]. A simple function which has these two asymptotic properties is \( g_2(s) = [h_2^2 + g_2(s)]^{1/n} \). (From test data fitting, the value \( n = 4 \) was found to be reasonable.) Thus, for any \( s/b \):

\[
\Delta U = \frac{V}{2} E_c \varepsilon_t^2; \quad V = \left( \frac{24k}{\pi(b-D)^2(2b+D)} \right)^n + \left( \frac{4}{\pi ks^2(ks + D)} \right)^{-1/n}
\]

is obtained.

Suppose now that the steel bars slip. The stress, \( \sigma_1 \), in concrete in the crosshatched regions of Fig. 1(g,i) before cracking no longer is \( E_c \varepsilon_t \) and must be figured out from the ultimate bond forces, \( F_b \), applied on concrete along the bar [Fig. 1(j)]. Their resultant is \( F_b s \), and so the average stress on the cross section, \( 01 \), in Fig. 1(j) or Fig. 1(g,i) is, approximately, for \( s >> b/2 \):

\[
\sigma_1 = F_b s h_3; \quad h_3 = \frac{4}{\pi(b^2 - D^2)}
\]

For \( s << b/2 \), we consider the average \( \sigma_1 \) at distance \( s/2 \) from the crack:

\[
\sigma_1 = F_b s g_3(s); \quad g_3(s) = \frac{1}{\pi ks(ks + D)}
\]

Here \( 1/h_1 \) and \( 1/g_1(s) \) represent the cross-sectional areas [Fig. 1(g,i)] over which \( \sigma_1 \) is distributed. Comparing Eqs. 11–12, we may now expect that, for any \( s/b \), \( \sigma_1 = F_b s \phi_3(s) \), in which \( \phi_3(s) \) is a smooth function which asymptotically approaches \( h_3 \) when \( s >> b/2 \), and \( g_3(s) \) when \( s << b/2 \) [Fig. 2(a)]. A simple function which exhibits these two asymptotic properties is \( \phi_3(s) = h_3 + g_3(s) \); this yields, for any \( s/b \), \( \sigma_1 = F_b s[h_3 + g_3(s)] \) or

\[
\sigma_1 = \frac{F_b s}{\pi} \left[ \frac{4}{b^2 - D^2} + \frac{1}{ks(ks + D)} \right]
\]

The loss of strain energy due to crack formation may now be expressed for any \( s/b \) as

\[
\Delta U = \frac{V}{2E_c} \sigma_1^2 \quad \text{(bond slip)} \quad \Delta W_b = \frac{V}{4E_c} \sigma_1^2 \quad \text{(bond slip, short cracks)}
\]

in which \( V = \) the same as already given in Eqs. 10.

The work, \( \Delta W_b \), consumed by fracture formation depends on the crack area. If the crack spacing is not sufficiently sparse, the cracks do not extend all the way to the boundary of the region. According to the inclined stress lines indicating the spread of the bond force [Fig. 1(i)], this occurs when \( ks + D/2 < b/2 \). Thus, for \( s << b - D/2k \):

\[
\Delta W_b = \frac{\pi}{4} (b^2 - D^2) \phi_3 \quad \text{(full length)}
\]

The condition for the formation of continuous cracks can now be stated as follows:

\[
\Delta U \geq \Delta W_b + \Delta W_s \quad \text{(bond slip)}
\]

in which \( \Delta W_s = 0 \) if there is no bond slip. The case = corresponds to static (slow) crack formation, and the case \( b/2 \) to dynamic crack formation in which the excess of the released strain energy goes into kinetic energy.

