SPECTRAL ANALYSIS OF RANDOM SHRINKAGE STRESSES IN CONCRETE

By Zdeněk P. Bážant,1 F. ASCE and Tongsheng Wang2

ABSTRACT: Random variation of environmental humidity is characterized by its power spectrum, and random variation of pore humidity and stresses in a half-space is analyzed under the assumption that the problem is linear (or linearized). The diffusion equation and the superposition integral for stress relaxation are used. The dependence of both the creep properties and the drying diffusivity on age is taken into account. Complex variable expressions for the frequency response functions of humidity and stress are obtained and are evaluated numerically. In contrast to nonaging structures, these functions depend on both the current age of concrete and the age when drying starts. The standard deviations of pore humidity and of stress exhibit oscillations about a drifting mean. For typical diffusivities of concrete, the solution is nonstationary for at least 50 yr, for environmental fluctuations whose period does not exceed one year, the fluctuations are not felt at depths over 20 cm below the surface.

Since, even for aging structures, the spectral densities of input and response are related algebraically, the spectral method is computationally more efficient than the impulse response function method, in which the autocorrelation functions of the input and the response are related by integrals.

INTRODUCTION

Random variation of environmental humidity or temperature has a significant effect on the stresses and deformations in concrete structures. The solution is greatly facilitated by considering, as an approximation, the material properties to be deterministic and linear. In the linear approximation, the problem is mathematically equivalent to that of random environmental temperature. For the case when the material exhibits no aging and the environmental input is a stationary random process, the response is also a stationary random process. Solution of a typical problem of this type, concerned with random thermal stresses in an infinitely long cylindrical elastic vessel, was pioneered by Heller (13) who used the method of power response spectra. His solution method, however, inapplicable to concrete because the drying diffusivity and creep strongly depend on age, making the response a nonstationary random process. Random shrinkage stresses in an aging viscoelastic vessel were solved in Ref. 16 by the method of impulse response functions. In this solution, the aging was considered for creep and elastic properties but not for the drying diffusivity of concrete.

The objective of this paper is to solve the problem of a half-space, and to develop a method of solution that takes into account also the effect of aging on drying diffusivity. The spectral approach is convenient because the environmental humidity variation has a few distinct dominat-
in which \( S_i \) and \( S_g \) are the spectral densities of the input and of the response (3). This relation differs from that for stationary response only by the dependence on \( t \) and \( t_0 \). The fact that this relation is algebraic makes the spectral approach computationally more efficient than the impulse response approach, for which the autocorrelation functions of the input and the response are related by integrals. This is especially so if the random input is sufficiently well characterized by only a few periodic components, as is the case for environmental humidity.

In the special case of a single frequency component, \( \omega_0 \), one has \( S_i(\omega) = \frac{1}{2\pi} \delta(\omega - \omega_0) \) and \( S_g(\omega) = \frac{1}{2\pi} \delta(\omega - \omega_0) \) in which \( \delta(\omega - \omega_0) = \) unit impulse = Dirac delta function. Eq. 2 implies Eq. 1 since \( \text{Var} [g(x,t,t_0)] = \int^\infty_{-\infty} S_g(\omega,x,t,t_0) \, d\omega/2\pi \) (compare Ref. 3 or 4).

**Humidity Variation in a Halfspace**

Consider the halfspace \( x \geq 0 \) whose surface humidity variation is prescribed [Fig. 1(a, b)]. The variation of humidity, \( h \), inside the halfspace may be assumed to be governed by the linear diffusion equation

\[
\frac{\partial h}{\partial t} = C(t) \frac{\partial^2 h}{\partial x^2} \tag{3}
\]

in which \( h \) = relative vapor pressure in the pores of concrete; and \( C(t) \) = the diffusivity of concrete which is strongly variable with the age, \( t \), of concrete, as experimentally demonstrated by Wierig (17,7) and others. Wierig's data may be approximated by

\[
C(t) = bt^{-m} \tag{4}
\]

in which material parameters \( b \) and \( m \) may be obtained by linear regression of log \( C \) versus log \( t \); \( m = 0.213 \); and \( b = 61.0 \) if \( t \) is given in days (Fig. 2). To make the solution easier, Eq. 4 is chosen in such a form that it can be inverted, i.e., \( t = (C/b)^{-1/m} \). The limit \( C \to \infty \) as \( t \to \infty \), implied by Eq. 4, is certainly unrealistic, but for typical lifetimes this does not matter.

