Wave Propagation in a Strain-Softening Bar: Exact Solution

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Abstract: A closed-form solution is given for a one-dimensional bar which undergoes strain softening (i.e., a gradual decline of stress to zero at increasing strain). It is shown that strain softening can occur in the interior of a body, and that the length of the strain-softening region tends to localize into a point, which agrees with what was previously shown by stability analysis for static situations. The stress in the strain-softening cross section drops to zero instantly, regardless of the shape of the strain-softening diagram, and the total energy dissipated in the strain-softening domain of the bar is found to vanish. Despite these unpleasant features, the problem apparently possesses a solution for certain boundary and initial conditions. However, the fact that the energy dissipation in the strain-softening process vanishes is not representative of the experimentally observed behavior of real strain-softening materials such as concrete or geomaterials.

Introduction

The strain softening of a material is the decline of stress at increasing strain (Fig. 1) and represents the process of progressive failure or damage. Since Hadamard's observation that the wave speed ceases to be real if the tangent modulus becomes negative (7), strain softening has been considered an unacceptable feature for a constitutive equation (12). The relevance of Hadamard's point to real materials, however, is questionable since it ignores the fact that, if the strain reverses, the tangent modulus becomes positive, even after strain softening, and that the strain-softening domain localizes so that the question of wave speed is irrelevant.

As is now well-documented by test results (compare Refs. 1–4), strain-softening zones of finite size can indeed be produced in certain materials, such as concrete, rocks, some soils and some composites. The fact that the physical source of strain softening in these zones are dispersed, highly localized defects, such as progressively developing microcracks, is an irrelevant objection. Structural analysts need a constitutive equation that approximates the behavior of the representative volume of a strain-softening material on the whole, in a smeared-out fashion. They have no hope of individually treating the localized defects, with all the attendant inhomogeneities of the microstructure (aggregate pieces, etc.).

By a static stability analysis which takes into account the fact that, at a strain decrease after strain softening, the tangent modulus is positive, it was shown that strain softening in the usual, classical (local) continuum always localizes into a layer of zero thickness (i.e., into a plane) (1,2). Similarly, strain softening was either assumed localized (6,9,10) or proven to localize (1) in bending problems. In these static analyses, the equilibrium path of the structure could be traced as well as it was traced in the static analysis of elastic buckling, and the problem formulation did not appear to be mathematically unreasonable.

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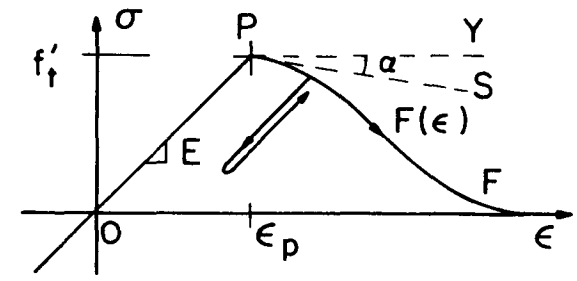

FIG. 1.—Stress-Strain Diagram with Strain Softening

Recently, the problem of strain softening in continuum dynamics has been intensely debated at some conferences (5,8,11), particularly in regard to large-scale finite element computations. It has been questioned whether strain softening in a continuum (of the usual, local type) is a sound concept from the mathematical point of view. This paper attempts to answer this question by analyzing a simple wave propagation problem for which an exact solution can be found. The analysis is expected to show whether a dynamic solution for given initial and boundary conditions exists. The solution can be used to check the convergence and accuracy of finite element programs with strain softening.

Problem Formulation

Consider a bar of length $2L$, with a unit cross section and mass ρ per unit length [Fig. 2(a)]. Let the bar be loaded by forcing both ends to move simultaneously outward, with constant opposite velocities of magnitude c. The boundary conditions are

$$\left.\begin{array}{ll}\text{For} & x = -L: \quad u = -ct \\ \text{For} & x = L: \quad u = ct\end{array}\right\} \quad (\text{for} \quad t \geq 0) \qquad (1)$$

