ZAMM · Z. angew. Math. u. Mech. 66 (1986) 1, 37–49

Bažant, Z. P.; Oh, B. H.

Efficient Numerical Integration on the Surface of a Sphere

Es werden verschiedene neue numerische Integrationsformeln auf der Oberfläche einer dreidimensionalen Kugel abgeleitet. Diese Formeln sind gegenüber den bereits existierenden insofern besser, als sie für den gleichen Approximationsgrad für Funktionen mit zentraler oder ebener Symmetrie weniger Integrationspunkte erfordern. Ferner wird eine allgemeine Methode zur Ableitung der Integrationsformeln bewiesen, die auf Kosten eines einem Computer zu überlassenden extensiven numerischen Aufwandes begriffliche Einfachheit erreicht. Bei dieser Methode werden die Koeffizienten der Integrationsformel aus einem System linearer algebraischer Gleichungen bestimmt, das direkt die Bedingungen dafür darstellt, daß eine gewisse Anzahl von Termen der dreidimensionalen Taylor-Reihenentwicklung der integrierten Funktion um den Kugelmittelpunkt verschwindet. Dagegen werden die unbekannten Lagen der Integrationspunkte aus einer Bedingung dafür bestimmt, daß der nächste Term (die nächsten Terme) der Entwicklung verschwindet, und wenn dieser nicht zum Verschwinden gebracht werden kann, aus einer Bedingung zur Minimierung der Größe dieses Terms (dieser Terme). Schließlich wird eine neue Optimalitätsbedingung der Integrationsformeln formuliert, die für den Integrationsfehler in gewissen Anwendungen von Bedeutung ist.

Several new numerical integration formulas on the surface of a sphere in three dimensions are derived. These formulas are superior to the existing ones in that for the same degree of approximation they require fewer integration points for functions with central or planar symmetry. Furthermore, a general method of deriving the integration formulas, which achieves conceptual simplicity at the expense of extensive numerical work left for a computer, is demonstrated. In this method, the coefficients of the integration formula are determined from a system of linear algebraic equations directly representing the conditions for a certain number of terms of the three-dimensional Taylor series expansion of the integrated function about the center of the sphere to vanish, while the unknown locations of the integration points are determined from a condition for the next term (or terms) of the expansion to vanish, and if it cannot be made to vanish, then from a condition for minimizing the magnitude of this term (or these terms). Finally, we formulate a new condition of optimality of the integration formulas which is important for the integration error in certain applications.

Выводятся некоторые новые формулы для численного интегрирования на поверхности сферы в трех размерности. Они лучше существующих еще формул в том смысле, что им надо меньше точек интегрирования для функций с центрально или плоской симметриями для такой-же степени приближения. Дальше доказывается общий метод для вывода формул интегрирования достигающие понятийную простоту за счет обширной численной работы оставленной ЭВМ. В этом методе коэффициенты формул интегрирования определяются непосредственно из системы линейных алгебраических уравнений. Эти уравнения представляют условия для того, что некоторое число термов трехмерного разложения в ряд Тэйлора интегрированной функции вокруг центра сфера обращается в нуль. Неизвестная локализация точек интегрирования определяется из условия для того, что следующий терм (или следующие термы) разложения обращается в нуль. Если невозможно совершать, что этот терм обращается в нуль , потом эти локализации определяются из условия для минимизации величину этого терма (этих терм). Наконец формулируется новое условие оптимальности для формул интегрирования, которое является важным для ошибки интегрирования в некоторых применениях.

Introduction

Some problems of physics and engineering require an accurate and efficient numerical integration over the surface of a sphere. One such problem is the determination of the relationship between the stress and strain tensors in a deformable material having nonlinear properties defined separately on planes of various orientation within the material.

The numerical integration can be carried out over a rectangular domain in the (θ, φ)-plane where θ and φ are the spherical angular coordinates. However, for application in finite element programs involving hundreds of elements and hundreds of loading (or time) steps, numerical integration over the surface of a sphere may have to be carried out million-times or more, and then the use of a rectangular (θ, φ) domain is inefficient since too many integration points are wastefully crowded near the pole of the spherical coordinates. Moreover, functions that are smooth and well behaved on the surface of a sphere are not such in the (θ, φ)-plane.

We seek Gauss-type quadratures, i.e., quadratures with optimally located points and optimal coefficients (weights). For an ideal formula, the integration points should obviously be distributed over the surface of the sphere as uniformly as possible. A perfectly uniform distribution is provided by the vertices or the face centroids of a regular polyhedron. No regular three-dimensional polyhedron has more than 20 vertices or faces, and, unfortunately, the corresponding 20-point 5th degree formula, due to Albrecht and Collatz [2, 1, 13], is not sufficiently accurate for the above-mentioned applications in their nonlinear range [2]. Higher degree formulas, involving a greater number of points, were derived by Albrecht and Collatz [2], Finden [7], Sobolev [12], McLaren [11], and Stroud [13]; they involve up to 240 points and are of degrees up to 14. A comprehensive listing of the formulas obtained prior to 1971 may be found in Stroud's book [13] (pp. 294–303). Lebedev [9, 10] recently derived certain formulas of degrees 19 and 23, involving over 200 integration points and exhibiting orthogonal symmetries. A review of the most recent work is given by Keats and Diaz [8].

In this paper, we present some new integration formulas which are superior in a certain sense to the existing formulas. We also demonstrate a particularly simple method of derivation of the formulas with certain optimal properties, which achieves conceptual simplicity at the cost of much numerical work, relying on the power of the computer.

Finally, we also formulate a new condition of optimality which is important for the actual error in certain applications.

Optimality of the integration formula

We are interested in a formula of the smallest possible error. This property is, however, difficult to quantify in general. Therefore, one usually seeks a formula which satisfies the following, more easily defined, optimality condition.