In the energy balance during crack formation, we should not forget to subtract the work consumed by a change, \( \Delta W_b \), in bond slip, if any. It may be expressed as \( \Delta W_b = F_b s \Delta \phi_3 \) in which \( \Delta \phi_3 = \) average change of slip. To estimate \( \Delta \phi_3 \), we sketch in Fig. 1(j) the distributions of the bond force, \( F_b \), and of the strains in concrete in the vicinity of the steel bar before and after crack formation. Since \( \varepsilon_t \) is assumed to remain constant during the crack formation, we may consider [according to Fig. 1(j)] that

\[
\Delta \phi_3 = \frac{\phi_3}{E_c} \frac{\Delta s}{s/2} \quad \text{in which } \phi_3 = \text{average strain}
\]

The foregoing equation gives an acceptable estimate for \( \Delta W_b \) only when the cracks are of full length. In the case of partial length, \( \sigma_1 \) varies within the region and the value of \( \sigma_1 \) at the line of symmetry is not a good approximation for \( \sigma_1 \). \( \Delta W_b \) may be expressed as \( \Delta W_b = F_b s \Delta \phi_3 dx \) in which \( \Delta \phi_3 = \) bond slip change from before cracking to after cracking [Fig. 1(i)]. Before cracking, \( F_b s \Delta \phi_3 dx \) must equal \( \Delta U \) given by Eq. 14 due to the requirement of energy conservation in the region of stressed concrete (considered elastic). From Fig. 1(j) we see that

\[
\Delta \phi_3 = \frac{1}{2} \frac{\Delta s}{s/2} = \frac{1}{2} \frac{\Delta s}{s/2} \quad \text{in which } \Delta s = \text{average strain}
\]

and, therefore, \( \Delta W_b = \Delta U/2 \) in which \( \Delta U \) is given by Eq. 14. So we have

\[
\Delta U = \Delta W_b = \frac{V}{4E_c} \sigma_1^2 \quad \text{(bond slip, short cracks)}
\]
Approximate Relation of Crack Spacing to Strain.—As the strain is increased, the first cracks appear at a certain strain and subsequently further cracks are produced, which causes the crack spacing to diminish as the strain grows. We are therefore interested in constructing the diagram of $s$ vs. $\epsilon_s$. At the beginning of loading the cracks may be supposed to be far apart and no bond slip to exist. The energy requirement for fracture to occur is that $\Delta U = \Delta W_f$ in which $\Delta W_f$ is given by Eq. 16. This yields, for the cracks cutting all the way across the width, $b$: 

$$\epsilon_s = \left[ \left( \frac{2k}{E_c} \right)^n \left( \frac{b^2}{k_s^2 (k_s + D)} \right)^{1/2} \right]$$

(no slip, full length) \dots \dots \dots \dots \dots (20)

in which $\epsilon_s = \max$ (Eq. 20, Eq. 2) \dots \dots \dots \dots \dots (22)

in which Eq. 2 prevails until a certain strain value, and then afterwards Eq. 20 prevails. In the cases where Eq. 2 prevails, the cracks form suddenly, with a snap (dynamic process), since the excess of $\Delta U$ over $\Delta W_f$ must go into kinetic energy [Fig. 2(b–d)].

At a certain, sufficiently large strain, the bond strength becomes exhausted and the bars slip within concrete. Considering cracks of full length, we now have $\Delta U$ given by Eqs. 13–14, $\Delta W_f$ by Eq. 16, and $\Delta W_b$ by Eq. 18. Eq. 17 then yields the condition

$$F(s) = \frac{V\sigma_1}{2E_c} - \frac{E_c^2 k_s^2}{2E_c} \sigma_1 - \frac{\pi}{4} (b^2 - D^2) k_s \geq 0$$

(full length) \dots \dots \dots \dots \dots (23)

This condition does not involve $\epsilon_s$, and so the solution is a fixed lower limit on the spacing of the cracks of full length $[s_{\text{min}}$ in Fig. 2(b–c)]. The left-hand side of Eq. 23 has the form $P(s)/s^2$ in which $P(s) = a sixth-order polynomial in s. The solution of Eq. 23 can be obtained by Newton iteration.

