Imbricate Continuum and Its Variational Derivation

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Abstract: The one-dimensional imbricate nonlocal continuum, developed in a previous paper in order to model strain-softening within zones of finite size, is extended here to two or three dimensions. The continuum represents a limit of a system of imbricated (overlapping) elements that have a fixed size and a diminishing cross section as the mesh is refined. The proper variational method for the imbricate continuum is developed, and the continuum equations of motion are derived from the principle of virtual work. They are of difference-differential type and involve not only strain averaging but also stress gradient averaging for the so-called broad-range stresses characterizing the forces within the representative volume of heterogeneous material. The gradient averaging may be defined by a difference operator, or an averaging integral, or by least-square fitting of a homogeneous strain field. A differential approximation with higher order displacement derivatives is also shown. The theory implies a boundary layer which requires special treatment. The blunt crack band model, previously used in finite element analysis of progressive fracturing, is extended by the present theory into the range of mesh sizes much smaller than the characteristic width of the crack band front. Thus, the crack band model is made part of a convergent discretization scheme. The nonlocal continuum aspects are captured by an imbricated arrangement of finite elements of the usual type.

Introduction

In the preceding paper of this issue (4), it was shown that finite-size strain-softening regions can be obtained with a new type of nonlocal continuum called the imbricate continuum. One-dimensional continuum equations have been obtained as a limit of a discrete model which consists of imbricated (regularly overlapping) elements. The element size is not reduced as the mesh is refined, but is kept fixed, while the cross section of the elements is reduced in proportion to the mesh size.

The purpose of this paper, based on a 1983 report (1) summarized in Ref. 2, is to develop for the imbricate continuum the proper variational method, to derive a multidimensional generalization, and to show approximation by derivatives. Consequences for the crack band model for finite element analysis of fracture of concrete or geomaterials will also be pointed out. All notations and definitions from the preceding paper (4) are retained.

Variational Method in One Dimension

To facilitate development of the variational technique for two or three dimensions, it is useful to formulate it first in one dimension. We want to derive Eqs. 19–20 of Ref. 4 from Hypothesis I of Ref. 4 using the principle of virtual work. (This is more generally applicable than the use of the principle of minimum potential energy, which could, of course, be applied just as well.)

According to Hypothesis I of Ref. 4, the incremental work, δW, depends on the local strain, ϵ, and on the mean strain, $\bar{\epsilon}$. Therefore, we have, for a bar of length L:

$$\delta W = c\,\delta W_0 + (1-c)\,\delta W_1 + \delta W_b - \int_0^L p(x)\,\delta u(x)\,dx = 0$$

$$\delta W_0 = \int_{l/2}^{L-l/2} \tau(x)\,\delta\epsilon(x)\,dx, \quad \delta W_1 = \int_{l/2}^{L-l/2} \sigma(x)\,\delta\bar{\epsilon}(x)\,dx \quad \text{(1)}$$

in which x = the length coordinate of the bar [see Fig. 2(*b–d*) of Ref. 4]; u = displacement; δW_b = the work of stresses done within segments of length l at each end; δW_1, δW_0 = the works of broad-range stresses σ and local stresses τ within the rest of the bar; $p(x)$ = distributed load; $\delta u(x)$, $\delta\epsilon(x)$, $\delta\bar{\epsilon}(x)$ are any kinematically admissible variations; and c = an empirical coefficient between 0 and 1, defined in Ref. 4. Substituting Eq. 4 of Ref. 4 for $\bar{\epsilon}$, and setting $\epsilon = \partial u/\partial x$, we have

$$\delta W_1 = \int_{l/2}^{L-l/2} \frac{\sigma(x)}{l}\left[\delta u\left(x+\frac{l}{2}\right) - \delta u\left(x-\frac{l}{2}\right)\right]dx,$$

$$\delta W_0 = \int_{l/2}^{L-l/2} \tau(x)\,\frac{\partial}{\partial x}\,\delta u(x)\,dx \quad \text{(2)}$$

Introducing new variables, $\xi = x + l/2$, $\eta = x - l/2$, and then renaming in separate integrals both ξ and η as x, we get

$$\delta W_1 = \frac{1}{l}\int_l^L \sigma\left(\xi-\frac{l}{2}\right)\delta u(\xi)\,d\xi - \frac{1}{l}\int_0^{L-l} \sigma\left(\eta+\frac{l}{2}\right)\delta u(\eta)\,d\eta$$

$$= \frac{1}{l}\int_l^L \sigma\left(x-\frac{l}{2}\right)\delta u(x)\,dx - \frac{1}{l}\int_0^{L-l} \sigma\left(x+\frac{l}{2}\right)\delta u(x)\,dx \quad \text{(3)}$$

After integrating by parts in δW_0, and substituting $p(x) = -\rho\,\partial^2 u(x)/\partial t^2$ (ρ = mass density) according to d'Alembert principle, we obtain

$$\delta W = -\int_l^{L-l}\left\{(1-c)\frac{1}{l}\left[\sigma\left(x+\frac{l}{2}\right) - \sigma\left(x-\frac{l}{2}\right)\right] + c\frac{\partial}{\partial x}\tau(x)\right.$$

$$\left. - \rho\frac{\partial^2 u(x)}{\partial t^2}\right\}\delta u(x)\,dx + \int_0^l F_1\,\delta u(x)\,dx + \int_{L-l}^L F_2\,\delta u(x)\,dx = 0 \quad \text{(4)}$$

in which F_1 and F_2 are certain functions which are zero for $l/2 \le x \le L - l/2$ and have effect only within end segments of length $l/2$. Since $\delta W = 0$ for any kinematically admissible $\delta u(x)$, the expression in the braces { } of the first integral must be zero. This yields the one-dimensional continuum equation of motion in Eq. 19 of Ref. 4.

Note that Eq. 4, as well as Eq. 19 of Ref. 4, applies only at points whose distances from the ends are at least l. The boundary conditions, which ensue from functions F_1 and F_2, do not refer just to the end points,

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Note.—Discussion open until May 1, 1985. Separate discussions should be submitted for the individual papers in this symposium. To extend the closing date one month, a written request must be filed with the ASCE Manager of Technical and Professional Publications. The manuscript for this paper was submitted for review and possible publication on January 16, 1984. This paper is part of the *Journal of Engineering Mechanics*, Vol. 110, No. 12, December, 1984. ©ASCE, ISSN 0733-9399/84/0012-1693/$01.00. Paper No. 19334.

but are evidently spread out over boundary segments of length l (which can be generalized in two or three dimensions as boundary layers). Setting up these boundary conditions in a discrete form as described in Ref. 4 is, however, simpler [see the arrangement of imbricate elements near the end of the bar in Fig. 2(*c*) of Ref. 4].

The foregoing variational derivation shows that, if there is a difference operator (or gradient-averaging operator) in the strain-displacement relation, the same operator must be applied to stress in the continuum equation of motion (or equilibrium). The fact that this equation can be derived from an energy principle is important. It means that the operators are symmetric and that discretization must lead to symmetric matrices (this is, of course, automatically implied if one begins with a finite element system, as has been done in Ref. 4). The foregoing properties, which are not shared with the existing nonlocal continuum theory, represent the characteristic features of the imbricate nonlocal continuum.

From now on we turn our interest to three or two dimensions.