in which x = length coordinate measured from the midlength; t = time; and $u(x,t)$ = displacement in direction x. Initially, at $t = 0$, the bar is undeformed and at rest ($u = \dot{u} = 0$). Due to symmetry, the problem is equivalent to a bar fixed at $x = 0$. The material of the bar has the stress-strain diagram OPF shown in Fig. 1, which exhibits elastic behavior with Young's modulus E up to strain ϵ_p at peak stress f'_t (strength), followed by a strain-softening curve $F(\epsilon)$—a positive monotonic continuous function which has a negative slope $F'(\epsilon)$ but, otherwise, an arbitrary shape, and which attains zero stress either at some finite strain or asymptotically for $\epsilon \to \infty$. The unloading ($\dot{\epsilon} < 0$) and reloading ($\dot{\epsilon} \geq 0$), up to the last previous maximum strain, is elastic with modulus E and, if the strain

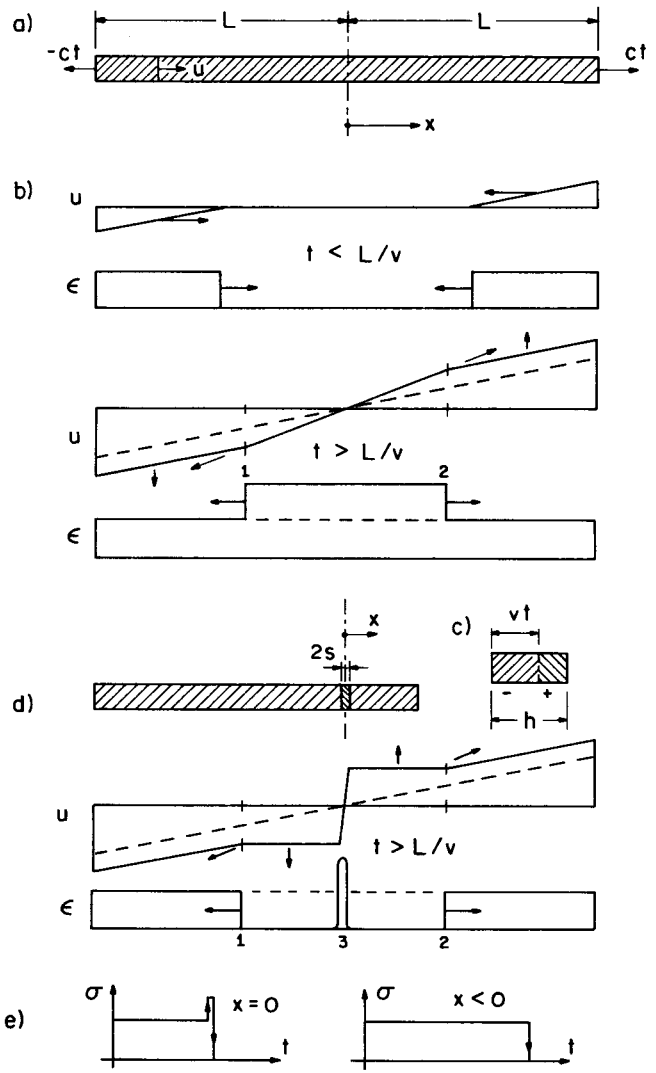

FIG. 2.—(*a*) Bar Analyzed; (*b*) Elastic Solution; (*c*, *d*, and *e*) Solution with Strain Softening

increases beyond this maximum, the (virgin) strain-softening diagram is followed.

SOLUTION

Suppose first that strains exceeding ϵ_p are never produced, i.e., the bar remains linearly elastic. The differential equation of motion is hyperbolic and reads

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad \text{with} \quad v^2 = \frac{E}{\rho} \qquad (2)$$

For the given boundary and initial conditions, the solution is

$$u = -c\left\langle t - \frac{x+L}{v}\right\rangle + c\left\langle t + \frac{x-L}{v}\right\rangle \qquad \left(\text{for} \quad t \le \frac{2L}{v}\right) \qquad (3)$$

in which the symbol $\langle\ \rangle$ is defined as $\langle A\rangle = A$ if $A > 0$ and $\langle A\rangle = 0$ if $A \le 0$, and v = wave velocity. The strain is

$$\epsilon = \frac{\partial u}{\partial x} = \frac{c}{v}\left[H\left(t - \frac{x+L}{v}\right) + H\left(t + \frac{x-L}{v}\right)\right] \qquad (4)$$

in which H denotes Heaviside step function. The strain consists of two tensile step waves of magnitude c/v, emanating from the ends of the bar and converging on the center. After the waves meet at the midpoint, the strain is doubled [Fig. 2(*b*)]. The stress is $\sigma = E\epsilon$.