Condition I: For the given number of integration points, the formula integrates exactly polynomials of the highest possible degree. This degree is called the *degree of the formula.*

In the case that various possible locations of integration points yield a formula of the same degree, then we seek a formula which also satisfies a second condition.

Condition II: The coefficient of the first nonzero term (truncation term) in the Taylor series expansion of the integral is minimum.

For many formulas, condition II cannot be applied because Condition I (or the requirement of maximum regularity in the location of integration points) fully defines the coefficients and locations of the integration points.

Symmetry properties: Comparing two different formulas of the same degree, the formula with a smaller number of integration points is normally preferable. Not always, however. Frequently, for example, the integrated function exhibits some symmetries. Consider the applications in continuum mechanics. Because stress or strain components on cross sections with normals of opposite but parallel orientation are equal, the integrated values in these applications are always the same for any two diametrically opposite points on the surface of the sphere. Therefore, formulas which are centrally symmetric, i.e. symmetric with regard to the center of the sphere, have a great advantage over those which are not.

Or the integrated function, for example, may be symmetric (or antisymmetric) with regard to one (or more) cartesian coordinate planes — a situation which is called the *full symmetry* if the formula is symmetric with regard to all three coordinate planes. In the aforementioned stress-strain calculations, this situation arises, e.g., for the states of plane stress, plane strain, or axisymmetric stress. In such situations, the formulas that possess the same type of symmetry have an advantage over the formulas that do not, since certain integration points may be deleted. So, a formula which in general involves more integration points may actually be more efficient in such a situation.

Even for two formulas that are both symmetric with regard to a plane, there may be a difference depending on whether the formula has any points on the symmetry plane. If it has not, then the reduction of the number of integration points due to a symmetric integrand is more significant.

Condition I, as well as II, does not, of course, guarantee minimum error. The error, however, depends on the type of function that is integrated. Let us call the *test function* some function which is typical of a given application and is not integrated by the integration formula exactly. It is interesting to see what happens if the set of integration points is subjected to a rigid-body rotation about the center of the sphere. If the formula were exact, then the resulting value of the integral would have to be the same for any rotation. The spread of the values obtained for all possible rigid body rotations is a good indication of the error of the formula. This leads us to formulate the following optimality condition.

Condition III: If, for a given test function, the values of the integral are calculated for all possible rigid-body rotations of the set of integration points about the center of the sphere, then the optimum formula is the one which gives the smallest difference between the maximum and minimum values of the integral.

The trouble with this optimality condition is that different integration formulas may be optimum for different test functions. Inevitably, this optimality condition must be restricted to a certain type of application. Experience from Refs. 3 and 4 indicates that a formula that is usually optimal for the given type of applications can be best identified from Condition III, and that Condition III is a practically important optimality condition.

Condition III seems capable of capturing the influence of coefficient values on the error. E.g., formulas with some negative coefficients usually have a large error and are generally undesirable [13], regardless of their degree. Approximately also, the smaller the ratio of the maximum to the minimum coefficient in the formula, the smaller the error.

Basic relations

The integral over the surface of a sphere of radius h of a $(2N + 2)$-times continuously differentiable function $u(x, y, z)$ may be expressed as [5]:

$$I = \frac{1}{4\pi h^2} \int\limits_S\int u(x, y, z)\,\mathrm{d}\sigma = \sum_{n=0}^{N} \frac{h^{2n}}{(2n+1)!} \Delta^n \bar{u} + R = \bar{u} + \frac{h^2}{3!} \Delta\bar{u} + \frac{h^4}{5!} \Delta^2\bar{u} + \frac{h^6}{7!} \Delta^3\bar{u} + \dots + R \tag{1}$$

in which S is the surface of the sphere in three dimensions; $\mathrm{d}\sigma$ is its area element; x, y, z are cartesian coordinates whose origin is at the center of the sphere; a superimposed bar denotes the values taken at the center of the sphere; R is a remainder whose order is $2N + 1$, and

$$\Delta^n\bar{u} = \left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^n u\right]_{x=y=z=0} \tag{2}$$

Equation 1 converts the integration problem to the evaluation of differential operators $\Delta^n\bar{u}$ at the center of the sphere. Consequently, to develop an integration formula it suffices to approximate the values of $\Delta^n\bar{u}$ by the values of u on the spherical surface. To this end, we may employ the Taylor series expansion:

$$u(x_1^\alpha, x_2^\alpha, x_3^\alpha) = \bar{u} + \frac{1}{1!}\bar{u}_{,i}x_i^\alpha + \frac{1}{2!}\bar{u}_{,ij}x_i^\alpha x_j^\alpha + \frac{1}{3!}\bar{u}_{,ijk}x_i^\alpha x_j^\alpha x_k^\alpha + \ldots \tag{3}$$

in which

$$x_1^\alpha = \cos\varphi_\alpha \sin\theta_\alpha\,, \qquad x_2^\alpha = \sin\varphi_\alpha \sin\theta_\alpha\,, \qquad x_3^\alpha = \cos\theta_\alpha\,. \tag{4}$$

Here the subscript or superscript α represents the number of the integration point on the spherical surface ($\alpha = 1, 2, 3, \ldots, n$); and φ_α, θ_α are its spherical coordinate angles. We use here numerical subscripts for the cartesian coordinates, $x_1 = x$, $x_2 = y$ and $x_3 = z$, and repetition of a latin subscript (not α) implies summation over 1, 2, 3. The latin subscripts preceded by a comma denote partial derivatives with regard to the respective coordinate.