Variational Derivation for Difference Operator

For the same reasons as before, we adopt Hypothesis I from Ref. 4 for three dimensions and assume that the stress depends on the change of distance between two points a fixed distance, l, apart [Fig. 1(*a*)]. Therefore, the stress depends on the mean displacement gradient, $D_i u_j$, which may be defined by a difference operator, D_i, as follows:

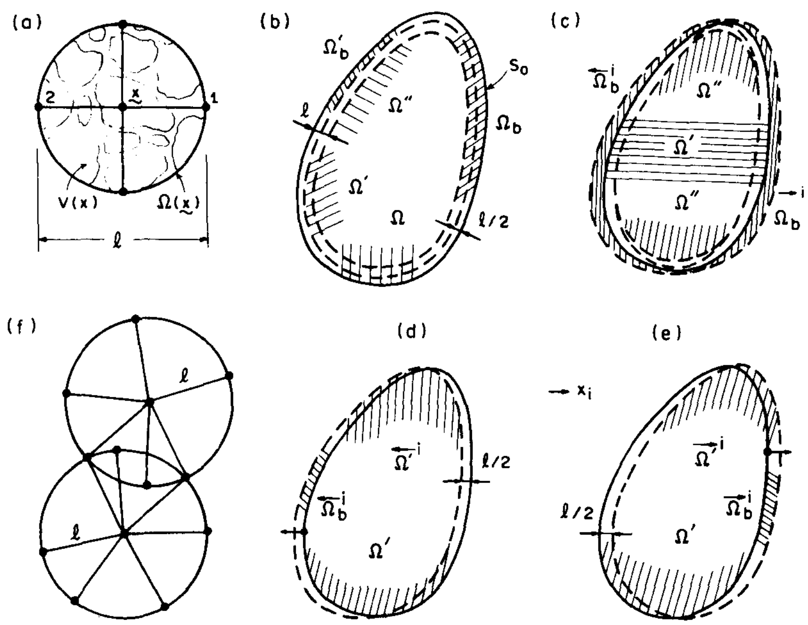

FIG. 1.—Difference Operator, Representative Volume, Various Domains of Integration, and Interaction of Points at Fixed Distance l

$$D_i u_j(\mathbf{x}) = \frac{1}{l}\left[u_j\left(\mathbf{x}+\frac{l}{2}\mathbf{i}\right) - u_j\left(\mathbf{x}-\frac{l}{2}\mathbf{i}\right)\right] a_i \quad (5)$$

in which $\mathbf{x} = (x_1, x_2, x_3)$ = Cartesian coordinate vector; $\mathbf{i}$ = unit vector of axis x_i; a_i = direction cosines of $\mathbf{i}$ = (1,0,0) or (0,1,0) or (0,0,1); u_i = Cartesian displacement components (i = 1, 2, 3); and l = the characteristic length which represents a material property, as in Ref. 4. For the sake of brevity of notation it is expedient to rewrite Eq. 1 as

$$D_i u_j = \frac{1}{l}(u_{j\to i} - u_{j\leftarrow i}) \quad (6)$$

in which the arrows in the subscripts denote shift operators defined as $u_{j\to i}(\mathbf{x}) = a_i u_j(\mathbf{x} + \mathbf{i}l/2)$ and $u_{j\leftarrow i}(\mathbf{x}) = a_i u_j(\mathbf{x} - \mathbf{i}l/2)$. It may be checked that the shift operator is linear and is commutative with differentiation or integration. Note that $u_{j\to i}$ and $u_{j\leftarrow i}$ are second-rank tensors; the subscripts following the arrow are tensorial and imply summation if repeated, just like the partial derivative subscripts preceded by commas.

The work of the variations $\delta(D_i u_j)$ per unit volume of the material may be written as $\sigma_{ij}\delta(D_i u_j)$ where repeated subscripts imply summation over 1, 2, 3 and σ_{ij} represents the components of the broad-range stress tensor, analogous to that defined in Ref. 4. Because tensor σ_{ij} must be symmetric, $\sigma_{ij}\delta(D_i u_j) = 1/2\,[\sigma_{ij}\,\delta(D_i u_j) + \sigma_{ji}\delta(D_i u_j)] = \sigma_{ij}\delta\bar{\epsilon}_{ij}$, in which

$$\bar{\epsilon}_{ij} = \frac{1}{2}(D_i u_j + D_j u_i) \quad (7)$$

This is the mean strain tensor, and we see it must be symmetric if σ_{ij} is symmetric.

According to Hypothesis I of Ref. 4, the variation of the total work in the body, δW, depends on $D_i u_j$, and for the same reasons as stated in Ref. 4, it also depends on the local strain, ϵ_{ij}. So

$$\delta W = (1-c)\,\delta W_1 + c\delta W_0 - \int_{\Omega'} f_i(\mathbf{x})\,\delta u_i(\mathbf{x})\,dV + \int_{\Omega_b'} \Phi_j\,\delta u_j\,dV$$

$$+ \int_{S_0} P_j\,\delta u_j\,dS = 0 \quad (8)$$

$$\delta W_0 = \int_{\Omega'} \tau_{ij}\,\delta\epsilon_{ij}\,dV,\quad \delta W_1 = \int_{\Omega'} \sigma_{ij}\,\delta(D_i u_j)\,dV \quad (9)$$

in which Ω' = the domain of the body Ω without a boundary layer of thickness, $l/2$ [Fig. 1(*b*)]; $dV = dx_1\,dx_2\,dx_3$ = volume element; δ = sign for a variation; δu_i = any kinematically admissible displacement variations; f_i = distributed volume forces; $c\delta W_0$ and $(1-c)\delta W_1$ are the works of the local stresses, τ_{ij}, and the broad-range stresses, σ_{ij}, within domain Ω'; subscripts preceded by a comma denote partial derivatives, e.g., $u_{i,j} = \partial u_i/\partial x_j$; Ω_b' = domain of boundary layer of thickness, $l/2$ [Fig. 1(*b*)]; S_0 = surface of the body [Fig. 1(*c*)]; dS = surface element; P_j =

applied surface forces; $\Phi_j\, \delta u_j$ = total work per unit volume within Ω_b' [Fig. 1(*c*)]; and coefficient c has the same meaning as in Ref. 1, characterizing the fraction of the material behaving in a local manner.

Let us now try to isolate δu_i in the integrand of δW_1. According to Eq. 6

$$\delta W_1 = \frac{1}{l}\left(\int_{\Omega'} \sigma_{ij}\, \delta u_{j\to i}\, dV - \int_{\Omega'} \sigma_{ij}\, \delta u_{j\leftarrow i}\, dV\right) \quad (10)$$

in which repeated subscripts (both i and j) imply summation over 1, 2, 3. By substitution of new variables, $x_i + l/2 = \xi_i$ and $x_i - l/2 = \eta_i$, one finds, similarly to Eq. 3, that

$$\int_{\Omega'} \sigma_{ij}\, \delta u_{j\to i}\, dV = \int_{\Omega'^{\rightarrow i}} \sigma_{ij\leftarrow i}\, \delta u_j\, dV \quad (11)$$

$$\int_{\Omega'} \sigma_{ij}\, \delta u_{j\leftarrow i}\, dV = \int_{\Omega'^{\leftarrow i}} \sigma_{ij\to i}\, \delta u_j\, dV \quad (12)$$

in which $\Omega'^{\rightarrow i}$ is a domain congruent to Ω' but shifted as a rigid body in the direction of x_i by distance $l/2$ [Fig. 1(*d*)]; $\Omega'^{\leftarrow i}$ is a domain congruent to Ω' but shifted as a rigid body against the direction x_i by distance $l/2$ [Fig. 1(*e*)]; and, if σ_{ij} is evaluated at $\mathbf{x}$, then $\sigma_{ij\to i}$ are the values of σ_{ij} at $\mathbf{x} + \mathbf{i}(l/2)$, and $\sigma_{ij\leftarrow i}$ are the values of σ_{ij} at $\mathbf{x} - \mathbf{i}(l/2)$.

The domains $\Omega'^{\rightarrow i}$ and $\Omega'^{\leftarrow i}$ are not identical; however, they have in common domain Ω'', obtained from domain Ω of the whole body by removing a boundary layer of thickness l (we assume Ω to be large enough so that Ω'' will not vanish) [see Fig. 1(*c*)]. We see that Ω'' is smaller than the intersection of domains $\Omega'^{\rightarrow i}$ and $\Omega'^{\leftarrow i}$, and we denote those parts of these two domains which are not included in Ω'' (i.e., disjunctions) as $\Omega_b^{\rightarrow i}$ and $\Omega_b^{\leftarrow i}$ [Fig. 1(*c*–*e*)]. Finally we may denote as Ω_b the domain of the boundary layer of thickness, l [Fig. 1(*b*)], and note that $\Omega_b^{\rightarrow i}$ and $\Omega_b^{\leftarrow i}$ are totally contained within Ω_b although they do not fill Ω_b completely [Fig. 1(*c*)].