The variation of humidity is a sum of a deterministic component, \( h_0(x,t) \), and a randomly fluctuating component of zero mean. The method to solve the first component, which is caused by the difference of the mean environmental humidity and the initial uniform humidity in concrete, is known. Therefore, we consider \( h(x,t) \) to be the difference of pore humidity from the mean environmental humidity, and the initial and boundary conditions for solving the fluctuating component of zero mean may then be written as

\[
\text{for } t = t_0, \quad x \leq 0: \quad h = 0; \quad \text{for } t \geq t_0, \quad x = 0: \quad h = e^{i\omega(t-t_0)} \tag{5}
\]

Here the amplitude of \( h \) is assumed as 1.0 since all results are proportional to the amplitude.

Despite aging, an analytical solution is rendered possible by substituting the variable

\[
\xi = \phi(t) = \int C(t) \, dt \tag{6}
\]

which may be called the reduced time. Assuming that \( \xi = 0 \) when \( t = t_0 \), Eq. 6 becomes

\[
\xi = \phi(t) = \int bt^{-m} \, dt = \frac{b}{m} t^m + C_0 \tag{7}
\]

in which \( C_0 = -t_0/b, \quad n = 1 - m \) \( \tag{8} \)

The inverse of Eq. 7 is

\[
t = \psi(\xi) = \left( \frac{b}{m} \right)^{1/m} \left( \frac{\xi + C_0}{b} \right)^{1/m} \tag{9}
\]
Noting that $\frac{\partial h}{\partial t} = \frac{\partial}{\partial \xi} \left( C(t) \frac{\partial h}{\partial \xi} \right)$, the field equation (Eq. 3) becomes

$$\frac{\partial h}{\partial t} = \frac{1}{\alpha^2} \frac{\partial^2 h}{\partial \xi^2} \text{.......................... (10)}$$

and the boundary conditions (Eq. 5) become

for $\xi = 0, \ x \geq 0: \ h = 0$; for $\xi \geq 0, \ x = 0: \ h = e^{i\omega(t-t_0)} \text{........... (11)}$

We see that the transformation of time reduces the problem to a diffusion problem with a constant diffusivity, equal to 1.0. According to Duhamel's theorem (10), the solution of this problem is

$$h(x,t) = e^{i\omega t} Y_h(\omega, x, t, t_0) \text{.......................... (12)}$$

$$Y_h(\omega, x, t, t_0) = \frac{x}{2\sqrt{\pi}} e^{-\frac{x^2}{4(\sqrt{1-\lambda})}} \int_0^t \left[ e^{\frac{x^2}{4(\sqrt{1-\lambda})}} \delta(\xi - \lambda)^{-\frac{3}{2}} d\lambda, \ \xi = \phi(t) \right] \text{.................................. (13)}$$

which can be verified by substitution into Eqs. 10 and 11. The actual humidity is the real part of $h(x,t)$, and function $Y_h$ is the frequency response of humidity.

For a general random input of given spectral density, the frequency response function given by Eq. 13 makes it possible to determine all the statistical characteristics of the response. In particular, according to Eq. 2, the standard deviation of humidity is

$$\sigma_h(x, t) = s_0|h(x,t)| = s_0|Y(\omega, x, t, t_0)| \text{.......................... (14)}$$

**Variation of Stress in Halfspace**

Changes of pore humidity produce local shrinkage strains, $\varepsilon_{sh}(x,t)$. Although the dependence of $\varepsilon_{sh}$ on $h$ is non-linear, it needs to be linearized to make an analytical solution feasible. A linearization is acceptable for humidities between 0.5 and 1.0, and may be written as

$$\varepsilon_{sh}(x,t) = \kappa_{sh} h(x,t) \text{.......................... (15)}$$

in which $\kappa_{sh}$ is a shrinkage coefficient (a real number), which depends on age $t$. This dependence has, however, been neglected in numerical calculations, for lack of experimental data, although the same type of solution is possible even when $\kappa_{sh}$ depends on $t$.