Method of derivation of integration formulas from conditions I and II

We first illustrate the method of derivation of an integration formula for which the point locations may be determined by maximizing symmetries of the set of integration points. As an example, we derive the 32 point formula of Finden [7], Sobolev [12] and McLaren [11]. This formula has 32 integration points (Fig. 1b), which consist of two groups: 20 points at the centroids of the faces of a regular icosahedron, and 12 points at its vertices (which coincide with the face centroids of a dual dodecahedron). It is well known that optimum formulas are obtained with a set of points of maximum possible regularity or symmetry, and the foregoing arrangement satisfies this condition. Each group of points is symmetric in that a certain permutation of point numbers is equivalent to a rigid body rotation of the group of points. Due to this, we know that within each such symmetric group of points the coefficients of the formula should be the same, which means that the function values at these points are simply summed. Therefore, we express the sums of the function values for each of the groups of 20 and 12 points, using three-dimensional Taylor series expansions about the center of the sphere (equation (3),) and we thus obtain

$$S_{20} = \sum_{\alpha=1}^{20} u^\alpha = 20\bar{u} + A_{ij}\bar{u}_{,ij} + B_{ijkl}\bar{u}_{,ijkl} + C_{ijklmn}\bar{u}_{,ijklmn} + D_{ijklmnpq}\bar{u}_{,ijklmnpq} + \ldots\,, \tag{5}$$

$$S_{12} = \sum_{\alpha=21}^{32} u^\alpha = 12\bar{u} + E_{ij}\bar{u}_{,ij} + F_{ijkl}\bar{u}_{,ijkl} + G_{ijklmn}\bar{u}_{,ijklmn} + H_{ijklmnpq}\bar{u}_{,ijklmnpq} + \ldots\,. \tag{6}$$

The numerical values of the coefficients A_{ij}, B_{ijkl}, ..., $H_{ijklmnpq}$ are calculated by a computer since these values are too numerous. Using a computer program, one may thus verify by numerical evaluation of all terms that many of these coefficients are zero, and that the only nonzero terms are

$$S_{20} = 20\bar{u} + c_1\Delta\bar{u} + c_2\Delta^2\bar{u} + c_3\Delta^3\bar{u} + c_4\Delta^4\bar{u} + c_5\Delta^5\bar{u} + \ldots\,, \tag{7}$$

$$S_{12} = 12\bar{u} + a_1\Delta\bar{u} + a_2\Delta^2\bar{u} + a_3\Delta^3\bar{u} + a_4\Delta^4\bar{u} + a_5\Delta^5\bar{u} + \ldots\,. \tag{8}$$

In the sums implied in each term of equations (5)—(6) by repetition of subscripts, we may add the terms related by permutation of subscripts; e.g.

$$B_{1112}\bar{u}_{,1112} + B_{1121}\bar{u}_{,1121} + B_{1211}\bar{u}_{,1211} + B_{2111}\bar{u}_{,2111} = (B_{1112} + B_{1121} + B_{1211} + B_{2111})\,u_{,1112}\,.$$

This complicates the evaluation of all those terms for which the numerical values of subscripts differ. It is, therefore, simpler to evaluate the coefficients c_1, c_2, c_3 and c_4 from the coefficients of those terms of equations (5)—(6) for which all subscripts are equal (e.g., 1,) since then the subscripts do not have different permutations. Consequently, we must have

$$\begin{aligned} c_1 &= A_{11}\,, \quad & c_2 &= B_{1111}\,, \quad & c_3 &= C_{111111}\,, \quad & c_4 &= D_{11111111}\,; \\ a_1 &= E_{11}\,, \quad & a_2 &= F_{1111}\,, \quad & a_3 &= G_{111111}\,, \quad & a_4 &= H_{11111111}\,. \end{aligned} \tag{9}$$

Expressing $\bar{u}$ from equations (7)—(8) and substituting it into equation (1), we then get the following approximations of the integral I:

$$I_{20} = \frac{1}{20}(S_{20} - c_1\Delta\bar{u} - c_2\Delta^2\bar{u} - c_3\Delta^3\bar{u} - c_4\Delta^4\bar{u}) + \frac{h^2}{3!}\Delta\bar{u} + \frac{h^4}{5!}\Delta^2\bar{u} + \frac{h^6}{7!}\Delta^3\bar{u} + \frac{h^8}{9!}\Delta^4\bar{u}\,, \tag{10}$$

$$I_{12} = \frac{1}{12}(S_{12} - a_1\Delta\bar{u} - a_2\Delta^2\bar{u} - a_3\Delta^3\bar{u} - a_4\Delta^4\bar{u}) + \frac{h^2}{3!}\Delta\bar{u} + \frac{h^4}{5!}\Delta^2\bar{u} + \frac{h^6}{7!}\Delta^3\bar{u} + \frac{h^8}{9!}\Delta^4\bar{u}\,. \tag{11}$$

Further we may verify by numerical evaluation that (for a unit sphere, $h = 1$):

$$\frac{1}{3!} - \frac{c_1}{20} = 0\,, \qquad \frac{1}{5!} - \frac{c_2}{20} = 0\,, \qquad \frac{1}{3!} - \frac{a_1}{12} = 0\,, \qquad \frac{1}{5!} - \frac{a_2}{12} = 0\,. \tag{12}$$

This means that the sums of the function values at the centroids of the faces of either an icosahedron or a dodecahedron approximate the integral I with an error of the order of h^6, as has been shown before (cf. [1, 2]).