An approximation to the solution of Eq. 23 can, however, be obtained by replacing Eq. 13 with the limiting value for sparse cracks, $\sigma_1 = 4F_s k_s/\pi (b^2 - D^2)$, and Eqs. 10 with $V = \pi(b - D)^2(2b + D)/2k_s$, and by neglecting $\Delta W_b$, then

$$s_{\text{min}} = \left( \frac{3\pi^2 k s b (D - b) (2b + D)}{4F_s k_s^2 (2b + D)} \right)^{1/2}$$

(full length) \dots \dots \dots \dots \dots (24)

If we want to calculate $s$ from Eq. 20, we need to solve a cubic equation. An approximate solution may be obtained by neglecting $D$, and we then get

$$s = \frac{8k_A}{(\pi k_s^2 (ks)^2 - (\epsilon_s)^3)^{1/3}} \qquad \frac{6k_A}{E_b} \quad \text{in which} \ A = \pi b^2/4.$$ When the ratio $\epsilon_s/\epsilon_o$ is large, the $\epsilon_o$ value may be neglected, of course, compared to $\epsilon_s$, and then $s$ is proportional to $\sqrt{A}$. This observation is interesting with regard to the well-known work of Gergely and Lutz (16,23). Using extensive statistical curve-fitting analysis of numerous test data rather than a mechanical analysis, they found that the crack width, $w$, is about proportional to $\sqrt{A}$. Since $w$ is roughly proportional to $s$ (see Eq. 32 in the sequel), Eqs. 25 now give theoretical support to Gergely and Lutz’s important classical result. This result, in turn, reinforces our theoretical arguments.

The continuous relation between $s$ and $\epsilon_s$ that we established (Eq. 20) does not mean that the crack spacing, $s$, varies continuously as the strain increases. This relation merely indicates the successive strain values at which the existing crack spacing is halved by the formation of intermediate cracks. The evolution of the crack system by successive halvings of its spacing is depicted by the descending staircase in Fig. 2(e), for which the bottom corners of the steps lie on the smooth curve of $s$ versus $\epsilon_s$. Because $s \rightarrow \infty$ at $\epsilon_s \rightarrow \epsilon_o$, the magnitude of the initial spacing can scatter very greatly with only small deviations from $\epsilon_o$. Therefore, the value of the initial spacing is essentially random, and for any point on the smooth $(s, \epsilon_s)$ curve there always exists an initial state (with strain as close to $\epsilon_o$ as desired) from which this point can be reached by successive halvings.

The foregoing solutions apply only for cracks of full length, cutting all across the width $b$. For short cracks (i.e., those of length less than $b$), different relations ensue. Consider first that the bars do not slip. Then, setting $\Delta U = \Delta W_f$, and substituting Eqs. 10 and 15, we find that the strain must equal or exceed the values for $s \leq b - D/2k$:

$$\epsilon_s = \left[ \frac{8k_A}{E_b} \left( \frac{6k_s (ks + D)}{(b - D)^2 (2b + D)} \right) \right]^{1/3} \quad \text{in which} \quad n = 4 \quad \text{is most reasonable. This equation is plotted in Fig. 2(f).}$$

We may note that for a given $\epsilon_s$ value there may exist two possible values of $s$ when $\epsilon_s > \epsilon_s$, and none when $\epsilon_s < \epsilon_s$ in which $\epsilon_s$ [Fig. 2(f)] is obtained by setting $\epsilon_s/\partial s = 0$. The solution is simple only when $n = 1$ [Fig. 2(f)], and then we obtain a cubic equation for the corresponding value, $s_1$. Furthermore, when also $D/b \ll 1$, Eq. 26 simplifies to

$$\epsilon_s = \left[ \frac{8k_A}{E_b} \left( \frac{3k_s^3 s^3}{b^3} + 1 \right) \right]^{1/3} \quad \text{(for} \ D/b \ll 1, \ n = 1) \quad \text{\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (27)$$

The condition, $\partial \epsilon_s/\partial s = 0$, then yields for $s$ and $\epsilon_s$ the solutions
\[ s_0 = \frac{b}{k} \quad (for \ D \ll b, \ n = 1) \]  

Now we may note that the limitation \( s \leq b/2k \) (\( D \approx 0 \)) is stricter than \( s \leq 0.550 b/k \). Therefore, the upper branch shown as the dashed portion in Fig. 2(f) never applies, and the same may be supposed to approximately hold for finite \( D \).

