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IMBRICATE CONTINUUM AND PROGRESSIVE FRACTURING OF CONCRETE AND GEOMATERIALS

Zdeněk P. Bažant*

SOMMARIO. Dopo una sintesi dei risultati recentemente ottenuti presso la Northwestern University sui modelli matematici della frattura dei materiali eterogenei, si affronta il problema di un modello atto a rappresentare l'effetto di strain-softening nel continuo. In un continuo locale classico la considerazione dello strain-softening conduce ad una risposta instabile non realistica, in quanto la rottura si localizza in uno strato di spessore tendente a zero, così come tende a zero l'energia dissipata. Mentre il continuo nonlocale classico non risolve il problema, la soluzione viene individuata introducendo un nuovo tipo di continuo nonlocale, denominato continuo embricato, che rappresenta il caso limite di un sistema di elementi finiti fra di loro parzialmente sovrapposti (embricati) aventi una certa dimensione caratteristica che costituisce una proprietà del materiale.

SUMMARY. After giving an overview of the recent results at Northwestern University on mathematical models for fracturing heterogeneous materials, the lecture addresses the problem of a continuum model for strain-softening. In a classical, local continuum, strain-softening leads to unrealistic unstable response, such that failure localizes into a layer of vanishing thickness and occurs at vanishing energy dissipation. While the classical nonlocal continuum does not resolve the problem, solution is found in the form of a new type of nonlocal continuum, called the imbricate continuum, which represents the limiting case of a system of overlapping (imbricated) finite elements of a certain fixed characteristic size that is a material property.

INTRODUCTION

Heterogeneous brittle materials such as concretes or rocks fail by progressive fracturing distributed over a finite-size zone within the material. In the continuum approximation, this zone is characterized by strain-softening, i.e., a stress-strain relation in which the maximum principal stress decreases at increasing strain. The purpose of the present lecture is to present an overview of some recent results obtained at Northwestern University in the use of strain-softening material models for the description of fracture of brittle heterogeneous materials, and to develop in detail one new approach consisting in a nonlocal material model, as proposed in Refs. 1 - 2.

* Professor of Civil Engineering and Director Center for Concrete and Geomaterials, Northwestern University, Evanston, Illinois, 60201, U.S.A.

STRAIN-SOFTENING AND BLUNT CRACK BAND MODEL

Due to the distributed nature of microcracking, as well as the fact that the path of a final crack is generally not smooth but highly tortuous, it is not unrealistic to model cracking by means of stress-strain relations, introducing strain-softening in which the major principal tensile stress is reduced to zero (Fig. 1). This approach is particularly convenient for finite element analysis, since it necessitates merely an adjustment in the incremental stiffness matrix of the finite element. In the form of sudden cracking, this approach was introduced in 1967 by Rashid [27, 19].

In the form of progressive strain-softening, this approach was applied to fracture mechanics in Ref. 4 - 5, in which the strain-softening triaxial stress-strain relation was introduced in the form

$$\boldsymbol{\epsilon} = \mathbf{D}\boldsymbol{\sigma} + \boldsymbol{\xi} \tag{1}$$

Here $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are the column matrices of the component of strain and stress, $\mathbf{D}$ is the 6 x 6 matrix of elastic constants, and $\boldsymbol{\xi} = (\xi_{11}, \xi_{22}, \xi_{33}, 0, 0, 0)^T$, where superscript T denotes a transpose and the numerical subscripts refer to cartesian coordinates x_i (i = 1, 2, 3); $\boldsymbol{\xi}$ is the column matrix of additional smeared-out strains due to cracking. The normal stresses are assumed to be uniquely related to their associated cracking strains (Fig. 1), i.e.,

$$\sigma_{11} = C(\xi_{11})\,\xi_{11}, \quad \sigma_{22} = C(\xi_{22})\,\xi_{22}, \quad \sigma_{33} = C(\xi_{33})\,\xi_{33} \tag{2}$$

in which C is the secant modulus which reduces to zero at

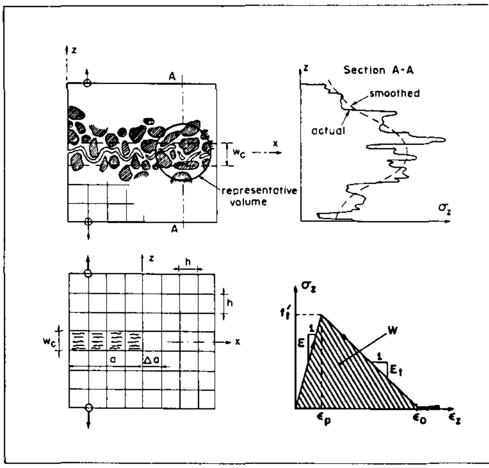

Fig. 1. Fracture in a Heterogeneous Material; Statistical Scatter of Stresses, Crack Band Model, and Strain-Softening Stress-Strain Diagram.

very large cracking strain and may be calibrated from direct tensile test data [21, 22, 23, 24, 27, 28]. Different algebraic relations are used for unloading. In this formulation it is assumed that cracking can occur only in three mutually orthogonal directions which are known in advance and do not rotate against the material once the cracking starts.