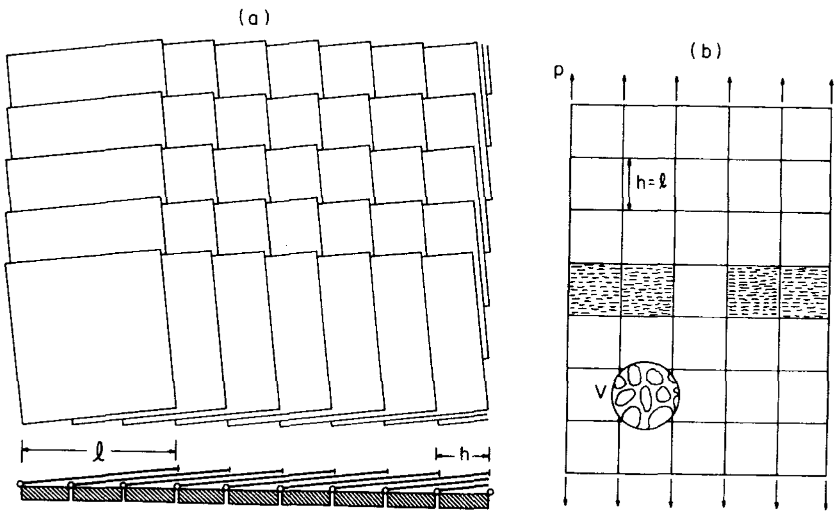

FIG. 2.—(*a*) Mesh of Square Imbricate Elements (Slightly Rotated to Permit Distinguishing Corners that Coincide); (*b*) Crack Band Model as Special Case when $h = l$

Using these notations, substituting Eqs. 11 and 12 into Eq. 10, and splitting each domain of integration into two subdomains as just mentioned, we may now write

$$\delta W_1 = \int_{\Omega''} \frac{1}{l}\,(\sigma_{ij\leftarrow i} - \sigma_{ij\to i})\, \delta u_j\, dV + \int_{\Omega_b} \phi_j\, \delta u_j\, dV \quad (13)$$

in which ϕ_j depend on shifted σ_{ij} within domains $\Omega_b^{\rightarrow i}$ and $\Omega_b^{\leftarrow i}$ and are zero within the rest of Ω_b [Fig. 1(*c*)]. Now, in view of Eq. 6, we may notice that Eq. 13 can be rewritten as

$$\delta W_1 = -\int_{\Omega''} D_i\, \sigma_{ij}\, \delta u_j\, dV + \int_{\Omega_b} \phi_j\, \delta u_j\, dV \quad (14)$$

The rest is routine. By Gauss' integral theorem, we have

$$\delta W_0 = \int_{\Omega'} \tau_{ij}\, \delta u_{i,j}\, dV = \int_{S'} \tau_{ij}\, \delta u_j\, n_i\, dS - \int_{\Omega'} \tau_{ij,i}\, \delta u_j\, dV \quad (15)$$

in which S' = the boundary surface of Ω'; n_i = the direction cosines of the unit normal of the surface; and dS = the surface element of S'. According to d'Alembert principle, we may also substitute $f_i = -\rho\ddot{u}_i$, in which ρ = mass density and $\ddot{u}_i = \partial^2 u_i/\partial t^2$, t = time. Thus, Eq. 8 takes the form

$$\delta W = \int_{\Omega''} [\rho\ddot{u}_j - (1-c)\, D_i\, \sigma_{ij} - c\tau_{ij,i}]\, \delta u_j\, dV + \int_{\Omega_b} \psi_j\, \delta u_j\, dV$$

$$+ \int_{\Omega_b'} \Phi_j\, \delta u_j\, dV + \int_{S_0} P_j\, \delta u_j\, dS + \int_{S'} \Psi_j\, \delta u_j\, dS \quad (16)$$

in which ψ_j, Ψ_j, Φ_j are certain functions independent of δu_j.

Now we should notice that none save the first of the integrals in Eq. 16 involve the interior domain Ω'', which excludes the boundary layer of thickness l. According to the principle of virtual work, Eq. 16 must hold for *any* kinematically admissible variation, $\delta u_j(\mathbf{x})$. Choosing $\delta u_j(\mathbf{x})$ to be zero outside Ω'', and nonzero and arbitrary within Ω'', it follows from the fundamental lemma of the variational calculus that the expression in parentheses in the first integral must vanish for all $\mathbf{x}$. This yields the continuum equation of motion:

$$(1-c)\, D_j\, \sigma_{ij} + c\tau_{ij,j} = \rho\ddot{u}_i \quad (17)$$

which is a partial difference-differential equation. Note that this equation applies only at points whose distance from the surface of the body is at least l, i.e., it does not apply within the boundary layer of thickness l.

The continuum equations of motion for the surface and the boundary layer of thickness l follow, in principle, from the second to fifth integrals in Eq. 16. However, their form appears quite complicated, and it is preferable to set up the discretization near the boundary directly, similar to the manner described in Ref. 4.

The foregoing variational analysis shows again that, in contrast to the existing nonlocal theory, the equation of motion must involve a difference rather than differential operator on the broad-range stresses, σ_{ij}. Note also that $D_k\,\sigma_{ij}$ is equivalent to the mean of stress gradient $\sigma_{ij,k}$ along a segment of length l in the direction x_k.

According to Hypothesis I in Ref. 4, σ_{ij} must depend on—and only on—$D_j u_i$ or $\bar{\epsilon}_{ij}$, the mean strain. Thus, we may write

$$\sigma_{ij} = \bar{C}_{ijkm}(\bar{\boldsymbol{\epsilon}})\,\bar{\epsilon}_{km} = \bar{C}_{ijkm}(\bar{\boldsymbol{\epsilon}})\,D_m u_k \quad \text{(18)}$$

in which $\bar{\boldsymbol{\epsilon}}$ is the mean strain tensor and $\bar{C}_{ijkm}$ are the broad-range secant moduli, depending in general on $\bar{\boldsymbol{\epsilon}}$. So Eq. 13 may be written as

$$(1-c)\,D_j\bar{C}_{ijkm}(\bar{\boldsymbol{\epsilon}})\,D_m u_k + c\frac{\partial}{\partial x_j}C_{ijkm}(\boldsymbol{\epsilon})\frac{\partial}{\partial x_m}u_k = \rho\ddot{u}_i \quad \text{(19)}$$

in which $C_{ijkm}(\boldsymbol{\epsilon})$ are the local secant moduli. Eq. 19 makes conspicuous the symmetric action of the difference operators.

Let us now determine the total stress S_{ij} resulting from the difference operator in Eq. 5. We consider a cubic mesh with step h in coordinates $x = x_1$, $y = x_2$, $z = x_3$, and use $I, J, K = 1, 2, \ldots$ as the subscripts of the nodal (mesh) planes normal to x, y and z, in order to make a distinction from tensorial subscripts i, j, k. We denote as $\sigma_{ij(IJK)}$ the broad-range stress based on the displacement differences between the face centroids or corners of a cube limited by coordinate planes x_I and x_{I+n}, x_J and x_{J+n}, and x_K and x_{K+n}. Noting that n^3 cubes overlap at one point, and that each cube must therefore have a weight $1/n^3$ (with $n^3 = l^3/h^3$), we find that the total stress at point (x_I, y_J, z_K) is

$$S_{ij(IJK)} = \frac{1-c}{l^3}\sum_{I'=I+1-n}^{I}\;\sum_{J'=J+1-n}^{J}\;\sum_{K'=K+1-n}^{K}\sigma_{ij(I'J'K')}h^3 + c\tau_{ij(IJK)} \quad \text{(20)}$$

Letting $h \to 0$, $n \to \infty$ ($h^3 = dx'\,dy'\,dz'$), we find that the continuum limit is

$$S_{ij}(x,y,z) = \frac{1-c}{l^3}\int_{x-l/2}^{x+l/2}\int_{y-l/2}^{y+l/2}\int_{z-l/2}^{z+l/2}\sigma_{ij}(x',y',z')\,dx'\,dy'\,dz'$$
$$+\,c\tau_{ij}(x,y,z) \quad \text{(21)}$$

Here the averaging integral extends over a cube of side l. This definition of S_{ij} implies some directional bias because the diagonals of the cube are longer than its edges.