Any combination such as $kI_{20} + (1 - k)\,I_{12}$, with any k, also approximates the integral with an error generally of the order of h^6. We now seek to find such k that the terms of the order of h^6 would cancel, i.e., such that

$$k\left(\frac{1}{7!} - \frac{c_3}{20}\right) + (1 - k)\left(\frac{1}{7!} - \frac{a_3}{12}\right) = 0\,. \tag{13}$$

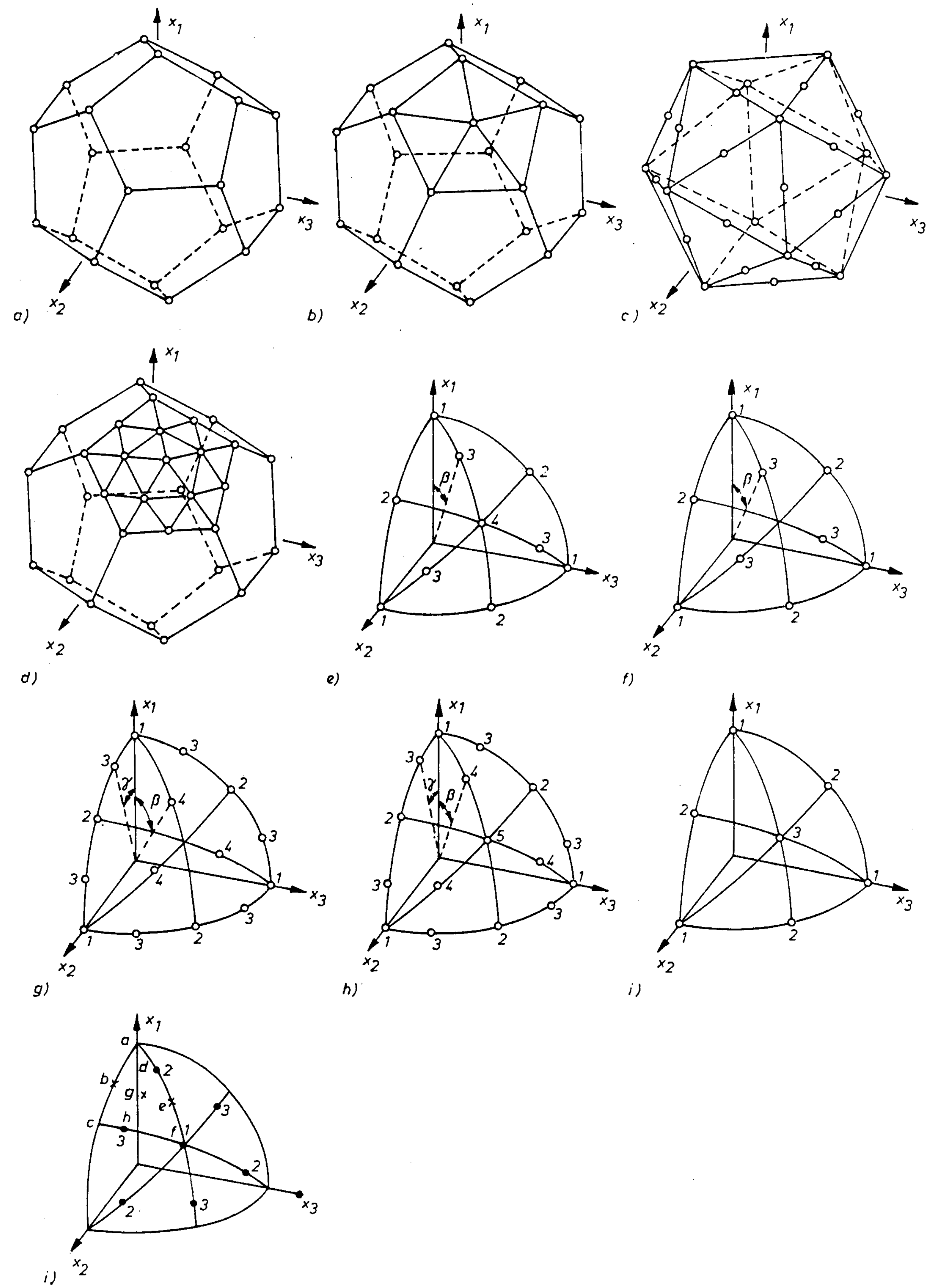


Fig. 1. a)—j)

The solution of this equation is $k = 0.642857142857$. This yields finally the numerical integration formula

$$I = kI_{20} + (1 - k)\, I_{12} = \frac{k}{20} S_{20} + \frac{1-k}{12} S_{12} + O(h^8) = B_2 S_{20} + B_1 S_{12} + O(h^8) = \sum_{\alpha=1}^{32} C_\alpha u^{(\alpha)} + O(h^8) \tag{14}$$

which approximates the integral I with an error of order of h^8, and B_2 and B_1 represent the coefficients (or weights) of the points of the integration formula located at the centroids of the faces of the icosahedron and the dual dodecahedron, respectively:

$$B_1 = 0.0297619047619\,, \qquad B_2 = 0.0321428571429\,. \tag{15}$$

According to previous works (cf. [6, 11, 10]), $B_1 = 25/840$ and $B_2 = 27/840$, which is the same. The direction cosines x_i^α for the integration points ($\alpha = 1, \ldots, 32$) are listed by STROUD [13, p. 300] and also in Refs. [3], [4] and [6].

Note that since integral I in (1) is divided by h^2, this formula integrates exactly a 9th degree polynomial, i.e., is of 9th degree. Generally, the degree of the formula, d, is higher by 1 than the truncation error $O(h^n)$ in the expansion of I ($d = n + 1$).

A similar derivation procedure may be employed for an 11th degree formula with 62 points which include all of the points of the previous formula (Fig. 1b) plus the centers of all faces of the icosahedron (cf. [13]); or for ALBRECHT's and COLLATZ' formula [2] of 7th degree with 26 points located at vertices, mid-edges and face centroids of an octahedron (Fig. 1i). It has been checked that for this 26-point formula the present method (computer program) yields exactly the same result (weights $w_1 = 0.0476190476190$, $w_2 = 0.0380952380952$, $w_3 = 0.0321428571429$ at points 1, 2, 3, Fig. 1i).