Since the shrinkage strains obtained from Eq. 15 are not compatible, shrinkage stresses are produced. They may be determined from the conditions that the normal stress, $\sigma_z$, and the normal strains, $\varepsilon_y$ and $\varepsilon_z$, vanish. For the case of an elastic halfspace, in which the shrinkage strains do not vary in the directions parallel to the surface, these conditions would yield the shrinkage stresses

$$\sigma_y = -\frac{E_c}{1-\nu} \kappa_{sh} h(x,t) \text{.......................... (16)}$$

in which $E_c = \text{Young's modulus of concrete}$; and $\nu$ = its Poisson ratio ($\sigma_z = 0$, $\sigma_x = \sigma_y$).
The basic premise of the present solution is complete linearity of the problem, including the diffusion equation, the shrinkage-humidity relation, and the stress-strain relation.

The present method of solution can be applied, in principle, for any structure for which an elastic solution, either analytical or numerical, is available. Note that the method holds even when \( R(t, t') \) depends also on spatial coordinate \( x \) (layered halfspace). This is because the conditions \( \sigma_x = 0 = \epsilon_x = 0 \) remain. However, Eq. 13 per se does not apply if diffusivity \( C(t) \) depends on \( x \).

It may be also noticed that an analogous solution is possible for thermal stresses. Random environmental humidity history is then replaced by random environmental temperature history, the diffusion equation for pore humidity by the heat conduction equation, and the shrinkage-humidity relation by the relation of thermal expansion to temperature. To apply the present solution to random thermal stresses is simply a case of that used in Ref. 9. These assumptions might be often unrealistic. For example, the thermal and shrinkage problems become coupled if moisture in the surface layer is subdivided by discrete times \( t_n \), which is given in Ref. 8 in the form of Dirichlet series, as specified in Ref. 8. Eq. 24 gives a good approximation of the drying diffusivity is \( C(t) \) typical for structural concretes \([6] \) but too high for dam concretes \([18] \), is the relaxation function obtained for Ross Dam concrete, which is given in Ref. 8 in the form of Dirichlet series, as specified in Ref. 8. Eq. 24 gives a good approximation of the drying diffusivity is \( C(t) \) typical for structural concretes \([6] \) but too high for dam concretes \([18] \), is the relaxation function obtained for Ross Dam concrete, which is given in Ref. 8 in the form of Dirichlet series, as specified in Ref. 8. Eq. 24 gives a good approximation of the drying diffusivity is \( C(t) \) typical for structural concretes \([6] \) but too high for dam concretes \([18] \), is the relaxation function obtained for Ross Dam concrete, which is given in Ref. 8 in the form of Dirichlet series, as specified in Ref. 8. Eq. 24 gives a good approximation of the drying diffusivity is \( C(t) \) typical for structural concretes \([6] \) but too high for dam concretes \([18] \), is the relaxation function obtained for Ross Dam concrete, which is given in Ref. 8 in the form of Dirichlet series, as specified in Ref. 8. Eq. 24 gives a good approximation of the drying diffusivity is \( C(t) \) typical for structural concretes \([6] \) but too high for dam concretes \([18] \), is the relaxation function obtained for Ross Dam concrete, which is given in Ref. 8 in the form of Dirichlet series, as specified in Ref. 8. Eq. 24 gives a good approximation of the drying diffusivity is \( C(t) \) typical for structural concretes \([6] \) but too high for dam concretes \([18] \), is the relaxation function obtained for Ross Dam concrete, which is given in Ref. 8 in the form of Dirichlet series, as specified in Ref. 8. Eq. 24 gives a good approximation of the drying diffusivity is \( C(t) \) typical for structural concretes \([6] \) but too high for dam concretes \([18] \), is the relaxation function obtained for Ross Dam concrete, which is given in Ref. 8 in the form of Dirichlet series, as specified in Ref. 8.
does not attain a stationary value within a 50-yr lifetime, due to both the small value of diffusivity and its decline with age.

The histories of stress (its real part) at various depths are plotted in Fig. 3(b). Due to the hardening of concrete with age, i.e., a decrease in
creep compliance with the age at loading, the stress amplitude keeps increasing at first (for the first few years), and then it declines as the stress relaxation and the decrease of diffusivity prevail.

The time variation of the standard derivation of stress, \( s_{os} \), is shown in Fig. 3(c). It is noteworthy again that \( s_{os} \) oscillates, except for the first few months. The amplitude, \( A_{os} \), and the average \( M_{os} \), of \( s_{os} \) [determined in a manner explained by Fig. 1(d)], are plotted in Fig. 5. The values of \( A_{os} \) increase with the decrease of \( x \), or of \( t_0 \). When the depth, \( x \), increases, \( M_{os} \) rapidly decreases, and the curves of \( M_{os} \) for different values of \( t_0 \) are getting closer. (It may be noted that the values of \( s_{os} \) for the thick-wall cylinder solved in Ref. 16 oscillate, too; the graphs in Ref. 16 do not show it, unfortunately, because they were plotted from calculated points that were spaced too far apart.)