As a generalization of Eqs. 34–37 of Ref. 4, the total energy consumed by damage within domain Ω'' [Fig. 1(*b*)] up to time t may be expressed as

$$W(t) = \int_0^t\int_{\Omega''} l\,\frac{1-c}{2}\,[d(C^u_{ijkm}\sigma_{ij}\sigma_{km}) + \sigma_{ij}\,d\epsilon_{ij}]\,dV\,dt' \quad \text{(22)}$$

in which C^u_{ijkm} are the unloading compliances which depend on the current strain tensor and are assumed to be constant during unloading and reloading.

Variants of Averaging Rule

The difference operator in Eq. 5 corresponds to taking displacements at the face centroids of a cube of edge length l centered at point $\mathbf{x}$. Alternatively, one could take the displacements at the corners of this cube, in which case

$$D_i u_m(\mathbf{x}) = \frac{1}{4l}\left\{u_m\left[\mathbf{x}+\frac{l}{2}(\mathbf{i}+\mathbf{j}+\mathbf{k})\right] + u_m\left[\mathbf{x}+\frac{l}{2}(\mathbf{i}-\mathbf{j}+\mathbf{k})\right]\right.$$
$$+\,u_m\left[\mathbf{x}+\frac{l}{2}(\mathbf{i}+\mathbf{j}-\mathbf{k})\right] + u_m\left[\mathbf{x}+\frac{l}{2}(\mathbf{i}-\mathbf{j}-\mathbf{k})\right]$$
$$-\,u_m\left[\mathbf{x}-\frac{l}{2}(\mathbf{i}+\mathbf{j}+\mathbf{k})\right] - u_m\left[\mathbf{x}-\frac{l}{2}(\mathbf{i}-\mathbf{j}+\mathbf{k})\right]$$
$$\left.-\,u_m\left[\mathbf{x}-\frac{l}{2}(\mathbf{i}+\mathbf{j}-\mathbf{k})\right] - u_m\left[\mathbf{x}-\frac{l}{2}(\mathbf{i}-\mathbf{j}-\mathbf{k})\right]\right\}a_i \quad \text{(23)}$$

in which $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors of coordinate axes x_i, x_j, x_k. This equation gives the same values as the displacement gradients at the center of a cubic finite element with 8 nodes at its corners. Although the procedure is more tedious, one may obtain from Eq. 23 the same continuum equation of motion (Eq. 17) using the same variational method as before. This proves that Eq. 23 approximates the same (imbricate) continuum as the simpler Eq. 5.

The mean strain is $\bar{\epsilon}_{ij} = (D_i u_j + D_j u_i)/2$ in which $D_i u_j$ has been defined by the operator in Eq. 5. This definition has the advantage of simplicity; however, it introduces a certain bias, since among all points at distance $l/2$ from point $\mathbf{x}$ [Fig. 1(*f*)] only a pair of two is used [Fig. 1(*a*)].

Noting that D_x in one dimension (4) is equivalent to a mean of the derivative (Eqs. 3 and 4 of Ref. 1), we may introduce an alternative definition

$$D_i u_j(\mathbf{x}) = \frac{1}{V}\int_{V(\mathbf{x})}\frac{\partial u_j(\mathbf{x}')}{\partial x_i'}\,dV' = \frac{1}{V}\int_{\Gamma(\mathbf{x})}u_j(\mathbf{x}')\,n_i(\mathbf{x}')\,dS \quad \text{(24)}$$

in which the last expression results from the application of Gauss' integral theorem. Here $V(\mathbf{x})$ = representative volume V centered at $\mathbf{x}$ [Fig. 1(*f*)]; dV' = element of volume V at location $\mathbf{x}'$; $\Gamma(\mathbf{x})$ = surface of $V(\mathbf{x})$ [Fig. 1(*f*)]; and n_i = direction cosines of an outward normal vector of surface $\Gamma(\mathbf{x})$. $D_i u_j$ may be called the mean displacement gradients. Note that $\partial u_j(\mathbf{x}')/\partial x_i'$ is equal to $\partial u_j(\mathbf{x}')/\partial x_i$ if $x_i' - x_i$ is kept fixed at partial differentiation.

For isotropic media, V should be considered as a sphere of diameter l', the value of which is close to the characteristic length l. Then, using spherical coordinates centered at point $\mathbf{x}$, we may write Eq. 24 as

$$D_i u_j(\mathbf{x}) = \frac{1}{V}\int_0^{l'/2}\int_0^{2\pi}\int_0^{\pi} \frac{\partial u_j(\mathbf{x}, r', \theta, \phi)}{\partial x_i} \sin\theta\, d\theta\, d\phi\, dr'$$

$$= \frac{1}{V}\int_0^{2\pi}\int_0^{\pi} u_j\left(\mathbf{x}, \frac{l'}{2}, \theta, \phi\right) n_i(\theta, \phi) \sin\theta\, d\theta\, d\phi \quad \text{(25)}$$

in which $V = \pi l'^3/6$ and $u_j(\mathbf{x}, r', \theta, \phi)$ = displacement at a point at distance r' from point $\mathbf{x}$ in the (θ, ϕ) direction. As a consequence of Eq. 24 and Eq. 7

$$\bar{\epsilon}_{km}(\mathbf{x}) = \frac{1}{V}\int_{V(\mathbf{x})} \epsilon_{km}(\mathbf{x}')\, dV' \quad \text{(26)}$$

Thus, $\bar{\epsilon}_{km}(\mathbf{x})$ actually represents a mean of $\epsilon_{km}(\mathbf{x}')$ taken over volume $V(\mathbf{x})$ centered at $\mathbf{x}$.

As another reasonable definition, $D_i u_j(\mathbf{x})$ may be defined as the gradient, $\bar{u}_{j,i}$, of the displacement field, $\bar{u}_i$, that has a uniform gradient and provides the least-square fit of actual displacements u_j with a uniform gradient field on the surface of a sphere of diameter l centered at point $\mathbf{x}$. Because the displacement gradient, $\bar{u}_{j,i}$, is homogeneous (uniform)

$$\bar{u}_j = \bar{u}_j + \bar{u}_{j,i}(x_i' - x_i) \quad \text{(27)}$$

in which $\bar{u}_{j,i} = \partial u_j/\partial x_i$ = const.; $\bar{u}_j$ = displacement at point x_i (center of sphere); and $\bar{u}_j$ = displacement at point x_i' anywhere within the sphere. We seek such $\bar{u}_{j,i}$ that $\Phi = \int_S (\bar{u}_j - u_j)^2 dS$ be minimum, with S being the surface of the sphere. The minimizing conditions are $\partial\Phi/\partial\bar{u}_{j,i} = 0$ for all i, j. Substituting for Φ and for $\bar{u}_j$, we obtain a system of 9 simultaneous algebraic equations for $\bar{u}_{j,k}$ (i, j = 1, 2, 3):

$$\bar{u}_{j,k}\int_S (x_i' - x_i)(x_k' - x_k)\, dS = \int_S (u_j - \bar{u}_j)(x_i' - x_i)\, dS \quad \text{(28)}$$

But these conditions can be greatly simplified. We may substitute $x_k' - x_k = n_k l'/2$ and $x_j' - x_j = n_j l'/2$, and then apply Gauss' integral theorem on both sides:

$$\bar{u}_{j,k}\frac{l'}{2}\int_V \frac{\partial}{\partial x_k'}(x_i' - x_i)\, dV' = \frac{l'}{2}\int_V \frac{\partial}{\partial x_j'}(u_j - \bar{u}_j)\, dV' \quad \text{(29)}$$

Now $\partial x_i/\partial x_k' = 0$; $\partial x_i'/\partial x_k' = \delta_{ik}$ = Kronecker delta; $\partial\bar{u}_j/\partial x_j' = 0$; and so $\bar{u}_{j,k}\int_V \delta_{ik}\, dV' = \int_V (\partial u_j/\partial x_i')\, dV'$, in which the left-hand side is $\bar{u}_{j,k}\delta_{ik} V$ or $\bar{u}_{j,i} V$. So we get

$$\bar{u}_{j,i} = \frac{1}{V}\int_V \frac{\partial u_j}{\partial x_i'}\, dV' \quad \text{(30)}$$

This is identical to Eq. 24. Thus, the least-square fitting of the displacement field over the surface of a sphere is equivalent to the averaging of displacement gradient. It may also be noted that least-square fitting over the volume of the sphere would yield a different result.