Let us now show the method of derivation of a formula for which, in contrast to the previous case (Fig. 1b), the locations of the integration points cannot be determined by maximizing the symmetries of the set of points. For the purpose of illustration, we consider now MCLAREN's fully symmetric 11th degree formula (cf. [11, 13]) with 50 points shown in Fig. 1e for one octant. In all other octants the integration points are located similarly. Altogether, we have four symmetric groups of like points which include six vertices of an octahedron, twelve mid-edges of the octahedron, eight octahedral directions, corresponding to the face centroids of the octahedron, and 24 points between the vertices and normals of the faces of the octahedron. In contrast to the previously considered formula, the optimum angular distances β of these 24 points from the octahedron vertices cannot be determined from symmetry conditions but must be found by other means.

For reasons of symmetry, the coefficients of the integration formula should be the same within each group of points, i.e., the integrand values for these points are simply summed. Therefore, using equation (3), we may again express the sums S_6, S_{12}, S_{24} and S_8 of the function values within each symmetric group of points, and from this we obtain the following approximations to the integral in (1) based on each of these groups of points taken separately;

$$I_6 = \frac{1}{6}(S_6 - a_1\Delta\bar{u} - a_2\Delta^2\bar{u} - a_3\Delta^3\bar{u} - a_4\Delta^4\bar{u} - a_5\Delta^5\bar{u}) + \\ + \frac{h^2}{3!}\Delta\bar{u} + \frac{h^4}{5!}\Delta^2\bar{u} + \frac{h^6}{7!}\Delta^3\bar{u} + \frac{h^8}{9!}\Delta^4\bar{u} + \frac{h^{10}}{11!}\Delta^5\bar{u}\,, \tag{16}$$

$$I_{12} = \frac{1}{12}(S_{12} - b_1\Delta\bar{u} - b_2\Delta^2\bar{u} - b_3\Delta^3\bar{u} - b_4\Delta^4\bar{u} - b_5\Delta^5\bar{u}) + \\ + \frac{h^2}{3!}\Delta\bar{u} + \frac{h^4}{5!}\Delta^2\bar{u} + \frac{h^6}{7!}\Delta^3\bar{u} + \frac{h^8}{9!}\Delta^4\bar{u} + \frac{h^{10}}{11!}\Delta^5\bar{u}\,, \tag{17}$$

$$I_{24} = \frac{1}{24}(S_{24} - c_1\Delta\bar{u} - c_2\Delta^2\bar{u} - c_3\Delta^3\bar{u} - c_4\Delta^4\bar{u} - c_5\Delta^5\bar{u}) + \\ + \frac{h^2}{3!}\Delta\bar{u} + \frac{h^4}{5!}\Delta^2\bar{u} + \frac{h^6}{7!}\Delta^3\bar{u} + \frac{h^8}{9!}\Delta^4\bar{u} + \frac{h^{10}}{11!}\Delta^5\bar{u}\,, \tag{18}$$

$$I_8 = \frac{1}{8}(S_8 - d_1\Delta\bar{u} - d_2\Delta^2\bar{u} - d_3\Delta^3\bar{u} - d_4\Delta^4\bar{u} - d_5\Delta^5\bar{u}) + \\ + \frac{h^2}{3!}\Delta\bar{u} + \frac{h^4}{5!}\Delta^2\bar{u} + \frac{h^6}{7!}\Delta^3\bar{u} + \frac{h^8}{9!}\Delta^4\bar{u} + \frac{h^{10}}{11!}\Delta^5\bar{u}\,. \tag{19}$$

It may be checked by a computer that the terms with h^2 cancel out, i.e., $-(a_1/6) + 1/3! = 0$. etc. We now multiply these equations by k_1, k_2, k_3 and $(1 - k_1 - k_2 - k_3)$, respectively, add them up, and write the conditions that in the resulting equation the terms of the orders h^4, h^6 and h^8 would cancel (Condition I). These conditions read

$$k_1\left(\frac{1}{5!} - \frac{a_2}{6}\right) + k_2\left(\frac{1}{5!} - \frac{b_2}{12}\right) + k_3\left(\frac{1}{5!} - \frac{c_2}{24}\right) + (1 - k_1 - k_2 - k_3)\left(\frac{1}{5!} - \frac{d_2}{8}\right) = 0\,, \tag{20}$$

$$k_1\left(\frac{1}{7!} - \frac{a_3}{6}\right) + k_2\left(\frac{1}{7!} - \frac{b_3}{12}\right) + k_3\left(\frac{1}{7!} - \frac{c_3}{24}\right) + (1 - k_1 - k_2 - k_3)\left(\frac{1}{7!} - \frac{d_3}{8}\right) = 0\,, \tag{21}$$

$$k_1\left(\frac{1}{9!} - \frac{a_4}{6}\right) + k_2\left(\frac{1}{9!} - \frac{b_4}{12}\right) k_3 + \left(\frac{1}{9!} - \frac{c_4}{24}\right) + (1 - k_1 - k_2 - k_3)\left(\frac{1}{9!} - \frac{d_4}{8}\right) = 0\,. \tag{22}$$

This is a system of three linear equations for k_1, k_2 and k_3. The coefficients, however, depend on the unknown angle, β, which is formed by the direction for the point of weight w_3 (Fig. 1e). We need a further condition for determining the optimal value of angle β, and for this purpose we consider the next term of the Taylor series expansion, which is of the order of h^{10}. For this term, the linear combination of equations (16)—(19) (with $h = 1$) furnishes the value

$$F_{10}(\beta) = k_1\left(\frac{1}{11!}-\frac{a_5}{6}\right) + k_2\left(\frac{1}{11!}-\frac{b_5}{12}\right) + k_3\left(\frac{1}{11!}-\frac{c_5}{24}\right) + (1-k_1-k_2-k_3)\left(\frac{1}{11!}-\frac{d_5}{8}\right). \tag{23}$$

There are now two possible cases: 1) Either function $F_{10}(\beta)$ has a zero point, or 2) it does not. In the first case, the value of β for which $F_{10}(\beta)$ would vanish would be the desired value of β. For the present arrangement of points, however, function $F_{10}(\beta)$ cannot be made to vanish for any β, as is indicated by plotting a graph from many calculated values of this function. Therefore, we have here the second case (optimizing condition II), and we search for solution β for which $F_{10}(\beta)$ becomes minimum.