A more sophisticated constitutive model which does not depend on this restriction has been developed in Ref. 6 This model is analogous to the slip theory of plasticity, however, the microstructure is considered to be constrained kinematically rather than statically. With such an approach, cracking can occur in all directions and the direction of the principal cracking strain may rotate during the deformation process.

In the classical formulation of finite element analysis of concrete structures, still employed in the current large computer programs, a strain-softening stress-strain relation, mostly one that exhibits a sudden stress drop, is used indiscriminately for an arbitrarily chosen finite element size. It has been demonstrated, however, that this approach is inconsistent, unobjective with regard to the analyst's choice of the element size. It yields greatly different results for different mesh sizes and coverges to a physically meaningless solution as the mesh size is refined to zero [7 - 9]. For failures of the brittle type, the load causing further failure extension converges to zero as the mesh size tends to zero, and so does the total energy consumed by failure of the structure.

It has been shown that these incorrect features can be eliminated by using an energy criterion rather than the stress-strain relation as the primary condition for the extension of the cracked band of finite elements. Considering a crack band of a single-element width, consisting of finite elements with a uniform strain distribution across the band, the fracture energy, i.e., the energy consumed per unit extension of the crack band (and per unit thickness), may be expressed as

$$G_f = w_c \int \sigma_{33}\, d\epsilon_{33} = \frac{w_c}{2} f_t'^2 \left(\frac{1}{E_0} - \frac{1}{E_t}\right) \tag{3}$$

in which w_c is the width of the crack band front, which must be considered as a material property, σ_{33} is the normal stress across the central plane of the crack band, f_t' is the direct tensile strength of the material, E_0 is the initial elastic Young's modulus, and E_t is the mean downward slope of the strain-softening segment of the stress-strain diagram, which is negative.

It was found that if w_c is taken as roughly 3-times the maximum aggregate size in concrete [5], or 5-times the grain size in rock [10], then the crack-band finite element model yields results which agree with all essential experimental evidence from fracture tests, both the maximum load data and the R-curves [20, 3]. However, since the cracking at the failure front always tends to localize into a single-element width, the use of the correct element size $h = w_c$ is essential. For large structures, such elements are impracticably small, and a larger element size needs to be used. It was shown [7 - 9, 5] that this leads to consistent results if the value of fracture energy G_f is preserved. This can be achieved by adjusting the value of E_t, or f_t', or both, in Eq. 3 in which w_c is replaced by h. So, one must use some equivalent strain-softening slope and some equivalent tensile strength, depending on the element size, if consistent results should be achieved.

On the other hand, one may imagine the width of the crack band to be reduced to zero, and if the strain-softening slope E_t is adjusted so as to keep the correct value of G_f, the model in the limit becomes equivalent to the use of the stress-displacement relation on the centerline of fracture. This limiting case of the crack band model coincides with the model of Hillerborg et al., [11, 21].

The adjustement of constitutive properties in order to achieve consistent results for different mesh sizes is, however, disconcerting from the fundamental, mathematical viewpoint. It appears as if the finite element model was approximating a continuum that is not uniquely defined. Obviously, in the crack band model we cannot say we are approximating the solutions for a classical, local continuum because for such a continuum the crack band can, of course, localize into a layer of zero thickness, while an adjustment of material properties is not permitted.

The source of the difficulty has been found [1, 2] to be the assumption that we deal with a local continuum, in which the stress at a certain point is a function of the strain at the same point. In the second part of the lecture, we now turn our attention to the development of a new type of continuum for which the aforementioned mathematical difficulties do not arise. At the same time we gain with this new type of continuum a means to resolve in detail the distributions of averaged stress and strain throughout the failure zone.

IMBRICATE NONLOCAL CONTINUUM

From the studies of Kröner, Krumhansl, Kunin, Levin, and others [12 - 16], it is known that in a statistically heterogeneous medium which is not in a macroscopically homogeneous state of strain, the averaged stress at a given point depends not only on the gradient of the averaged displacements at that same point, but also on the gradient of the averaged displacements within a certain finite neighborhood of the point. A continuum model with these properties is, therefore, called nonlocal. We will now use the idea of nonlocality to derive in a consistent manner the appropriate constitutive model. To introduce a particularly simple form of interaction with the displacements in the neighborhood of a given point, we postulate the following hypothesis.