Variational Derivation for Integral Averaging Operator

Let us now demonstrate that the definition of mean strain by an averaging integral (Eq. 24) yields the same continuum equation of motion as before. Consider, more generally than Eq. 24, an averaging integral with a weighting function, $\alpha(\mathbf{r})$, as used in the classical nonlocal theory:

$$D_i u_j(\mathbf{x}) = \frac{1}{V}\int_{V(\mathbf{x})} \frac{\partial u_j(\mathbf{x}')}{\partial x_i'} \alpha(\mathbf{r})\, dV' \quad \text{(31)}$$

$$\epsilon_{ij}(\mathbf{x}) = \frac{1}{V}\int_{V(\mathbf{x})} \epsilon_{ij}(\mathbf{x})\, \alpha(\mathbf{r})\, dV' = \frac{1}{2}[D_i u_j(\mathbf{x}) + D_j u_i(\mathbf{x})] = H\epsilon_{ij}(\mathbf{x}) \quad \text{(32)}$$

in which H = averaging operator; $\mathbf{r} = \mathbf{x}' - \mathbf{x}$ = radius vector from the center of volume V; and $\int_V \alpha(\mathbf{r})\, dV' = 1$. The variation of the total work in the body of domain Ω may be again expressed by Eq. 8, in which δW_0 is the same as in Eq. 9, while δW_1 may now be expressed as

$$\delta W_1 = \int_{\Omega'} \sigma_{ij}\delta\bar{\epsilon}_{ij}\, dV = \int_{\Omega'} \sigma_{ij}(\mathbf{x})\frac{1}{V}\int_{V(\mathbf{x})} \frac{\partial \delta u_j(\mathbf{x}')}{\partial x_i'} \alpha(\mathbf{r})\, dV'(\mathbf{x}')\, dV(\mathbf{x}) \quad \text{(33)}$$

in which Ω' is the domain of the body after removing the boundary layer of thickness $l/2$ [Fig. 1(*b*)]. Eq. 33 may be transformed as

$$\delta W_1 = \frac{1}{V}\int_{V(\mathbf{x})} \left\{\int_{\Omega'} \sigma_{ij}(\mathbf{x}' - \mathbf{r})\frac{\partial \delta u_j(\mathbf{x}')}{\partial x_i'} \alpha(\mathbf{r})\, dV(\mathbf{x}' - \mathbf{r})\right\} dV'(\mathbf{x}') \quad \text{(34)}$$

Here we may apply Gauss' integral theorem to the integral over domain Ω':

$$\delta W_1 = \frac{1}{V}\int_{V(\mathbf{x})} \left\{-\int_{\Omega'} \frac{\partial[\sigma_{ij}(\mathbf{x}' - r)\,\alpha(\mathbf{r})]}{\partial x_i'} \delta u_j(\mathbf{x}')\, dV(\mathbf{x}' - \mathbf{r}) + \int_{S'} \ldots dS'\right\} dV'(\mathbf{x}')$$

$$= -\frac{1}{V}\int_{\Omega'}\int_{V(\mathbf{x})} \delta u_j(x')\frac{\partial[\sigma_{ij}(\mathbf{x})\,\alpha(\mathbf{r})]}{\partial x_i'}\, dV'(\mathbf{x}')\, dV(\mathbf{x}) + \int_{S'} \ldots \quad \text{(35)}$$

in which the integral over the surface S' of domain Ω' need not be written in detail. Now we may set $\partial[\sigma_{ij}(\mathbf{x})\alpha(\mathbf{r})]/\partial x_i' = \partial[\sigma_{ij}(\mathbf{x})\alpha(\mathbf{r})]/\partial x_i$ because $\mathbf{r}$ is fixed at partial differentiation (and $x_i' = x_i + r_i$). Then, in the double volume integral, we may mutually interchange x_i with x_i'. After the interchange of variables, we imagine first integrating over x_i' at fixed x_i, and then over x_i. (In the old variables, this means integrating first over x_i at fixed x_i', i.e., taking first all volumes $V(\mathbf{x})$ which include a fixed point $\mathbf{x}'$, and then repeating this for all $\mathbf{x}'$.) Consequently, as a result of the interchange of variables, the domain of integration Ω' changes to Ω^* by shifts such that for all the integrand values corresponding to the same $\mathbf{r}$ (at various $\mathbf{x}'$), the domain Ω' is shifted by $-\mathbf{r}$ as a rigid body. All these shifted domains $\Omega^*(\mathbf{r})$ lie within the domain Ω of the whole body. The fixed domain common to all the shifted domains $\Omega^*(\mathbf{r})$ for all possible $\mathbf{r}$ is the domain Ω'' shown in Fig. 1(*b*), i.e., the domain left after removing a boundary layer of thickness l from the whole body. Note that the shifts of Ω' happen here in all directions, while in the previous derivation for the difference operator (Eqs. 8–14) the shifts occurred only in the coordinate directions.

After the interchange of variables and domain shifts, Eq. 35 becomes

$$\delta W_1 = -\int_{\Omega''} \delta u_j(\mathbf{x}) \left\{ \frac{1}{V} \int_{V(x)} \frac{\partial[\sigma_{ij}(\mathbf{x}')\,\alpha(\mathbf{r})]}{\partial x_i} dV'(\mathbf{x}') \right\} dV(\mathbf{x})$$

$$- \int_{\Omega_b} \ldots + \int_{S'} \ldots \qquad (36)$$

in which the integrals over Ω_b and S' are quite complicated but need not be written out. According to Eqs. 8 and 15, the total energy variation is

$$\delta W = \int_{\Omega''} \left[\rho \ddot{u}_i - \frac{1-c}{V} \int_{V(\mathbf{x})} \frac{\partial[\sigma_{ij}(\mathbf{x}')\alpha(\mathbf{r})]}{\partial x_i} dV'(\mathbf{x}') \right.$$

$$\left. - c \frac{\partial \tau_{ij}}{\partial x_i} \right] \delta u_j(\mathbf{x})\, dV + \ldots \qquad (37)$$

in which the omitted terms consist of integrals that do not include the interior of Ω''.

Since, according to the principle of virtual work, Eq. 37 must hold true for any kinematically admissible displacement variation $\delta u_j(\mathbf{x})$ (and since the unwritten integrals over the boundary surface and the boundary layer can be made to vanish), the bracketed term in Eq. 37 must vanish. This yields again the same continuum equation of motion as before (Eqs. 17–19), but with a different definition of operator D_i (Eq. 31).

Eq. 17 may also be written in the form

$$S_{ij,j} = \rho \ddot{u}_i \qquad (38)$$

in which $S_{ij} = (1-c)\,\bar{\sigma}_{ij} + c\tau_{ij}$ (39)

$$\bar{\sigma}_{ij}(\mathbf{x}) = H\sigma_{ij}(\mathbf{x}) = \frac{1}{V}\int_{V(\mathbf{x})} \sigma_{ij}(\mathbf{x}')\,\alpha(\mathbf{r})\,dV'(\mathbf{x}') = H\bar{C}_{ijkm} H\epsilon_{km} \qquad (40)$$

H is the averaging operator initially defined by Eq. 32, and $\bar{\sigma}_{ij}$ is the mean broad-range stress tensor. In view of Eq. 38, S_{ij} must be the total stress, and indeed its definition (Eqs. 39–40) is consistent with Eq. 21 representing the limit of a discrete model. Note that Eqs. 17 and 38–40 differ from the continuum equation of motion in the existing nonlocal theory, which is written as $\sigma_{ij,j} = \rho\ddot{u}_i$, although the same strain averaging (Eq. 32) is assumed at the outset.