Obviously, if $c/v \le \epsilon_p/2$, the assumption of elastic behavior holds for $x \le 2Lv$, i.e., until the time each wave-front runs the entire length of the bar. If, however, $\epsilon_p/2 < c/v \le \epsilon_p$, the previous solution applies only for $t < L/v$ and the midpoint cross section ($x = 0$) enters the strain-softening regime at $t = L/v$, i.e., when the wave fronts meet at the midpoint.

Let us examine the behavior in a very small neighborhood of the interface between the elastic and the strain-softening region, imagining a segment (control volume) of length h [Fig. 2(*c*)] that is fixed within the material and contains the interface at the distance Vt from the left end of the bar, where V denotes velocity of the interface. The displacements at points x just to the right of the interface are $u^+ = U + \epsilon^+(x - Vt)$, and those just to the left of it are $u^- = U + \epsilon^-(x - Vt)$, in which U = displacement at the interface, and ϵ^+, ϵ^- = strains just to the right and to the left of the interface. Differentiating, we get the material velocities (material derivatives of u) $\dot{u}^+ = \dot{U} - V\epsilon^+$ and $\dot{u}^- = \dot{U} - V\epsilon^-$ just to the right and to the left of the interface. The equation of motion of the small element h, fixed within the material, may be stated as follows: the rate of the linear momentum of element h equals the total force applied on this element. Thus, we have

$$\frac{\partial}{\partial t}\left[\int_0^{Vt} \rho(\dot{U} - V\epsilon^-)dx + \int_{Vt}^{h} \rho(\dot{U} - V\epsilon^+)dx\right] = \sigma^+ - \sigma^- \qquad (5)$$

from which we get the jump relation, relating the jumps in stress and strain:

$$\sigma^+ - \sigma^- = \rho V^2(\epsilon^+ - \epsilon^-) \qquad (6)$$

Now consider again the whole bar and assume that $\epsilon_p/2 < c/v < \epsilon_p$. Then the waves are elastic until they meet at time $t_1 = L/v$. Strain softening begins immediately at time t_1 at $x = 0$ (midpoint). Due to symmetry, a centrally located strain-softening segment of length $2s$ with initial value $s = 0$ is created in the middle of the bar at $t = t_1$ [Fig. 2(*d*)].

It might be of interest to mention what happens if the step waves are considered as the limiting cases of strain waves with a wave-front that rises very sharply but continuously. Then the stress at $x = 0$ would rise to $\sigma = f_t'$ before strain softening begins, and so an elastic stress wave of magnitude $f_t' - c/v$, superimposed on the incoming wave of magnitude c/v, would pass through the midpoint ($x = 0$) from right to left. However, this passing elastic wave would become in the limit a finite magnitude stress pulse of an infinitely short duration, because the strain softening is reached instantly in the limit. The strain-energy of such a pulse is zero; thus it can have no effect on the bar and need not be considered in our solution.

The differential equation of motion within the strain-softening segment $2s$ has the form $\partial\sigma/\partial x = \rho\partial^2 u/\partial t^2$. Since $\partial\sigma/\partial x = (\partial\sigma/\partial\epsilon)\partial\epsilon/\partial x = F'(\epsilon)\partial^2 u/\partial x^2$, we have

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0, \quad \text{with} \quad \alpha^2 = \frac{-F'(\epsilon)}{\rho} \qquad (7)$$

Because $F'(\epsilon) < 0$, this equation is *elliptic*, which means that interaction over finite distances is immediate; however, that need not be a problem as long as s remains infinitely small.