HYPOTHESIS I. In the smoothing continuum of a statistically heterogeneous material, there are two stresses at each point of coordinate vector $\mathbf{x}$: the local stress $\tau(\mathbf{x})$, which is a function of the displacement gradient $\partial \mathbf{u}/\partial \mathbf{x}$, and the broad-range stress $\sigma(\mathbf{x})$, which is a function of the relative displacements at two opposite points located symmetrically at distances $\ell/2$, in which ℓ is a certain given characteristic length characterizing the maximum size of the inhomogeneities in the material (Fig. 2).

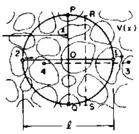

Fig. 2. Characteristic Volume of a Heterogeneous Material (the broad range stress at point 0 depends on the relative displacements at Points 1 and 2 but not on those at Points 3 and 4).

Let us first explore the consequences of this hypothesis in one dimension. We consider a one-dimensional bar of length coordinate x, with displacements u. According to Hypothesis I, the local stress depends on the local strain ϵ and the broad-range stress depends on the mean strain $\bar{\epsilon}$ which are expressed as follows

$$\epsilon(x) = \frac{\partial u(x)}{\partial x}, \quad \bar{\epsilon}(x) = D_x u(x) = \frac{1}{\ell}\left[u\left(x + \frac{\ell}{2}\right) - u\left(x - \frac{\ell}{2}\right)\right] \tag{4}$$

The last equation defines difference operator D_x. According to Hypothesis I, incremental work, δW, depends on the local strain, ϵ, and also on the mean strain, $\bar{\epsilon}$. Therefore, we have, for a bar of length L,

$$\delta W = c\delta W_0 + (1-c)\,\delta W_1 + \delta W_b - \int_0^L p(x)\delta u(x)\,\mathrm{d}x = 0,$$

$$\delta W_0 = \int_{\ell/2}^{L-\ell/2} \tau(x)\,\delta\epsilon(x)\,\mathrm{d}x,$$

$$\delta W_1 = \int_{\ell/2}^{L-\ell/2} \sigma(x)\,\delta\bar{\epsilon}(x)\,\mathrm{d}x, \tag{5}$$

in which x is the length coordinate of the bar (see Fig. 2); u = displacement, δW_b is the work of stresses done within segments of length ℓ at each end; δW_1, δW_0 are the works of broad-range stresses σ and local stresses τ within the rest of the bar; $p(x)$ = distributed load; $\delta u(x)$, $\delta\epsilon(x)$, $\delta\bar{\epsilon}(x)$ are any kinematically admissible variations; and c is a given empirical coefficient indicating the distribution of strain energy between the total strains and the mean strains. Substituting Eq. 4 for $\bar{\epsilon}$ and ϵ, we have

$$\delta W_1 = \int_{\ell/2}^{L-\ell/2} \frac{\sigma(x)}{\ell}\left[\delta u\left(x + \frac{\ell}{2}\right) - \delta u\left(x - \frac{\ell}{2}\right)\right]\mathrm{d}x,$$

$$\delta W_0 = \int_{\ell/2}^{L-\ell/2} \tau(x)\,\frac{\partial}{\partial x}\,\delta u(x)\,\mathrm{d}x. \tag{6}$$

Introducing new variables $\xi = x + \ell/2$, $\eta = x - \ell/2$, and then renaming in separate integrals both ξ and η as x, we get

$$\delta W_1 = \frac{1}{\ell}\int_{\ell}^{L} \sigma\left(\xi - \frac{\ell}{2}\right)\delta u(\xi)\,\mathrm{d}\xi - \frac{1}{\ell}\int_0^{L-\ell} \sigma\left(\eta + \frac{\ell}{2}\right)\delta u(\eta)\,\mathrm{d}\eta =$$
$$= \frac{1}{\ell}\int_{\ell}^{L} \sigma\left(x - \frac{\ell}{2}\right)\delta u(x)\,\mathrm{d}x - \frac{1}{\ell}\int_0^{L-\ell} \sigma\left(x + \frac{\ell}{2}\right)\delta u(x)\,\mathrm{d}x. \tag{7}$$