The stresses produced by humidity fluctuations are rather high and far exceed the tensile strength. Thus, the present solution, based on assuming linearity, is applicable only when the permanent loads or pre-stress, or both, produce superimposed compressive stresses that reduce the combined tensile stress peaks below the strength limit. Otherwise, tensile strain-softening would have to be taken into account, which would make the problem nonlinear.

**Conclusions**

1. The age-dependence of material properties causes the frequency response functions of pore humidity and stress, as well as their standard deviations, to depend on time, both on the current time and on the age at the start of drying. The equations governing the frequency response functions are of integral or differential type in time, rather than algebraic.

2. The time variations of the standard deviation of pore humidity and of stress exhibit oscillations about a drifting mean.

3. For the usual values of drying diffusivity, the solution is nonstationary for at least a 50-yr lifetime, and for environmental fluctuations whose period does not exceed 1 yr, the fluctuations are not felt at depths over 20 cm (for a concrete of typical diffusivity).

4. The use of the spectral method, instead of the impulse response method, is more efficient numerically since the spectral densities of input and of response are related algebraically.

5. The present linear analysis is applicable only when the permanent loads or pre-stress, or both, reduce the stress peaks below the tensile strength limit.

**Acknowledgment**

Partial financial support under U.S. National Science Foundation Grant No. CEE-8303148 is gratefully acknowledged. Mary Hill is thanked for her expert secretarial assistance.

**Appendix I.—Stress Fluctuation at Surface**

Assuming the pore humidity at the surface, \( x = 0 \), to be equal to the environmental humidity, we can express the stress, \( \sigma_s \), at the surface analytically if the moduli \( E_μ(t') \) in Eq. 24 for \( R(t,t') \) are expressed also as a Dirichlet series (5), i.e.

\[
E_μ(t') = E_0μ + \sum_{j=1}^{n} E_μ e^{-iμt'/T_j}; \quad E_s(t') = E_0s + \sum_{j=1}^{n} E_μ e^{-iμt'/T_j},
\]

in which \( E_μ \ldots E_s \) are constants (whose determination from test data is described in Ref. 5). The values of \( T_j (j = 1, 2 \ldots) \) may be chosen to be the same as the values of \( T_μ (μ = 1, 2 \ldots) \). Substituting \( h(0,t') = s_{os} e^{iω(t-t_0)} \) and Eq. 24 with Eq. 25 into Eq. 17, in which \( ν = \text{const}., \ k_{sh} = \text{const.} \), we may easily integrate term by term and obtain

\[
σ_s(t) = [σ_1 + σ_2 e^{iω(t-t_0)} + σ_3 e^{-iωt} + σ_4 e^{iω(t-t_0)} - L_j(2ω)] s_{os} \ldots (26)
\]

**Appendix II.—References**


**Appendix III.—Notation**

The following symbols are used in this paper:

- $A$ = amplitude of environmental humidity;
- $C$ = drying diffusivity in concrete;
- $E$ = expectation;
- $E_c, E_r$ = creep operator and Young's modulus of concrete;
- $E_n(t')$ = relaxation moduli of Maxwell chain at time $t'$ ($\mu = 1, 2, \ldots, m$);
- $h$ = complex variable whose real part is pore humidity in pores of concrete;
- $R(t, t')$ = relaxation function of concrete;
- $s_{A_h}$ = standard deviation of amplitude of environmental humidity;
- $s_{R_h}$ = standard deviation of humidity;
- $s_{\sigma_o}$ = standard deviation of stress;
- $t$ = current time;
- $t'$ = age of concrete;
- $x$ = spatial coordinate for points in structure;
- $Y$ = frequency response function;
- $\nu$ = Poisson's ratio of concrete;
- $\sigma_x, \sigma_y, \sigma_z$ = normal stresses;
- $\epsilon_x, \epsilon_y, \epsilon_z$ = normal strains;
- $\epsilon_{sh}$ = shrinkage strain; and
- $\omega$ = circular frequency of component of environmental humidity.