Discrete Approximation by Finite Elements

We have shown that various definitions of operator D_i (Eqs. 5, 23, 24, 25, 31) lead to the same continuum equations of motion. Therefore, the discrete finite element approximation may also be the same for all these definitions. The finite element approximation is immediately obvious for the difference operator that uses the function values at the corners of a cube (Eq. 23). This operator gives the values of strain at the center of the cubical finite element of side l. Thus, the finite element model of two-dimensional or three-dimensional continuum is as follows.

We consider a square or cubic mesh with step h and we anchor at each node all possible imbricated (regularly overlapping) square or cube elements of size $l = nh$ which bridge all nodes between the corners except the nodes coinciding with the corners [Fig. 2(*a*)] (n is a given integer). The square imbricate elements may be generated by placing into each node of the mesh of size h the lower left corner of one element of size l. Each square of the mesh, which also represents a local element of size h, is overlapped by n^2 imbricate elements of size l [Fig. 2(*a*)], and so the cross section of each of these imbricate elements must be considered to be $(1-c)/n^2$. For a cubic mesh in three dimensions, the imbricate cube's "cross section" would be $(1-c)/n^3$ by similar reasoning.

As for the treatment of boundaries, all imbricate elements may be first generated ignoring the boundaries. Then, those elements that stick out beyond the boundary surface must be cut at the boundary and attached to the boundary nodes, similar to the manner shown in Fig. 2(*c*) of Ref. 4.

It might seem that the number of imbricate elements is vastly larger than the number of mesh size elements. However, their numbers are in fact the same (except for a small difference due to the boundaries). This is because in two dimensions, each square element of size l overlaps n^2 elements of size h, and a similar situation applies in three dimensions.

In the foregoing arrangement, each node is attached to 4 local square elements of size h, and to 4 imbricate square elements of size l [Fig. 2(*c*)]. The nodal equilibrium equation for each direction involves a sum of 4 nodal forces from the local elements plus a sum of 4 nodal forces from the imbricate elements. Obviously, the standard procedure may be used to assemble the total nodal forces.

The continuum limit ($h \to 0$ at fixed l) of the finite element model just described can be used as still another derivation of Eqs. 17 or 18. This approach may also lead to more general formulations. One could postulate several layers, each one consisting of identical imbricated elements with a different length l for each layer and a different constitutive relation for each layer (e.g., steep softening, milder softening, almost no softening, etc.). One could even introduce a continuous distribution of such layers with a continuous distribution of l from 0 to ∞ and a continuous distribution of constitutive parameters.

Relation to Blunt Crack Band Theory and Determination of l

When the mesh size h is chosen equal to size l of the representative volume, the finite element model of the imbricate continuum becomes equivalent to that of the classical, local continuum. Meshes with $h > l$ make no sense for the imbricate continuum model because the characteristic length cannot be resolved. The local continuum theory, therefore, is a proper basis for the finite element analysis if $h \geq l$. On the other hand, finite element calculations based on the local continuum theory make no sense when $h < l$, and the imbricate continuum should be used.

The transition from the classical, local continuum to the imbricate continuum during mesh refinements is shown in Fig. 3, where arrangements 1–3 correspond to the classical case, for which the crack band theory has been set up, while arrangements 4–5 correspond to the imbricate continuum, which allows the extension of the crack band theory to finer subdivisions. (The nodes above each other in Fig. 3 are assumed to have the same horizontal displacement.)

In previous works (see Refs. 3, 5, 6, 10–12, 17, 19, 21, and 26 in Ref. 4) it has been shown that a finite element model of the usual type, which involves a single-element-wide band of strain-softening (progressively cracking) elements of a fixed size, w_c, in relation to the aggregate size [Fig. 2(*b*)], is capable of closely describing essentially all fracture test data available in the literature, particularly their deviations from linear elastic fracture mechanics. It has been also shown that, by using an energy criterion or an equivalent strength criterion for the onset of fracture (strain-softening), proper convergence can be achieved for meshes with $h \geq l$.

From the mathematical viewpoint, however, convergence could not be demonstrated for element sizes smaller than l and approaching zero. The present formulation allows this to be done, thus resolving the detail

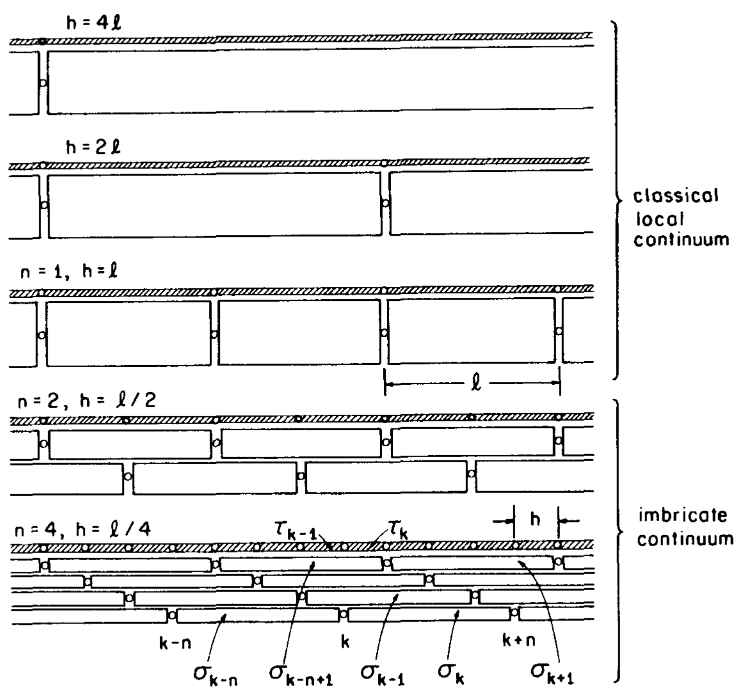

FIG. 3.—Subsequent Refinements of One-Dimensional Discretization and Transition from Finite Elements for Local Continuum to Those for Imbricate Nonlocal Continuum

of the strain and displacement fields inside the fracture process zone. The fact that the size of the strain-softening region is, for the imbricate continuum, fixed and determined by the characteristic length l—as the calculations in Ref. 1 confirm—justifies the use in the crack band theory of a preferred element size and of a single-element width at the crack band front [Fig. 2(*b*)].

The question of how to determine l is vital, since it is the only parameter which cannot be found by testing homogeneously strained specimens. Recognizing that for $h = l$ the imbricate continuum becomes essentially equivalent to the crack band model, we may determine approximately the characteristic length, l, as the effective width, w_c, of the crack band front. This width can be obtained by linear regression analysis of the test data from fracture tests of geometrically similar specimens of widely different sizes, supplemented by tensile strength data (3). Previous analysis (see Ref. 17 of Ref. 4) yielded $w_c \simeq 3\,d_a$ in which d_a = maximum aggregate size of concrete. So as a crude estimate, we may use

$$l = 3d_a \qquad (41)$$

Differential Approximation of Imbricate Continuum

The imbricate continuum may be also approximately described by differential equations. For this purpose, expand the displacement and its gradient at point $\mathbf{x}'$ into Taylor series centered at point $\mathbf{x}$:

$$u_i(\mathbf{x}') = u_i(\mathbf{x}) + u_{i,k}(\mathbf{x})\,r_k + \frac{1}{2!}\,u_{i,km}(\mathbf{x})\,r_k r_m + \frac{1}{3!}\,u_{i,kmn}(\mathbf{x})\,r_k r_m r_n + \dots \qquad (42)$$

$$u_{j,i}(\mathbf{x}') = u_{j,i}(\mathbf{x}) + u_{j,ik}(\mathbf{x})\,r_k + \frac{1}{2!}\,u_{j,ikm}(\mathbf{x})\,r_k r_m + \dots \qquad (43)$$

in which $r_k = x'_k - x_k$. Consider first the difference operator in Eq. 5. If Eq. 42, truncated after the cubic term, is substituted into Eq. 5 along with $r_i = \pm a_i l/2$ in which a_i = direction cosines of axis x_i, it follows that $D_i u_j = [u_{j,k} a_k + (1/6)\,u_{j,kmn} a_k a_m a_n\,(l/2)^2]\,a_i$. Noting that $a_i a_k = \delta_{ik}$ = Kronecker delta, we thus obtain the approximations