The solution may be obtained iteratively. We choose some value of angle β, calculate coefficients k_1, k_2, k_3 from equations (20)—(22), and evaluate $F_{10}(\beta)$ from (23). Then we repeat it for other values of angle β and use Newton's method to find the β-value which yields minimum $F_{10}(\beta)$ (or $\mathrm{d}F_{10}(\beta)/\mathrm{d}\beta = 0$). In this manner, we find that $k_1 = 0.076190476190 5$, $k_2 = 0.270899470899$, $k_3 = 0.484160052910$, and

$$\beta = 25.2394018206° . \tag{24}$$

Subsequently, the numerical integration formula that we have been seeking is obtained as a linear combination of equations (16)—(19):

$$\begin{aligned} I &= k_1 I_6 + k_2 I_{12} + k_3 I_{24} + (1-k_1-k_2-k_3)\, I_8 = \\ &= \frac{k_1}{6} S_6 + \frac{k_2}{12} S_{12} + \frac{k_3}{24} S_{24} + \frac{1-k_1-k_2-k_3}{8} S_8 + O(h^{10}) = B_1 S_6 + B_2 S_{12} + B_3 S_{24} + B_4 S_8 + O(h^{10}) = \\ &= \sum_{\alpha=1}^{50} C_{(\alpha)} u^{(\alpha)} + O(h^{10}) \end{aligned} \tag{25}$$

in which $B_1 = 0.0126984126984$, $B_2 = 0.0225749559033$, $B_3 = 0.0201733355379$, $B_4 = 0.0210937500000$. This agrees with McLaren's results: $B_1 = 9216/725760$, $B_2 = 16384/725760$, $B_3 = 14641/725760$, $B_4 = 15309/725760$. Since the integral I in (1) is divided by h^2, this formula integrates exactly an 11th degree polynomial.

In contrast to the standard procedure (cf. [2, 13], the method of derivation just illustrated does not use the theory of orthogonal polynomials, which is rather complicated in more than one dimension. Also, we do not need to set up here a polynomial whose roots would give the coefficients of the formula. Instead, these coefficients are solved from a system of linear equations such as equations (20)—(22) (or equation (13)), which may be supplemented (if the optimum locations of some points are not known) by the condition that a certain additional function (or functions), representing the next term of the expansion, should either vanish or be minimized.

New formulas and their discussion

Using the method just illustrated, it is quite easy to derive, with the help of a computer, various new integration formulas. The following new formulas have been set up:

1) A 42-point, 9th degree, fully symmetric formula (Table 1, Fig. 1f);
2) A 66 point, 11th degree, fully symmetric formula (Table 2, Fig. 1g);
3) A 74-point, 13th degree, fully symmetric formula (Table 3, Fig. 1h);
4) A 42-point, 9th degree formula without full symmetry (Table 4, Fig. 1c);
5) A 122-point, 13th degree formula without full symmetry (Table 5, Fig. 1d; note that the point arrangement resembles Buckminster Fuller's geodesic domes, popular with architects).

All these formulas are centrally symmetric. The tables list only half of the points; for the other half, all elements of the direction vectors have opposite signs while the weights are the same.

One existing formula, due to McLaren [13, p. 302], which is of 14th degree and involves 72 points, appears, according to Condition I, to be superior to the present 74-point and 122-point formulas, which are of 13th degree. This McLaren's formula, however, is not centrally symmetric, and so the present formulas, which require only half as many points (i.e. 37 and 61) for centrally symmetric functions are superior in the sense of Condition I for such functions.

The 50-point McLaren's formula [13, p. 300] is the 9th degree formula which involves, among the existing formulas, the smallest known number of points for a fully symmetric arrangement of points (i.e., symmetric with regard to all cartesian coordinate planes). The present 9th degree fully symmetric formula involves 42 points (fig. 1f), and is therefore superior in the sense of Condition I.

The degree of the formula is, however, only a crude indicator of its accuracy. Tests of the type stated in Condition III revealed (cf. [4]), for example, that the 66-point 11th degree formula is just about equally accurate for the aforementioned stress-strain calculations as the 74-point 13th degree formula (both being orthogonally symmetric), and that Finden's 9th degree 32-point formula [13, p. 299] is quite inferior to the 9th degree 42-point and 50-point formulas.

A comparison based on Condition III has been made for problems in continuum mechanics mentioned in the introduction (cf. [4]), in which the test function mentioned in Condition III is represented by the uniaxial stress components on planes of various orientations. Consider a point of an inelastic material for which the strain components on a plane of any orientation are the resolved components of the same strain tensor. Further assume that, for increasing strain, there is a unique nonlinear relation between the normal stress and the normal component of the so-called microstress on any such plane, and that this relation has a peak stress after which the stress declines to zero at increasing strain (which is called strain-softening). For unloading (decreasing strain) on any plane we assume

Table 1. Direction cosines and weights for 2 × 21 points (degree 9, orthogonal symmetries, Fig. 1f)