After integrating by parts in δW_0, and substituting $p(x) = -\rho\,\partial^2 u(x)/\partial t^2$ according to d'Alembert principle, we obtain

$$\delta W = -\int_{\ell}^{L-\ell}\left\{(1-c)\,\frac{1}{\ell}\left[\sigma\left(x + \frac{\ell}{2}\right) - \sigma\left(x - \frac{\ell}{2}\right)\right] + c\,\frac{\partial}{\partial x}\,\tau(x) - \rho\,\frac{\partial^2 u(x)}{\partial t^2}\right\}\delta u(x)\,\mathrm{d}x + \int_0^{\ell} F_1\,\delta u(x)\,\mathrm{d}x + \int_{L-\ell}^{L} F_2\,\delta u(x)\,\mathrm{d}x = 0 \tag{8}$$

in which F_1 and F_2 are certain functions which are zero for $\ell/2 \leqslant x \leqslant L - \ell/2$ and have effect only within end segments of length $\ell/2$. Since $\delta W = 0$ for any kinematically admissible $\delta u(x)$, the expression in the braces { } of the first integral must be zero. This yields the one-dimensional continuum equation of motion:

$$(1-c)\,D_x\sigma(x) + c\,\frac{\partial}{\partial x}\,\tau(x) = \rho\,\frac{\partial^2 u(x)}{\partial t^2}$$

$$\left(\text{for } \frac{1}{2} \leqslant x \leqslant L - \frac{\ell}{2}\right) \tag{9}$$

in which

$$\sigma(x) = \bar{E}D_x\,u(x), \qquad \tau(x) = E\,\frac{\partial}{\partial x}\,u(x) \tag{10}$$

Here E and $\bar{E}$ are the (secant) elastic moduli for the local stress τ and the broad-range stress σ. These moduli can depend on ϵ and $\bar{\epsilon}$.

Note that Eq. (8), as well as Eq. (9), applies only at points whose distances from the ends are at least ℓ. The boundary conditions, which ensue from functions F_1 and F_2, do not refer just to the end points, but are evidently spread out over boundary segments of length ℓ (which can be generalized in two or three dimensions as boundary layers). Setting up these boundary conditions in a discrete finite element form is, however, quite simple [1, 2].

Now we turn our interest to three dimensions. According to Hypothesis I, the stress depends on the mean displacement gradient which may be defined as

$$D_i u_j(\mathbf{x}) = \frac{1}{\ell}\left[u_j\left(\mathbf{x} + \frac{\ell}{2}\mathbf{i}\right) - u_j\left(\mathbf{x} - \frac{\ell}{2}\mathbf{i}\right)\right] a_i \tag{11}$$

in which $\mathbf{x} = (x_1, x_2, x_3)$ = cartesian coordinate vector, $\mathbf{i}$ = unit vector of axis x_i, a_i = direction cosines of $\mathbf{i} = (1, 0, 0)$ or $(0, 1, 0)$, or $(0, 0, 1)$, u_i = cartesian displacement components (i = 1, 2, 3), D_i = difference operator defined by Eq. 11, and ℓ is the characteristic length, which represents a material property. For the sake of brevity of notation, it is convenient to rewrite Eq. 1 as

$$D_i u_j = \frac{1}{\ell}(u_{j\to i} - u_{j\leftarrow i}) \tag{12}$$

in which the arrows in the subscripts denote shift operators defined as $u_{j\to i}(\mathbf{x}) = a_i u_j(\mathbf{x} + \mathbf{i}\,\ell/2)$ and $u_{j\leftarrow i}(\mathbf{x}) = a_i u_j(\mathbf{x} - \mathbf{i}\,\ell/2)$. It may be checked that the shift operator is linear and is commutative with differentiation or integration, except for some pathological cases of little practical interest. Note that the subscripts following the arrows are tensorial and imply summation if repeated, just like the partial derivative subscripts preceded by commas.

The work of the variations $\delta(D_i u_j)$ per unit volume of the material may be written as $\sigma_{ij}\delta(D_i u_j)$ where repeated subscripts imply summation over 1, 2, 3, and σ_{ij} represent the components of the broad-range stress tensor, analogous to that in Eq. 10. Because σ_{ij} must be symmetric,

$$\sigma_{ij}\delta(D_i u_j) = \frac{1}{2}[\sigma_{ij}\delta(D_i u_j) + \sigma_{ji}\delta(D_i u_j)] = \sigma_{ij}\delta\bar{\epsilon}_{ij} \tag{13}$$

in which

$$\bar{\epsilon}_{ij} = \frac{1}{2}(D_i u_j + D_j u_i). \tag{14}$$

This is the mean strain tensor, and we see it must be symmetric if σ_{ij} is.