As mentioned before, the length, \( L \), of these short cracks when they form cannot exceed a certain limit (approximately \( L \leq ks/2 \), in which \( k \approx 0.7 \)). Thus, as long as the bond slip does not occur, the pattern of cracks produced by successive halvings of the spacing as the strain increases looks like that shown in Fig. 2(g). The intermediate cracks are getting shorter and shorter and the cracks are getting denser and denser. This type of behavior, as sketched in Fig. 2(g), was experimentally observed by Broms (11,12) in his study of bending cracks in beams. A fundamental theoretical explanation, with exact rules, governing this behavior was obtained through stability analysis of parallel crack systems (6,10).

Suppose now that the bar slips; for simplicity consider the case \( s \ll b/2 \). Then, substituting Eq. 15 and Eq. 19 with \( \sigma_1 \) according to Eqs. 12, and with \( V = \pi s^2 (ks + D)/4 \) into Eq. 17, we obtain

\[ s \leq s_0 = \frac{F_x^2}{16\pi^2 k^2 E_s' b} \]  

which represents the upper bound on possible crack spacing for short cracks. In the case where this bound is smaller than the bound in Eq. 23 for full-length cracks, cracks of partial length can never form. In the opposite case, they will.

The bound, \( s_0 \) (Eq. 29), must be compared further with the strength bound, \( s_1 \), in Eq. 7. If \( s_1 < s_0 \), then the strength criterion governs and the bound, \( s_0 \), in Eq. 7 can never be reached.

**Crack Opening Width.—**If the concrete between the cracks were without stress, then the crack opening width would be \( w = s \varepsilon_s \). Due to the tensile straining of concrete, this is merely an upper bound, i.e., \( w \leq s \varepsilon_s \). To estimate \( w \), we must now subtract from \( s \varepsilon_s \) the deformation due to the tensile stress in concrete. For the crack width near the surface of the bars, we may thus write

\[ w = s \varepsilon_s - \frac{\left( s - \frac{b - D}{k} \right) \sigma_1}{E_c} \]  

and for \( s \leq b - D/k \):

\[ w = s \varepsilon_s \]  

**Numerical Example.—**To illustrate our formulation, consider the following example: \( b = 4 \) in. (10.16 cm); \( D = 0.75 \) in. (1.905 cm); \( E_s = 29 \times 10^6 \) psi (200,000 MPa); \( E_s = 3 \times 10^6 \) psi (20,690 MPa); \( f; = 400 \) psi (2.758 MPa); and \( k = 0.7 \). We estimate the fracture energy from a new formula, \( f' = (2.77 + 0.0214 f; ) f;^2 d_s / E_s \) (developed in Ref. 3), in which \( d_s \) is the maximum aggregate size. Considering \( d_s = 1 \) in., we thus have \( f' = 0.602 \) lb/in. (105.3 N/m). The calculated results for crack spacing and crack width are shown in Fig. 3(a,b). The values of the crack spacing are obtained by solving Eqs. 10 and 16 for the full-length cracks. The minimum crack spacing expressed by Eq. 24 is shown by the dash-dot line in Fig. 3(a). The lower value of the minimum crack spacing can be obtained by using a larger bond force [see the dash-dot line at bottom of Fig. 3(a)]. The crack width is calculated from Eqs. 31–32 and is shown in Fig. 3(b).

Now we consider the case when \( b = 6 \) in. (15.24 cm), with the other data remaining the same. The calculated curves for this case are shown in Fig. 3(c,d) in the same manner as in Fig. 3(a,b).

**Extension to Beams in Flexure.—**Although we assumed the initial stress before cracking to be uniform, we may apply our formulation in an approximate sense also to situations where the initial stress is nonuniform. This is the case in beams under flexure [Fig. 2(h)]. To find an equivalent situation within our assumptions, it is perhaps best to assume that the strain energy of concrete within area \( A_0 \) below the neutral axis [Fig. 2(h)] is equal to the strain energy within an equivalent area, \( A_t \), in which the strain is uniform and the same as the actual steel strain. This requirement yields \( \int f' E_s \varepsilon^2 B dz/2 = E_s \varepsilon_c B b/2 \), in which \( B \) = beam width, \( b \) = the depth of equivalent area \( A_t \) [Fig. 2(h)], \( \varepsilon \) = actual strains due to bending, and \( z \) = a coordinate. Since \( \varepsilon = \varepsilon_z / c_y \), we obtain

![FIG. 3.—Diagrams of Spacing or Crack Width versus Strain Calculated from Typical Numerical Examples (1 in. = 25.4 mm)](image)
in which $c_1, c_3 =$ distances from the neutral axis to the bottom face and to the steel bar axis. This value of $b$ may then be used in the preceding formulation, considering the bar to be in the center of depth $b$.