α	x_1^α	x_2^α	x_3^α	C_α
1	1	0	0	0.0265214244093
2	0	1	0	0.0265214244093
3	0	0	1	0.0265214244093
4	0.707106781187	0.707106781187	0	0.0199301476312
5	0.707106781187	−0.707106781187	0	0.0199301476312
6	0.707106781187	0	0.707106781187	0.0199301476312
7	0.707106781187	0	−0.707106781187	0.0199301476312
8	0	0.707106781187	0.707106781187	0.0199301476312
9	0	0.707106781187	−0.707106781187	0.0199301476312
10	0.387907304067	0.387907304067	0.836095596749	0.0250712367487
11	0.387907304067	0.387907304067	−0.836095596749	0.0250712367487
12	0.387907304067	−0.387907304067	0.836095596749	0.0250712367487
13	0.387907304067	−0.387907304067	−0.836095596749	0.0250712367487
14	0.387907304067	0.836095596749	0.387907304067	0.0250712367487
15	0.387907304067	0.836095596749	−0.387907304067	0.0250712367487
16	0.387907304067	−0.836095596749	0.387907304067	0.0250712367487
17	0.387907304067	−0.836095596749	−0.387907304067	0.0250712367487
18	0.836095596749	0.387907304067	0.387907304067	0.0250712367487
19	0.836095596749	0.387907304067	−0.387907304067	0.0250712367487
20	0.836095596749	−0.387907304067	0.387907304067	0.0250712367487
21	0.836095596749	−0.387907304067	−0.387907304067	0.0250712367487

$\beta = 33.2699078510°$ in Fig. 1f.

Table 2. Direction cosines and weights for 2 × 33 points (degree 11, orthogonal symmetries, Fig. 1g)

α	x_1^α	x_2^α	x_3^α	C_α
1	1	0	0	0.00985353993433
2	0	1	0	0.00985353993433
3	0	0	0	0.00985353993433
4	0.707106781187	0.707106781187	0	0.0162969685886
5	0.707106781187	−0.707106781187	0	0.0162969685886
6	0.707106781187	0	0.707106781187	0.0162969685886
7	0.707106781187	0	−0.707106781187	0.0162969685886
8	0	0.707106781187	0.707106781187	0.0162969685886
9	0	0.707106781187	−0.707106781187	0.0162969685886
10	0.933898956394	0.357537045978	0	0.0134788844008
11	0.933898956394	−0.357537045978	0	0.0134788844008
12	0.357537045978	0.933898956394	0	0.0134788844008
13	0.357537045978	−0.933898956394	0	0.0134788844008
14	0.933898956394	0	0.357537045978	0.0134788844008
15	0.933898956394	0	−0.357537045978	0.0134788844008
16	0.357537045978	0	0.933898956396	0.0134788844008
17	0.357537045978	0	−0.933898956394	0.0134788844008
18	0	0.933898956394	0.357537045978	0.0134788844008
19	0	0.933898956394	−0.357537045978	0.0134788844008
20	0	0.357537045978	0.933898956394	0.0134788844008
21	0	0.357537045978	−0.933898956394	0.0134788844008
22	0.437263676092	0.437263676092	0.785875915868	0.0175759129880
23	0.437263676092	0.437263676092	−0.785875915868	0.0175759129880
24	0.437263676092	−0.437263676092	0.785875915868	0.0175759129880
25	0.437263676092	−0.437263676092	−0.785875915868	0.0175759129880
26	0.437263676092	0.785875915868	0.437263676092	0.0175759129880
27	0.437263676092	0.785875915868	−0.437263676092	0.0175759129880
28	0.437263676092	−0.785875915868	0.437263676092	0.0175759129880
29	0.437263676092	−0.785875915868	−0.437263676092	0.0175759129880
30	0.785875915868	0.437263676092	0.437263676092	0.0175759129880
31	0.785875915868	0.437263676092	−0.437263676092	0.0175759129880
32	0.785875915868	−0.437263676092	0.437263676092	0.0175759129880
33	0.785875915868	−0.437263676092	−0.437263676092	0.0175759129880

$\beta = 38.1982375056°$, $\gamma = 20.9490144149°$ in Fig. 1g.

Table 3. Direction cosines and weights for 2 × 37 points (degree 13, orthogonal symmetries, Fig. 1h)

α	x_1^α	x_2^α	x_3^α	C_α
1	1	0	0	0.0107238857303
2	0	1	0	0.0107238857303
3	0	0	0	0.0107238857303
4	0.707106781187	0.707106781187	0	0.0211416095198
5	0.707106781187	−0.707106781187	0	0.0211416095198
6	0.707106781187	0	0.707106781187	0.0211416095198
7	0.707106781187	0	−0.707106781187	0.0211416095198
8	0	0.707106781187	0.707106781187	0.0211416095198
9	0	0.707106781187	−0.707106781187	0.0211416095198
10	0.951077869651	0.308951267775	0	0.00535505590837
11	0.951077869651	−0.308951267775	0	0.00535505590837
12	0.308951267775	0.951077869651	0	0.00535505590837
13	0.308951267775	−0.951077869651	0	0.00535505590837
14	0.951077869651	0	0.308951267775	0.00535505590837
15	0.951077869651	0	−0.308951267775	0.00535505590837
16	0.308951267775	0	0.951077869651	0.00535505590837
17	0.308951267775	0	−0.951077869651	0.00535505590837
18	0	0.951077869651	0.308951267775	0.00535505590837
19	0	0.951077869651	−0.308951267775	0.00535505590837
20	0	0.308951267775	0.951077869651	0.00535505590837
21	0	0.308951267775	−0.951077869651	0.00535505590837
22	0.335154591939	0.335154591939	0.880535518310	0.0167770909156
23	0.335154591939	0.335154591939	−0.880535518310	0.0167770909156
24	0.335154591939	−0.335154591939	0.880535518310	0.0167770909156
25	0.335154591939	−0.335154591939	−0.880535518310	0.0167770909156
26	0.335154591939	0.880535518310	0.335154591939	0.0167770909156
27	0.335154591939	0.880535518310	−0.335154591939	0.0167770909156
28	0.335154591939	−0.880535518310	0.335154591939	0.0167770909156
29	0.335154591939	−0.880535518310	−0.335154591939	0.0167770909156
30	0.880535518310	0.335154591939	0.335154591939	0.0167770909156
31	0.880535518310	0.335154591939	−0.335154591939	0.0167770909156
32	0.880535518310	−0.335154591939	0.335154591939	0.0167770909156
33	0.880535518310	−0.335154591939	−0.335154591939	0.0167770909156
34	0.577350269190	0.577350269190	0.577350269190	0.0188482309508
35	0.577350269190	0.577350269190	−0.577350269190	0.0188482309508
36	0.577350269190	−0.577350269190	0.577350269190	0.0188482309508
37	0.577350269190	−0.577350269190	−0.577350269190	0.0188482309508