According to Hypothesis I, the variation of the total work in the body, δW, depends on $D_i u_j$, and for the same reasons as stated in (1), it also depends on the local strain, ϵ_{ij}. So,

$$\delta W = (1-c)\,\delta W_1 + c\,\delta W_0 - \int_{\Omega'} P_i(\mathbf{x})\,\delta u_i(\mathbf{x})\,\mathrm{d}V + \int_{\Omega'_b} \phi_j \delta u_j\,\mathrm{d}V + \int_{S_0} P_j\,\delta u_j\,\mathrm{d}S = 0 \tag{15}$$

$$\delta W_0 = \int_{\Omega'} \tau_{ij}\delta\epsilon_{ij}\,\mathrm{d}V,$$

$$\delta W_1 = \int_{\Omega'} \sigma_{ij}\delta(D_i u_j)\,\mathrm{d}V, \tag{16}$$

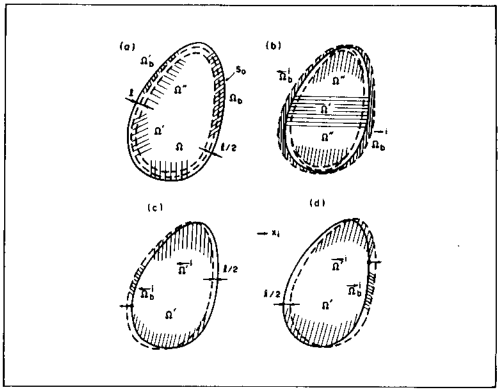

Fig. 3. Illustration of Various Domains of Integration.

in which Ω' is the domain of the body Ω without a boundary layer of thickness $\ell/2$ (Fig. 3a), $\mathrm{d}V = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$ = volume element, δ = sign for a variation, δu_i = any kinematically admissible displacement variations, p_i = distributed volume forces, $c\delta W_0$ and $(1-c)\delta W_1$ are the works of the local stresses τ_{ij} and the broad-range stresses σ_{ij} within domain Ω'; subscripts preceded by a comma denote partial derivatives, e.g., $u_{i,j} = \partial u_i/\partial x_j$; Ω'_b = domain of boundary layer of thickness $\ell/2$ (Fig. 3a); S_0 = surface of the body (Fig. 3b), $\mathrm{d}S$ = surface element; P_j = applied surface forces, $\phi_j\delta u_j$ = total work per unit volume within Ω'_b (Fig. 3b), and coefficient c characterizes the fraction of the material behaving in a local manner.

Let us now try to isolate δu_i in the integrand of δW_1. According to Eq. 11

$$\delta W_1 = \frac{1}{\ell}\left(\int_{\Omega'} \sigma_{ij}\delta u_{j\to i}\,\mathrm{d}V - \int_{\Omega'} \sigma_{ij}\delta u_{j\leftarrow i}\,\mathrm{d}V\right) \tag{17}$$

where repeated subscripts (both i and j) imply summation over 1, 2, 3. By substitution of new variables $x_i + \ell/2 = \xi_i$ and $x_i - \ell/2 = \eta_i$, one finds, similarly to Eq. 7, that

$$\int_{\Omega'} \sigma_{ij}\delta u_{j\to i}\,\mathrm{d}V = \int_{\overset{\to i}{\Omega'}} \sigma_{ij\leftarrow i}\,\mathrm{d}u_j\,\mathrm{d}V, \tag{18}$$

$$\int_{\Omega'} \sigma_{ij}\delta u_{j\leftarrow i}\,\mathrm{d}V = \int_{\overset{\leftarrow i}{\Omega'}} \sigma_{ij\to i}\,\delta u_j\,\mathrm{d}V \tag{19}$$

where $\overset{\to i}{\Omega'}$ is a domain congruent to Ω' but shifted as a rigid body in the direction of x_i by distance $\ell/2$ (Fig. 3c), $\overset{\leftarrow i}{\Omega'}$ is a domain congruent to Ω' but shifted as a rigid body against the direction x_i by distance $\ell/2$ (Fig. 3d); and if σ_{ij} is evaluated at $\mathbf{x}$ then $\sigma_{ij\to i}$ are the values of σ_{ij} at $\mathbf{x} + \mathbf{i}(\ell/2)$, and $\sigma_{ij\leftarrow i}$ are the values of σ_{ij} at $\mathbf{x} - \mathbf{i}(\ell/2)$.