**Comparison with Test Data**

To verify our formulation, we show in Figs. 4–6 comparisons with the test data available in the literature [13, 14, 18–20, 22]. The parameters of our formulation which correspond to the plotted curves are listed in Table 1. For all the figures, $k = 0.7$ was used as it was found to give the best results.

In our comparisons with the test data, we must be cautious in interpreting the observed crack spacing. For a regularly spaced crack system, the crack spacing may decrease with increasing strain only by halvings, but the reported data [13, 14] indicate a series of much smaller decreases in the crack spacing (Fig. 4). This is due to the fact that the observed crack system was no doubt somewhat irregularly spaced, and that the overall bar extension was not the same between each two cracks (due to variation of the bending stress along the beam near its ends). Such irregularities, although inevitable, are not covered by our theory. Since the overall observed decrease of crack spacing is less than one half, it would make no sense to use our formula for the $s$ vs. $e$, curve (Eqs. 20, 21, 25), and so we must content ourselves by fitting the test data in Fig. 4 (14, 13) merely with the formula for the constant limiting (minimum) values of crack spacing (Eq. 24). The corresponding crack widths may be calculated by using Eqs. 31–32. A possible effect of transverse reinforcement, which we neglect, may also cause deviations from our theory.

Fig. 4 shows the comparisons of our theory for the crack spacing with the test data by Clark (14), and Chi and Kirstein (13). Fig. 5(a–d) shows a comparison with the average crack width data by Clark, indicating an increase of crack width with the increase of steel strain. Fig. 5(e–h) shows crack width comparisons for the data by Chi and Kirstein. Fig. 6 shows the comparisons with similar data by Kaar and Mattock (20), Hognestad (18), and Mathey and Watstein (22).

We see that despite various rather crude approximations, our theory compares satisfactorily with test data and better than the previously developed empirical formulas. Since our theory was developed theoretically, it should also have broader applicability, i.e., it should be applicable even for cases outside the range of existing test data.

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**FIG. 4.—Comparison of Calculated Crack Spacing with Test Data by: (a–d) Clark (1956); (e–h) Chi and Kirstein (1958)**

**FIG. 5.—Comparison of Calculated Average Crack Width with Test Data by: (a–d) Clark (1956); (e–h) Chi and Kirstein (1958)**

---

$t = \frac{c_1^3}{3c_3^3}$ ....................................................... (33)
i.e., the vertical deviations of data points, have to be a straight line \( Y = a + bX \) with \( a = 0, b = 1 \). So, a linear regression analysis may be applied. The difference of \( a \) from 0 and of \( b \) from 1 then indicates possible improvement in the optimum calibration of our theory [which is seen to be negligible for Fig. 7(a)]. The errors, i.e., the vertical deviations of data points, \( Y_i \), from the regression line, \( Y' = a + bX \), indicate how good the theory is. The coefficient of variation of the errors, defined as \( \sigma = s/\bar{Y} \), in which \( s \) = standard error, \( s^2 = \sum(Y_i - Y')^2/(n - 2) \), and \( \bar{Y} = \sum Y_i/n \) (\( i = 1, 2, \ldots, n \)), is found to be \( \sigma = 0.120 \).

Another useful plot may be constructed by plotting \( Y = w_i/\epsilon_i \) vs. \( X = \epsilon_i \), in which \( \epsilon_i \) = steel strain [Fig. 7(b)]. The coefficient of variation of errors in this type of regression is found to be \( \sigma = 0.126 \).

The 95% confidence limits corresponding to \( \sigma \) are plotted as the dashed curves in Fig. 7(a,b). These curves are hyperbolas, but due to the large size of our statistical samples they are almost straight, in which case the 95% confidence limits relative to mean \( \bar{Y} \) are approximately 1.96 \( \sigma \).