$$D_i u_j = u_{j,i} + \lambda^2 u_{j,ikk} = (1 + \lambda^2 \nabla^2)\,\frac{\partial u_j}{\partial x_i} \qquad (44)$$

in which ∇^2 is the Laplace operator and

$$\lambda^2 = \frac{1}{24}\,l^2 \qquad (45)$$

Second, consider the three-dimensional gradient averaging operator on a sphere of diameter l' (Eq. 24). Substituting Eq. 43, truncated after third derivatives, into Eq. 24, we see that all even derivatives of u_j cancel out, due to symmetry of volume $V(\mathbf{x})$, and we obtain

$$D_i u_j(\mathbf{x}) = u_{j,i}(\mathbf{x}) + \frac{1}{2!}\,A_{km}\,u_{j,ikm}(\mathbf{x}) + \frac{1}{4!}\,B_{kmpq}\,u_{j,ikmpq}(\mathbf{x}) + \dots \qquad (46)$$

in which $A_{km} = \frac{1}{V}\int_r r_k r_m dV = \frac{l'^2}{20}\delta_{km}$ .... (47)

$$B_{kmpq} = \frac{1}{V}\int_V r_k r_m r_p r_q dV \quad (48)$$

Thus, neglecting terms higher than second degree, we again obtain the approximation in Eq. 44, but with

$$\lambda^2 = \frac{1}{40} l'^2 \quad (49)$$

For the two-dimensional case, gradient averaging is carried out on a circle of diameter l', and Eq. 47 then yields $A_{km} = \delta_{km} l'^2/32$. Eq. 44 is obtained again, but with

$$\lambda^2 = \frac{1}{32} l'^2 \quad (50)$$

Comparing Eqs. 45, 49 and 50, we may further conclude that various gradient averaging operators lead to the same differential approximation of the difference operator if

$$\text{for a sphere:} \quad l' = \sqrt{\frac{5}{3}}\, l = 1.291\, l$$

$$\text{for a circle:} \quad l' = \sqrt{\frac{4}{3}}\, l = 1.155\, l \quad (51)$$

in which characteristic length l is defined as the side of a cube or a square. The operators satisfying Eq. 51 are, therefore, approximately equivalent.

Similar differential approximations may be obtained for $D_i\sigma_{ij}$. It then follows from Eqs. 17–19 that the field equations for the differential approximation of the imbricate continuum have the form

$$S_{ij,j} = \rho \ddot{u}_i \quad (52)$$

$$S_{ij} = (1-c)(1+\lambda^2\nabla^2)\,\sigma_{ij} + c\tau_{ij} \quad (53)$$

$$\sigma_{ij} = C_{ijkm}(\bar{\epsilon})\,\bar{\epsilon}_{km}, \quad \tau_{ij} = C_{ijkm}(\epsilon)\,\epsilon_{km} \quad (54)$$

$$\bar{\epsilon}_{km} = (1+\lambda^2\nabla^2)\,\epsilon_{km} \quad (55)$$

$$\epsilon_{km} = \frac{1}{2}(u_{k,m} + u_{m,k}) \quad (56)$$

For $\lambda \to 0$ we recover the usual relations for local continuum.

Eqs. 52–56 may also be derived directly from the principle of virtual work, which may now be written as

$$\delta W = (1-c)\int_\Omega \sigma_{ij}\delta\bar{\epsilon}_{ij}dV + c\int_\Omega \tau_{ij}\delta\epsilon_{ij}dV - \int_\Omega f_i\delta u_i dV$$

$$-\int_S p_i\delta u_i dS = 0 \quad (57)$$

Using Gauss' integral theorem, we obtain

$$\int_\Omega \sigma_{ij}\delta\bar{\epsilon}_{ij}dV = \int_\Omega (\sigma_{ij}\delta u_{i,j} + \lambda^2\sigma_{ik}\delta u_{i,kjj})\,dV = \int_S \sigma_{ij}\delta u_i n_j dS$$

$$-\int_\Omega \sigma_{ij,j}\delta u_i dV + \int_S \lambda^2\sigma_{ik}\delta u_{i,kj} n_j dS + \delta W_a \quad (58)$$

in which

$$\delta W_a = -\int_\Omega \lambda^2\sigma_{ik,j}\delta u_{i,kj}dV = \int_\Omega \lambda^2\sigma_{ik,jj}\delta u_{i,k}dV - \int_S \lambda^2\sigma_{ik,j}n_j\delta u_{i,k}dS \quad (59)$$

Substituting Eqs. 58–59 and Eq. 15 into Eq. 57, setting $f_i = -\rho\ddot{u}_i$, and grouping the volume and surface integrals, we can bring Eq. 57 to the form $\int F_a\delta u_i dV + \int F_b\delta u_i dS + \int F_c\delta u_{i,k}dS + \int F_d\delta u_{i,kj}dS = 0$. To satisfy it for any kinematically admissible δu_i, it is necessary (according to the fundamental lemma of variational calculus) that $F_a = 0$ in Ω, and that either δu_i or F_b, either $\delta u_{i,k}$ or F_c, and either $\delta u_{i,kj}$ or F_d vanish at the surface. Thus, we obtain Eq. 52 for the interior of domain Ω, and the following boundary conditions on surface S:

Either $\delta u_i = 0$ or $S_{ij}n_j = (1-c)(1+\lambda^2\nabla^2)\,\sigma_{ij}n_j + c\tau_{ij}n_j = p_i$ ..... (60)

Either $\delta u_{i,j} = 0$ or $\sigma_{ij,k}n_k = 0$ .... (61)

Either $\delta u_{i,jk}n_k = 0$ or $\sigma_{ij} = 0$ .... (62)

There are more boundary conditions than in the local continuum theory because the differential equations are of a higher order. From Eqs. 52 and 60 we may also conclude that tensor S_{ij} must represent the differential approximation of the total stress defined by Eq. 21 or Eqs. 39–40.

For the special case of uniaxial stress in a bar of length L, the virtual work principle requires that

$$\delta W = \int_0^L [(1-c)\,\sigma\delta\bar{\epsilon} + c\tau\delta u' - f\delta u]\,dx - [p\delta u]_0^L = 0 \quad (63)$$

in which primes denote derivatives with respect to $x = x_1$; and $u = u_1$, $\sigma = \sigma_{11}$, and $\tau = \tau_{11}$. Substituting $\delta\bar{\epsilon} = \delta u' + \lambda^2\delta u'''$, $p = -\rho\ddot{u}$, integrating by parts, and applying the fundamental lemma of variational calculus, we obtain the differential equation

$$\frac{\partial}{\partial x}\left[(1-c)\left(\sigma + \lambda^2\frac{\partial^2\sigma}{\partial x^2}\right) + c\tau\right] = \rho\frac{\partial^2 u}{\partial t^2} \quad (64)$$

which is a special case of Eqs. 52–56, and also the boundary conditions at $x = 0$ or $x = L$:

Either $\delta u = 0$ or $S = (1-c)(\sigma + \lambda^2\sigma'') + c\tau = p$ .... (65)

$$\text{Either} \quad \delta u' = 0 \quad \text{or} \quad \sigma' = 0 \qquad (66)$$

$$\text{Either} \quad \delta u'' = 0 \quad \text{or} \quad \sigma = 0 \qquad (67)$$

which are recognized as a special case of Eqs. 60–62. Note that S in Eq. 65 represents the approximation of the total uniaxial stress.

The natural boundary conditions in Eqs. 60–62 or Eqs. 65–67 depend on our tacit assumption that, in writing Eq. 57, Eq. 54 gives the correct mean strain approximation even for points within the boundary layer for which the representative volume actually reaches outside the body surface. If a correction to the work expression $\sigma_{ij}\,\delta\bar{\epsilon}_{ij}$ were to be introduced for points in the boundary layer in Eq. 57, then the natural boundary conditions would be different.