One possible solution of Eq. 7 is

$$u = [a(t - t_1) + \epsilon_p]x \qquad (\text{for} \quad -s \le x \le s,\ t > t_1) \qquad (8)$$

in which a is some constant. This solution, implying a uniform strain distribution, can adequately describe the strain-softening segment as long as s is infinitely small. Note that Eq. 8 satisfies the symmetry condition, and also the condition $\partial u/\partial x = \epsilon_p$ at $t = t_1$.

Check that, for $-v(t - t_1) \le x < -s$ (i.e., outside segment s, on its left), Eq. 2 is solved by the following expression:

$$u = c\left(\frac{x + L}{v} - t\right) + f(\xi) = c\left(\frac{2x}{v} - \xi\right) + f(\xi), \quad \xi = t - \frac{L - x}{v} \qquad (9)$$

in which f is an arbitrary function describing a wave propagating toward the left. By differentiation of Eq. 9

$$\epsilon = \frac{\partial u}{\partial x} = \frac{1}{v}[c + f'(\xi)] \quad \text{for} \quad -v\left(t - \frac{L}{v}\right) < x < -s \qquad (10)$$

in which $f'(\xi) = df(\xi)/d\xi$.

Now we must set up the interface conditions for displacements and stresses at $x = -s$. Continuity of displacement requires, for $t > (L + s)/v$, that

$$f(\xi_1) - c\left(\xi_1 + \frac{2s}{v}\right) = -[a(t - t_1) + \epsilon_p]s, \quad \xi_1 = t - \frac{L + s}{v} \qquad (11)$$

The interface stresses must satisfy the jump condition in Eq. 6. The interface at the left end of segment s can be either stationary (constant s, $s \to 0$) or it can move to the left at velocity $V = \dot{s}$. It cannot move to the right since the softening segment would not then exist, and there would be no strain softening.

Suppose that the interface moves to the left. Then the material points are entering the strain-softening regime as the interface moves through them. Therefore, σ^- on the left of the interface must be equal to the strength f_t'. From this we know that $\sigma^- \ge \sigma^+$. At the same time, $\epsilon^+ > \epsilon^-$ because the strain must be larger than ϵ_p inside the strain-softening segment, and must not be larger than ϵ_p outside it. Thus, we see from Eq. 6 that $V^2 < 0$ if the interface moves. This, however, is impossible, since V must not be imaginary. The only remaining possibility is that the interface does not move ($V = 0$) and s remains infinitesimal.

Therefore, according to Eq. 6

$$\sigma^+ = \sigma^- \quad (\text{at} \quad x = s) \qquad (12)$$

i.e., the stress must be continuous at $x = s$ (strain-softening boundary) for $t > t_1 + s/v$, although the strain is discontinuous. It follows that $E\epsilon^- = F(\epsilon^+)$. Substituting Eq. 10 here and calculating ϵ^+ from Eq. 8 as $\partial u/\partial x$, we get (for $t > t_1 + s/v$):

$$\frac{E}{v}[c + f'(\xi_1)] = F(\epsilon^+), \quad \epsilon^+ = a(t - t_1) + \epsilon_p \qquad (13)$$

Eliminating a from Eqs. 11 and 13, we find

$$\epsilon^+ = \frac{1}{s}\left[c\left(\xi_1 + \frac{2s}{v}\right) - f(\xi_1)\right] \qquad (14)$$

Now, since the strain-softening segment must remain infinitely short, i.e., $s \to 0$, we have $\epsilon^+ \to \infty$, and consequently $F(\epsilon^+) \to 0$. Therefore, $c + f'(\xi_1) = 0$, i.e., $f(\xi_1) = -c\xi_1$ or $f(\xi) = -c\xi$. Hence

$$f(\xi) = -c\left\langle t - \frac{L - x}{v}\right\rangle \qquad (15)$$

Consequently, the complete solution for $0 \le t \le 2L/v$ and $x < 0$ is

$$u = -c\left\langle t - \frac{x + L}{v}\right\rangle - c\left\langle t - \frac{L - x}{v}\right\rangle \qquad (16)$$

$$\text{and} \quad \epsilon = \frac{c}{v}H\left(t - \frac{x + L}{v}\right) - \frac{c}{v}H\left(t - \frac{L - x}{v}\right) \qquad (17)$$