We see from Fig. 7 that our theory achieves a satisfactory agreement with test data.

As for the errors in the crack spacing, we may characterize them by the coefficient of variation, \( \omega \), of the population of the values of \( \epsilon_i - s \), \( \epsilon_i \) and \( s_i \) being the measured and theoretical spacings. From all the data points in Fig. 4 we get \( \omega = 0.145 \). We should note, however, that a statistical comparison of the errors in spacing based on Fig. 4 is of limited relevance since, as we mentioned before, the spacing data available involve some irregular spacings, do not have a sufficient range to show the halving of spacing, and do not permit showing the changes of regular spacing with steel strain. Because of probable irregular spacing, the first and the last data points in Fig. 4 were not included in the calculation of \( \omega \), given previously.

To calculate the fracture energy, \( \psi \), the recently developed new formula from Ref. 3 has been used.

The values of ultimate bond force, \( F' \), obtained as the optima for fitting the data (Table I) may be analyzed to see whether they show some systematic pattern and can be approximately predicted. Assuming that \( F' \) depends on the compressive strength, \( f' \), of concrete, and carrying out a least-square regression analysis of the optimum values of \( F' \) listed in Table 1, we can obtain the following approximate formula for the ultimate bond force per unit length of a deformed steel bar:

![Table 1 - Parameters for Test Data](image)

<table>
<thead>
<tr>
<th>Test Series</th>
<th>( E_i ), in thousand pounds per square inch</th>
<th>%t, in pounds per inch</th>
<th>( f_i ), in pounds per inch</th>
<th>( d_i ), in inches per inch</th>
<th>( F' ), in pounds per inch</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Clark</td>
<td>No. 1 3,212</td>
<td>0.505</td>
<td>385</td>
<td>1.0</td>
<td>3,876</td>
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<tr>
<td></td>
<td>2 3,192</td>
<td>0.411</td>
<td>357</td>
<td>1.0</td>
<td>3,499</td>
</tr>
<tr>
<td></td>
<td>3 3,457</td>
<td>0.412</td>
<td>367</td>
<td>1.0</td>
<td>3,199</td>
</tr>
<tr>
<td></td>
<td>4 1,912</td>
<td>0.393</td>
<td>290</td>
<td>1.0</td>
<td>3,783</td>
</tr>
<tr>
<td>2. Chi, Kirstein</td>
<td>No. 1 4,277</td>
<td>0.684</td>
<td>476</td>
<td>1.0</td>
<td>5,950</td>
</tr>
<tr>
<td></td>
<td>2 3,713</td>
<td>0.661</td>
<td>447</td>
<td>1.0</td>
<td>6,426</td>
</tr>
<tr>
<td></td>
<td>3 3,422</td>
<td>0.611</td>
<td>422</td>
<td>1.0</td>
<td>6,391</td>
</tr>
<tr>
<td></td>
<td>4 1,500</td>
<td>0.204</td>
<td>207</td>
<td>1.0</td>
<td>2,771</td>
</tr>
<tr>
<td>3. Kaar, Mattock</td>
<td>No. 1 2,891</td>
<td>0.287</td>
<td>259</td>
<td>1.5</td>
<td>3,470</td>
</tr>
<tr>
<td></td>
<td>2 3,149</td>
<td>0.304</td>
<td>273</td>
<td>1.5</td>
<td>3,504</td>
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<tr>
<td></td>
<td>3 3,028</td>
<td>0.524</td>
<td>329</td>
<td>1.5</td>
<td>3,868</td>
</tr>
<tr>
<td></td>
<td>4 3,120</td>
<td>0.217</td>
<td>240</td>
<td>1.5</td>
<td>3,735</td>
</tr>
<tr>
<td>4. Hognestad</td>
<td>No. 1 2,514</td>
<td>0.109</td>
<td>170</td>
<td>1.5</td>
<td>2,830</td>
</tr>
<tr>
<td></td>
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<td>0.328</td>
<td>286</td>
<td>1.5</td>
<td>4,173</td>
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<tr>
<td>5. Mathey, Watstein</td>
<td>No. 1 3,246</td>
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<td>1.5</td>
<td>3,226</td>
</tr>
<tr>
<td></td>
<td>2 3,272</td>
<td>0.259</td>
<td>303</td>
<td>1.0</td>
<td>3,187</td>
</tr>
</tbody>
</table>