Let us now examine the effect of considering a nonuniform, but continuous and smooth weighting function, $\alpha(\mathbf{r})$, as in Eq. 31. Introducing in Eq. 31 again the Taylor series expansion (Eq. 43), truncated after the quadratic term, one finds that

$$D_i u_j = (1 + \bar{\alpha}\lambda^2\nabla^2)\frac{\partial u_i}{\partial x_j} \qquad (68)$$

$$\text{in which} \quad \bar{\alpha} = \frac{1}{V}\int_V \alpha(\mathbf{r})\, r_k r_k\, dV' \qquad (69)$$

Coefficient $\bar{\alpha}$, however, appears only in its product with λ^2, and never alone. Therefore, as far as the differential approximation of the third order is concerned, the use of nonuniform weights, $\alpha(\mathbf{r})$, is not more general in any respect. Parameter $\bar{\alpha}$ cannot be calibrated by tests, since only the product $\bar{\alpha}\lambda^2$ can be obtained (on the basis of fracture tests). So it does not matter which smooth weighting function is chosen, and the simplest choice is $\alpha(\mathbf{r}) = 1/V = \text{const}$. A nonuniform smooth function, $\alpha(\mathbf{r})$, would make a difference only for differential approximations of a higher order than that just shown (higher than third).

Note the symmetric action of operator $(1 + \lambda^2\nabla^2)$ in Eqs. 53–55. The foregoing variational derivation (Eqs. 57–64) again proves that if this operator is applied to local strains, ϵ_{ij} (Eq. 55), it must also be applied to stresses, σ_{ij}, in the differential equation of motion (Eq. 53). Replacing Eq. 52 with the usual equation of motion, $\sigma_{ij,j} = \rho\ddot{u}_i$, would be incorrect.

For mean strains we have the approximation

$$\bar{\epsilon}_{ij} = \epsilon_{ij} + \frac{1}{2}\lambda^2(u_{i,jkk} + u_{j,ikk}) \qquad (70)$$

It is interesting to compare it with the well-known theories of the Cosserat continuum with couple stresses, or the micropolar continuum with rotation gradients (5,6). These theories employ only the first and second derivatives of displacements, while here we employ the first and third derivatives and skip the second derivatives. Moreover, here we do not need to associate with the higher displacement derivatives any special type of stress tensor of a higher rank, such as the couple stress tensor used in the previous theories. The internal forces in the imbricate nonlocal continuum are fully characterized by stress tensors of the second rank. Instead of higher-rank tensors, we have the imbrication, or the

Laplacian in the differential approximation.

Let us now check stability of the differential approximation, restricting attention to the one-dimensional case and to linearly elastic properties. Eqs. 52–56 or Eq. 64 furnish the differential equation of motion

$$(1 - c)\left(1 + \lambda^2\frac{\partial^2}{\partial x^2}\right)^2\frac{\partial^2 u}{\partial x^2} + c\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 u}{\partial t^2} \qquad (71)$$

in which $x = x_1$; $u = u_1$; and E = Young's modulus. We seek a solution of the form $u = A\exp[i\omega(x - vt)]$ in which v = wave velocity and ω = frequency ($i^2 = -1$). Substitution in Eq. 71 provides the relation

$$v^2 = [(1 - c)(1 - \lambda^2\omega^2)^2 + c]\frac{E}{\rho} \qquad (72)$$

For reasons of stability, the wave velocity v for $E > 0$ must always be real and positive, and so stability requires that $c > 0$, which is the same result as in Ref. 4.

Note that if the operator $(1 + \lambda^2\nabla^2)$ on stress were deleted from the equation of motion (from Eqs. 52–53), we would get $v^2\rho/E = c + (1 - c)(1 - \lambda^2\omega^2)$, which cannot avoid complex values of v from occuring for any $c < 1$ if ω is sufficiently large. This again demonstrates that if the gradient averaging operator is applied to strains, it must also be applied to stress in the differential equation of motion, or else instability results.

The discrete finite element approximation could also be based on the differential form (Eqs. 52–56). In this case, the connectivity of the elements would be as usual, i.e., the imbricated finite element structure would be unnecessary. However, since third displacement derivatives are present, the finite elements would have to involve cubic polynomials for their displacement distribution functions. Such finite elements are unwieldy and complicated, as is evident in the bending of thin shells. Therefore, it seems preferable for finite element modeling to use the imbricated elements as previously described. The usual element types may then be used, and the nonlocal character is taken into account merely by properly defining the integer matrix that gives the node numbers for each element number.

Finally, let us examine the alternative symmetric nonlocal formulation from Eqs. 26–31 of Ref. 4 in which the local term is not used while the weighting function has a spike with Dirac delta function $\delta(\mathbf{r})$, i.e., $\alpha(\mathbf{r}) = c\delta(\mathbf{r}) + (1 - c)/V$ in V. Then the differential approximation of the weighted gradient averaging operator is found to be

$$D_i u_j = (1 - c)(1 + \lambda^2\nabla^2)\frac{\partial u_j}{\partial x_i} + c\frac{\partial u_j}{\partial x_i} = (1 + \lambda'^2\nabla^2)\frac{\partial u_j}{\partial x_i} \qquad (73)$$

in which $\lambda'^2 = (1 - c)\lambda^2$. Because Eq. 73 is of the same form as Eq. 44, this case is equivalent to the differential approximation of the imbricate continuum (Eqs. 52–56) with $c = 0$, i.e., with no local term. Such an approximation is not stable. A higher-order approximation would have to replace Eq. 73 in order to achieve a differential approximation that is stable. It further follows that the alternative symmetric nonlocal formulation in Eqs. 27–31 of Ref. 4 is asymptotically close to instability (up to third-order terms in the Taylor series expansion of displacement) and is thus questionable for numerical analysis.

Conclusions

1. Postulating that the stress depends on the change of distance between a pair of points lying at a certain fixed distance, l, apart, one can use variational calculus to derive three-dimensional continuum equations of motion of the imbricate nonlocal continuum developed in a preceding paper (4) for one dimension. These equations are of difference-differential or integro-differential type. Equations of the same form result when the strain is defined by averaging the displacement gradient.
2. There exists a boundary layer which requires special treatment.
3. If strains are defined by displacement gradient averaging, then the continuum equation of motion must involve averaging of stress gradients or must be stated in terms of gradients of the total stresses obtained by averaging the stresses that appear in the constitutive equation. Otherwise, the operators and corresponding finite element stiffness matrices would be nonsymmetric (even if the continuum is elastic).
4. The averaging may be defined most simply by a difference operator, but more generally by an averaging integral for partial derivatives over the volume of a sphere, or by least-square fitting of the derivatives on the surface of this sphere. The latter two definitions are equivalent.
5. The averaging operator may be approximated by the operator $(1 + \lambda^2\nabla^2)$ where λ = material constant. A differential approximation of the imbricate nonlocal continuum is thus obtained. Then the mean strains are given by differential expressions involving up to third derivatives of displacements, and the continuum equation of motion becomes a sixth-order differential equation in displacements. Although higher displacement derivatives are present, it is not necessary to introduce couple stresses or rotation gradients, and only second-rank stress tensors are needed to fully characterize macroscopic (smoothed) internal forces.
6. Various possible definitions of the mean strain may be considered approximately equivalent if they have the same differential approximation.
7. The nonlocal continuum aspect may be simply modeled by an imbricated arrangement of ordinary finite elements. The imbricate continuum could be also modeled by unimbricated finite elements based on a higher-order differential approximation.
8. The blunt crack band model used in finite element analysis of progressive fracturing is obtained as the special case of the imbricate continuum discretization for sufficiently large mesh sizes. The discretization of imbricate continuum may, on the other hand, be used to extend the blunt crack band model to arbitrarily fine meshes and to make possible crack bands more than single-element wide. The blunt crack band model becomes part of a convergent discretization scheme, and thus is put on a firm continuum foundation. Because the size of the imbricate elements is fixed and does not approach zero in the continuum limit, the energy consumed by damage (strain-softening) up to final failure converges to a finite value as the mesh is refined, while for finite element models based on the classical local continuum, it converges to zero.

9. The characteristic length, l, may be calibrated on the basis of maximum load fracture tests of geometrically similar specimens of different sizes (3).

Acknowledgment

Financial support under AFOSR Grant No. 83-0009 to Northwestern University is gratefully acknowledged. Thanks are due to Prof. Ted Belytschko for various stimulating discussions, and to Mary Hill for her excellent secretarial assistance.

Appendix.—References

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