The domains $\overset{\to i}{\Omega'}$ and $\overset{\leftarrow i}{\Omega'}$ are not identical, however they have in common domain Ω'' that is obtained from the domain Ω of the whole body by removing a boundary layer of thickness ℓ (we assume any cross section of Ω

to be thicker than 2ℓ); see Fig. 3b. We see that Ω'' is smaller than the intersection of domains $\vec{\Omega}'^{i}$ and $\overleftarrow{\Omega}'^{i}$, and we denote those parts of these two domains which are not included in Ω'' (i.e. disjunctions) as $\vec{\Omega}_b^{i}$ and $\overleftarrow{\Omega}_b^{i}$ (Fig. 3c-d). Finally we may denote as Ω_b the domain of the boundary layer of thickness ℓ (Fig. 3a) and note that $\vec{\Omega}_b^{i}$ and $\overleftarrow{\Omega}_b^{i}$ are totally contained within Ω_b although they do not fill Ω_b completely (Fig. 3b).

Using these notations, substituting Eqs. 18 and 19 into Eq. 17, and splitting each domain of integration into two subdomains as just mentioned, we may now write

$$\delta W_1 = \int_{\Omega''} \frac{1}{\ell}(\sigma_{ij\leftarrow i} - \sigma_{ij\rightarrow i})\delta u_j \,\mathrm{d}V + \int_{\Omega_0} \phi_j \,\delta u_j \,\mathrm{d}V \tag{20}$$

where ϕ_j depend on shifted σ_{ij} within domains $\vec{\Omega}_b^{i}$ and $\overleftarrow{\Omega}_b^{i}$ and are zero within the rest of Ω_b (Fig. 3b). Now, in view of Eq. 12, we may notice that Eq. 20 can be rewritten as

$$\delta W_1 = -\int_{\Omega''} D_i\sigma_{ij}\,\delta u_j \,\mathrm{d}V + \int_{\Omega_b} \phi_j\,\delta u_j \,\mathrm{d}V. \tag{21}$$

The rest is routine. By Gauss integral theorem, we have

$$\delta W_0 = \int_{\Omega'} \tau_{ij}\delta u_{i,j}\,\mathrm{d}V = \tag{22}$$

$$= \int_{S'} \tau_{ij}\delta u_j n_i \,\mathrm{d}S - \int_{\Omega'} \tau_{ij,j}\delta u_j \,\mathrm{d}V$$

where S' is the boundary surface of Ω', n_i are the direction cosines of the unit normal of the surface, and $\mathrm{d}S$ is the surface element of S'. According to d'Alembert principle, we may also substitute $p_i = -\rho\ddot{u}_i$ where ρ = mass density and $\ddot{u}_i = \partial^2 u_i/\partial t^2$, t = time. Thus, Eq. 15 takes the form

$$\delta W = \int_{\Omega''} [\rho\ddot{u}_j - (1-c)\,D_i\sigma_{ij} - c\tau_{ij,i}]\delta u_j\,\mathrm{d}V + \tag{23}$$

$$+ \int_{\Omega_b} \psi_j\delta u_j\,\mathrm{d}V + \int_{\Omega'_b} \phi_j\,\delta u_j\,\mathrm{d}V +$$

$$+ \int_{S_0} P_j\delta u_j\,\mathrm{d}S + \int_{S'} \Psi_j\delta u_j\,\mathrm{d}S$$

where ψ_j, Ψ_j, ϕ_j are certain functions independent of δu_j.

Now we should notice that the integrals in Eq. 23 other than the first one do not involve the interior domain Ω'' which excludes the boundary layer of thickness ℓ. Eq. 23 must hold for *any* kinematically admissible variation $\delta u_j(\mathbf{x})$. Choosing $\delta u_j(\mathbf{x})$ to be zero outside Ω'', and nonzero and arbitrary within Ω'', it follows from the fundamental lemma of the variational calculus that the expression in parentheses in the first integral must vanish for all $\mathbf{x}$. This yields the continuum equation of motion

$$(1-c)\,D_j\sigma_{ij} + c\,\tau_{ij,j} = \rho\,\ddot{u}_i \tag{24}$$

which is a partial difference-differential equation. Note that this equation applies only at points whose distance from the surface of the body is at least ℓ, i.e., it does not apply in the boundary layer of thickness ℓ.

The continuum equations of motion for the surface and the boundary layer of thickness ℓ follow, in principle, from the second to fifth integrals in Eq. 23. However, their form appears quite complicated, and it is preferable to set up the discretization near the boundary directly, similarly as described in Ref. 19.