For the right half of the bar, $x > 0$, a symmetric solution applies. For $x \to 0^-$ the displacements are $u = -2c\langle t - L/v\rangle$, and for $x \to 0^+$ they are $u = 2c\langle t - L/v\rangle$. So, after time $t_1 = L/v$, the displacements develop a discontinuity at $x = 0$, with a jump of magnitude $4c\langle t - L/v\rangle$. Therefore, the strain near $x = 0$ is $\epsilon = 4c\langle t - L/v\rangle\delta(x)$ in which $\delta(x)$ = Dirac's delta function. This expression for ϵ satisfies the condition that $\int_{-s}^{s}\epsilon dx = 4c\langle t - L/v\rangle$ for $s \to 0$.

The complete strain field for $x \le 0$ and $0 \le t \le 2L/v$ is

$$\epsilon = \frac{c}{v}\left[H\left(t - \frac{x + L}{v}\right) - H\left(t - \frac{L - x}{v}\right) + 4\langle vt - L\rangle\delta(x)\right] \qquad (18)$$

This solution is sketched in Fig. 2(*d*) and Fig. 2(*e*). It may be checked

that no unloading occurs within the strain-softened material (at $x = 0$) as supposed.

A subtle question remains in regard to the possibility of unloading (i.e., strain reversal). We tacitly implied no unloading and showed that a solution exists. Can we find another solution if unloading is assumed to occur after the start of strain softening? Apparently we cannot. Indeed, for unloading, the tangent modulus becomes positive; thus the equation of motion in segment 2s becomes hyperbolic, which means that the wave arriving from the right is transmitted to the left across the midpoint. But then the superposition of the converging waves yields an increase of strain, which contradicts the assumption of unloading. Thus, our solution for $c < v\epsilon_p/2$ (Eq. 16) appears to be unique.

Properties of Solution

Some properties of the solution are rather interesting. It does not depend on the shape of the strain-softening diagram; the result is the same for strain-softening stress-strain diagrams of very mild slope or very steep slope, and even for a vertical stress drop in the σ-ϵ diagram. The latter case is equivalent to having a bar that splits in the middle at time $t = L/v$, with each half having a free end at $x = 0$; indeed, this behavior leads to the same solution.

Consider the dependence of the solution on the boundary conditions. When c is just slightly smaller than $v\epsilon_p/2$, the solution is given by Eq. 3, and when c is just slightly larger than $v\epsilon_p/2$, the solution is given by Eq. 16, which differs from Eq. 3 by a finite amount for $t > L/v$. Thus, an infinitely small change in the boundary condition can lead to a finite change in the response. Hence, the solution of the dynamic problem for a strain-softening material does not depend continuously on the boundary conditions, i.e., the response may be termed unstable (for $c = v\epsilon_p/2$).

The solution also exhibits a discontinuous dependence on the parameters of the stress-strain diagram. Compare the solutions for stress-strain diagram OPS in Fig. 1, for which the magnitude of the downward slope PS is as small as desired but nonzero, and for the stress-strain diagram OPY in Fig. 1, for which the straight line PY is horizontal and represents plastic yielding. For the former, the present solution applies. For the latter, plastic case, Eq. 9 also applies but must be subjected to the boundary condition $E\,\partial u/\partial x = f_t'$ at $x = -s$. The resulting solution is well-known and is entirely different from the present solution. Thus, the response is discontinuous in regard to the strain-softening slope E_t as $E_t \to 0$.

As we see from our solution, strain softening cannot happen within a finite segment of the bar. It is confined to a single cross-sectional plane of the bar, at which the strains become infinite within an instant (i.e., an infinitely short time interval after the start of softening), regardless of the shape and slope of the strain-softening part of the σ-ϵ diagram. Strain softening cannot happen as a field in the usual classical type of a deformable continuum as considered here. The parameters of the strain-softening diagram, such as the slope $F'(\epsilon)$, cannot be considered as characteristic properties of a classical continuum, since they have no effect on the solution.

This conclusion agrees with that drawn in 1976 on the basis of static stability analysis of strain localization (1–3). To circumvent the phenomenon of localization into a single cross-sectional plane, one would have to postulate some special type of continuum in which the strains cannot localize into a single cross-sectional plane (a nonlocal continuum can do that).