Note: psi = 6,895 N/m²; lb/in. = 175.1 N/m; in. = 25.4 mm; ksi = 1,000 psi. All numbers except \( d_i \) had to be estimated, in order to get optimum fit, because they were not reported by the experimentalists.
We may now compare two possible crack locations: the cracks running through the bar crossings [Fig. 2(j)], and the cracks running in the middle between these crossings [Fig. 2(l,m)]. Comparing the areas of the regions of relief of \( \sigma_y \) for both cases (vertically crosshatched regions), we see that their area is somewhat smaller for the second of these two cases [Fig. 2(l,m)]. Therefore, the strain energy release, \( \Delta U \), for cracks running through the bar crossings appears to be larger, and so this is the type of cracks which should form [Fig. 2(j,k)]. This now restricts the possible spacings to \( s = nb/\sqrt{2} \) in which \( n = 1, 2, 3, \ldots \) We note however that \( \Delta U \) per crack is the same for all these crack spacings.

Therefore, cracks of the densest spacing [Fig. 2(k)]:

\[
s = \frac{b}{\sqrt{2}} \tag{35}
\]

should form immediately when the minimum necessary axial strain, \( \epsilon_\alpha \), in the bars is reached. For this spacing, the stress, \( \sigma_\alpha \), is completely relieved from all concrete [see the vertically shaded areas in Fig. 2(k)], and so \( \Delta U = (E_c \epsilon_\alpha/2)sb\sqrt{2}b_3 \) per mesh of bars, in which \( b_3 = \) thickness of the plate. As for bond slip, none can occur before the first cracking (since the stress is uniform). Assuming that no bond slip occurs during cracking, the consumed energy is \( \Delta W_c = b\sqrt{2}b_3\gamma_c \), and setting \( \Delta U = \Delta W_c \) we get

\[
\epsilon_\alpha = 1.68 \sqrt{\frac{\gamma_c}{E_c b}} \tag{36}
\]

If there is bond slip, then \( \Delta W_s > 0 \), and so a larger strain is required to produce these cracks.

Subsequent increase of strain cannot, according to our analysis, produce further continuous skew cracks. It can, however, produce shorter, discontinuous cracks of the type already analyzed [Fig. 2(n)]. These are likely not to be parallel to the continuous cracks.

There are many more interesting questions with regard to skew cracks under biaxial stress, but they are beyond the scope of this study.

### Summary and Conclusions

The spacing and width of cracks in a parallel crack system is approximately analyzed using the energy criterion of fracture mechanics as well as the strength criterion. Following previous works on rocks (8,10), the energy criterion is applied here in a novel way—integrally, considering the formation of the entire crack as one event. For cracks that are not long enough compared to the aggregate size, this approach appears to be more realistic than the usual fracture analysis which pertains to energy balance at infinitesimal crack length increments (i.e., balance of energy release and consumption rates). The energy criterion involves not only the release of strain energy and the energy consumed to produce the crack, but also the energy consumed by bond slip during cracking (if any).

The energy criterion indicates that the crack spacing is a function of the axial strain of the bars and also depends on bar spacing, bar diameter, fracture energy of concrete, and its elastic modulus. Both the energy and strength criteria yield a minimum strain necessary to pro-
duce any cracks. The energy criterion and the bond slip conditions also yield a lower bound on possible spacing of continuous cracks. The rules for the formation of shorter, partial length cracks are also set up. Approximate expressions for the crack width at, and away, from the bars are derived.

Numerical comparisons indicate satisfactory agreement with existing test data, and also lend theoretical support to one aspect of the empirical Gergely-Lutz formula previously obtained by statistical regression analysis of test data. Finally, formation of skew cracks in a biaxially stressed and biaxially reinforced plate is analyzed and a crack spacing formula is derived for one typical case.

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APPENDIX.—REFERENCES