According to Hypothesis I, σ_{ij} must depend on and only on $D_j u_i$ or $\bar{\epsilon}_{ij}$, the mean strain. Thus, we may write

$$\sigma_{ij} = \bar{C}_{ijkm}(\bar{\boldsymbol{\epsilon}})\bar{\epsilon}_{km} = \bar{C}_{ijkm}(\bar{\boldsymbol{\epsilon}})D_m u_k \tag{25}$$

in which $\bar{\boldsymbol{\epsilon}}$ is the mean strain tensor and $\bar{C}_{ijkm}$ are the broad-range secant moduli, depending in general on $\bar{\boldsymbol{\epsilon}}$. So we have

$$(1-c)D_j\bar{C}_{ijkm}(\bar{\boldsymbol{\epsilon}})D_m u_k + c\,\frac{\partial}{\partial x_j}\,C_{ijkm}(\boldsymbol{\epsilon})\,\frac{\partial}{\partial x_m}\,u_k = \rho\,\ddot{u}_i \tag{26}$$

in which $C_{ijkm}(\boldsymbol{\epsilon})$ are the local secant moduli. Eq. 26 makes conspicuous the symmetric action of the difference operators.

DISCUSSION OF RESULTS

From the foregoing variational analysis we see that the continuum equation of motion must involve a difference operator rather than a differential operator on the broad-range stresses σ_{ij}. In this regard, the present formulation of nonlocal continuum differs significantly from the classical nonlocal continuum theory, in which the following continuum equation motion is implied

$$\frac{\partial}{\partial x_j}\,C_{ijkm}(\bar{\boldsymbol{\epsilon}})\,D_m u_k = \rho\ddot{u}_i \tag{27}$$

In this equation, the local stress term is missing, which may be shown to cause certain periodic instabilities, and the operators on the left hand side are nonsymmetric, i.e., a differential operator for stresses is mixed with a difference averaging operator for displacement [1]. This feature causes certain fundamental difficulties if the classical nonlocal continuum theory [12 - 18] is used as a basis for finite element analysis. Even if the material is assumed elastic, the finite element stiffness matrices are then obtained non-symmetric, which is certainly an unacceptable feature, and may be a source of spurious instability. Indeed, it was only after the classical nonlocal continuum theory was found unworkable as a model for strain-softening, that the new concept of the imbricate continuum was developed.

In Ref. 1 it has been shown that the continuum equation of motion (Eq. 26) for the imbricate continuum may be obtained as a limit of the difference equilibrium equations for a system of identical finite elements of size ℓ which are overlapped (or imbricated) as the mesh is refined. Hence

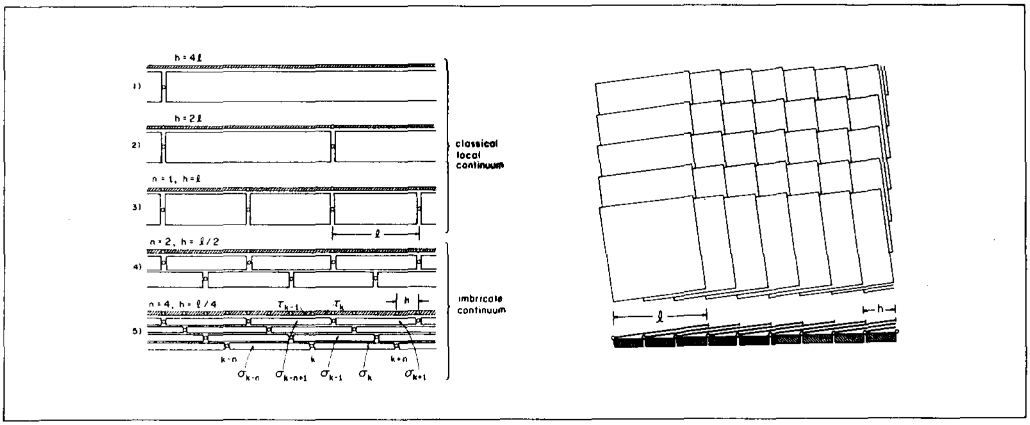

Fig. 4. Illustration of Mesh Refinement for a One-Dimensional Imbricate Continuum, and of Element Arrangement for a Two-Dimensional Imbricate Continuum (the elements are slightly rotated for the purpose of illustration).

the term imbricate. Conversely, it may be shown that the finite element model of the continuum defined by Eq. 26 is simply a system of imbricated finite elements of fixed size ℓ, arranged as illustrated for one-dimension and two-dimensions in Fig. 4. Note from this figure that, as the mesh is refined, the finite elements bridge one or more intermediate nodes. If the finite element size h is larger than the characteristic length ℓ, then the finite element model of the imbricate continuum becomes identical to that for the classical local continuum. For these cases ($h > \ell$), adjustment of the constitutive properties must be used in the analysis, as is done in the crack band model, or else consistent results would not be obtained. This adjustment is not done when the mesh size h is less than the characteristic length ℓ. So, we have a true continuum model which allows refining the mesh to zero.