We should observe, too, that strain softening consumes or dissipates no energy because the volume of the strain-softening region is zero while the energy density in the strain-softening domain is finite, not infinite. This is also confirmed by the fact that, for $t > L/v$, the present solution is the same as the elastic solution for a bar that is split initially in the middle into two bars, for which the mechanical energy (potential plus kinetic) must be conserved.

Finally, we should note that strain softening can be produced in the interior of a body, and not just at the boundaries, as has been recently suggested. To produce strain softening, it is also not necessary to have a symmetric problem, e.g., we may enforce velocity $c = c_1 = -0.6\epsilon_p/v$ at the left end, and $c = c_2 = 0.9\epsilon_p/v$ at the right end, and strain softening would again be produced at $x = 0$.

Conclusion

Strain softening in a classical (local) continuum is not a mathematically meaningless concept. A solution exists for given initial and boundary conditions. However, the hypothesis that strain softening may occur in a classical (local) continuum is not representative of known strain-softening materials (concrete, geomaterials, some composites), in which strain softening consumes finite energy and strain-softening regions of finite size are observed experimentally.

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Appendix.—References

1. Bažant, Z. P., "Instability, Ductility and Size Effect in Strain-Softening Concrete," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 102, No. EM2, 1976, pp. 331–344 (Discussions, Vol. 103, No. EM2, pp. 357–358, No. EM4, pp. 775–777; Vol. 104, No. EM2, pp. 501–502).
2. Bažant, Z. P., and Panula, L., "Statistical Stability Effects in Concrete Failure," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 104, No. EM5, 1978, pp. 1195–1212.
3. Bažant, Z. P., "Crack Band Model for Fracture of Geomaterials," *Proceedings of the 4th International Conference on Numerical Methods in Geomechanics*, Z. Eisenstein, ed., Vol. 3, Edmonton, Canada, 1982, pp. 1137–1152.
4. Bažant, Z. P., and Oh, B. H., "Crack Band Theory for Fracture of Concrete," *Materials and Structures* (RILEM, Paris, France), Vol. 16, 1983, pp. 155–177.
5. Bergan, P. G., "Record of the Discussions on Numerical Modeling," *Mechanics of Geomaterials and Concrete*, Z. P. Bažant, ed., *Proceedings of the IUTAM W. Prager Symposium*, held at Northwestern University, at Evanston, Ill., in Sept., 1983, John Wiley and Sons, London, England, 1985 (in press).

6. Darvall, P. L., "Critical Softening of Hinges in Indeterminate Beams and Portal Frames," *Civil Engineering Transactions*, The Institution of Civil Engineers, Australia, 1983, pp. 199–209.
7. Hadamard, J., *Leçons sur la propagation des ondes* (Chapter VI), Hermann et Cie, Paris, France, 1903.
8. Hegemier, G. A., and Read, H. E., "Some Comments on Strain-Softening," University of California, San Diego, Calif., (Manuscript, 19 pages) (presented at the DARPA-NSF Workshop, held at Northwestern University, Evanston, Ill., Oct., 1983).
9. Maier, G., "On Structural Instability due to Strain-Softening," *IUTAM Symposium on Instability of Continuum Systems*, held at Herrenhalb, Germany, Sept., 1969, Springer Verlag, West Berlin, Germany, 1971, pp. 411–417.
10. Maier, G., Zavelani, A., and Dotreppe, J. C., "Equilibrium Branching due to Flexural Softening," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 99, No. EM4, Aug., 1973, pp. 897–901.
11. Sandler, I., and Wright, J., "Summary of Strain-Softening," *Theoretical Foundations for Large Scale Computations of Nonlinear Material Behavior*, Preprints of the DARPA-NSF Workshop, held at Northwestern University, Evanston, Ill., Oct., 1983, S. Nemat-Nasser, ed., pp. 225–241.
12. Truesdell, C., and Noll, W., "The Nonlinear Theories of Mechanics," *Encyclopedia of Physics*, Vol. III/3, S. Flügge, ed., Springer Verlag, West Berlin, Germany, 1965.