To assure convergence and stability, the local stress-strain relation may not exhibit strain-softening, although it may exhibit plasticity and other nonlinear behavior. Strain-softening must be modeled exclusively with the broad-range stress-strain relation [1, 2].

Fig. 5 reproduces some of the results of explicit dynamic finite element calculations from Ref. 1. Analyzed is wave propagation in a strain-softening bar of length L, subjected at both ends to outward constant velocity v beginning at time $t = 0$. This loading produces step waves of strain propagating inward. When the waves meet at midlength, strain suddenly increases and strain-softening is produced. When this problem is analyzed with the usual finite element method for local continuum, it is found that strain-softening is always limited to a single finite element and so the width of the strain-softening zone reduces to zero as the element mesh is refined. Consequently, the energy W consumed by failure decreases with decreasing mesh size and approaches zero

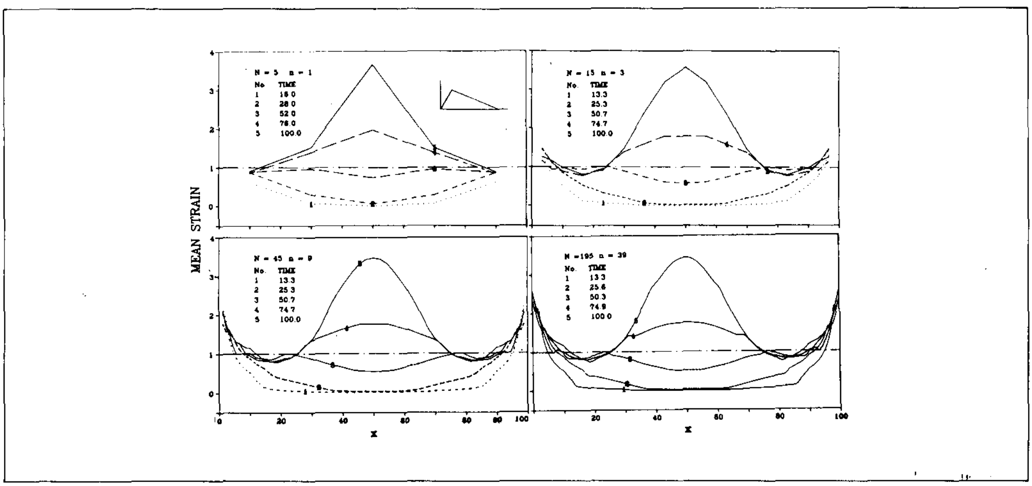

Fig. 5. Numerical Results of Bazant, Chang, and Beltyschko [1] for Wave Propagation in a Strain-Softening Bar.

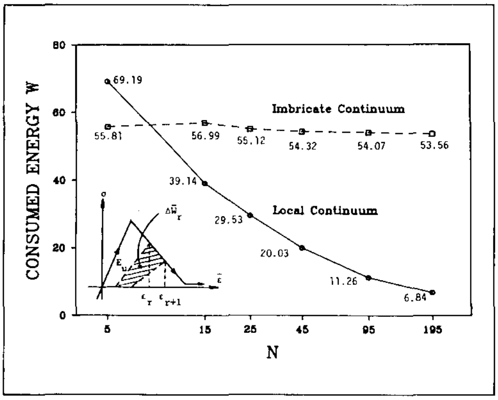

Fig. 6. Energy Consumed by Failure Caused by Wave Propagation in a One-Dimensional Bar.

as the mesh size tends to zero (see Fig. 6). It is also found that for the finite element model of local continuum, the solution depends discontinuously on the prescribed end velocities, as well as on the slope E_T of the strain-softening branch. The solution, however, converges to a unique, exact solution, although this solution is physically unrealistic.

By contrast, for the present imbricate continuum, the solution converges to a solution exhibiting a strain-softening zone of a finite size. Also, the energy consumed by failure converges to a finite value, as shown in Fig. 6. The characteristic length in these computations was considered as $\ell = L/5$.

CONCLUSIONS

Failure of heterogeneous materials which exhibit progressive microcracking can be analyzed with continuum models characterized by strain-softening. However, the continuum cannot be considered in the classical, local sense. Rather, one must use a nonlocal continuum of a type called imbricate. The finite element model of this continuum converges to a physically realistic solution as the mesh is refined; the energy consumed by failure approaches a finite value and the size of the strain-softening zone is finite. For finite element sizes equal to or larger than the characteristic length of the imbricate continuum, the finite element model is equivalent to the previously proposed crack band theory. Using elements smaller than the characteristic length, one can obtain resolution of the averaged stress and strain fields within strain-softening zones.

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