$\beta = 28.2929697104°$, $\gamma = 17.9960403883°$ in Fig. 1h.

Table 4. Direction cosines and weights for 2 × 21 points (degree 9, no orthogonal symmetries, Fig. 1c)

α	x_1^α	x_2^α	x_3^α	C_α
1	0.187592474085	0	0.982246946377	0.0198412698413
2	0.794654472292	−0.525731112119	0.303530999103	0.0198412698413
3	0.794654472292	0.525731112119	0.303530999103	0.0198412698413
4	0.187592474085	−0.850650808352	−0.491123473188	0.0198412698413
5	0.794654472292	0	−0.607061998207	0.0198412698413
6	0.187592474085	0.850650808352	−0.491123473188	0.0198412698413
7	0.577350269190	−0.309016994375	0.755761314076	0.0253968253968
8	0.577350269190	0.309016994375	0.755761314076	0.0253968253968
9	0.934172358963	0	0.356822089773	0.0253968253968
10	0.577350269190	−0.809016994375	−0.110264089708	0.0253968253968
11	0.934172358963	−0.309016994375	−0.178411044887	0.0253968253968
12	0.934172358963	0.309016994375	−0.178411044887	0.0253968253968
13	0.577350269190	0.809016994375	−0.110264089708	0.0253968253968
14	0.577350269190	−0.5	−0.645497224368	0.0253968253968
15	0.577350269190	0.5	−0.645497224368	0.0253968253968
16	0.356822089773	−0.809016994375	0.467086179481	0.0253968253968
17	0.356822089773	0	−0.934172358963	0.0253968253968
18	0.356822089773	0.809016994375	0.467086179481	0.0253968253968
19	0	−0.5	0.866025403784	0.0253968253968
20	0	−0.5	−0.866025403784	0.0253968253968
21	0	1	0	0.0253968253968

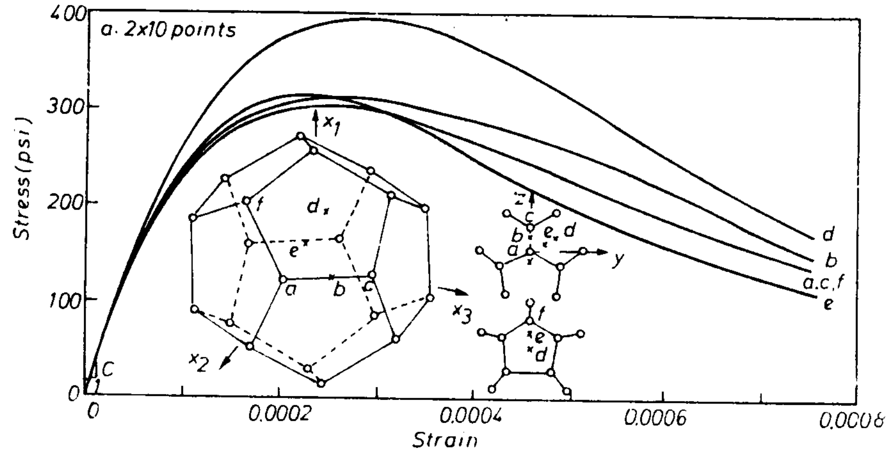
a linear stress-strain diagram from the point of reversal, with an unloading slope equal to the initial elastic slope. The macroscopic uniaxial stress is a certain integral over the microstresses from the planes of all orientations (cf. [3]). This integral is evaluated by some numerical integration formula, and the direction of each integration point of the formula is associated with a plane normal to it.

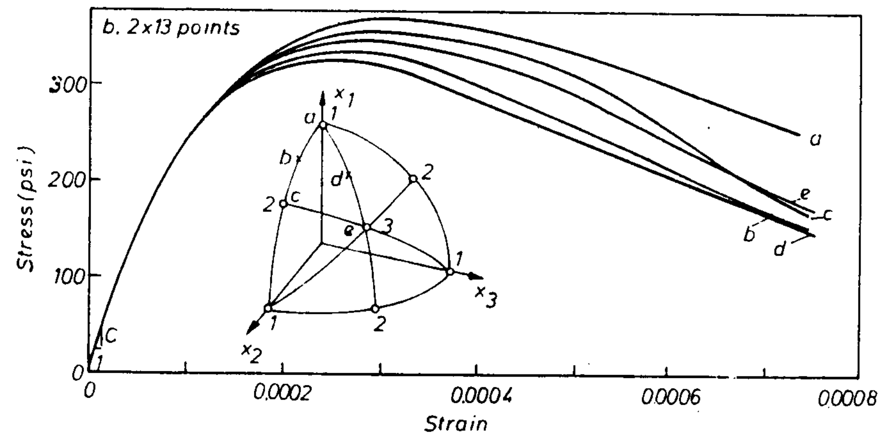
Table 5. Direction cosines and weights for 2×61 points (degree 13, no orthogonal symmetries, Fig. 1d).

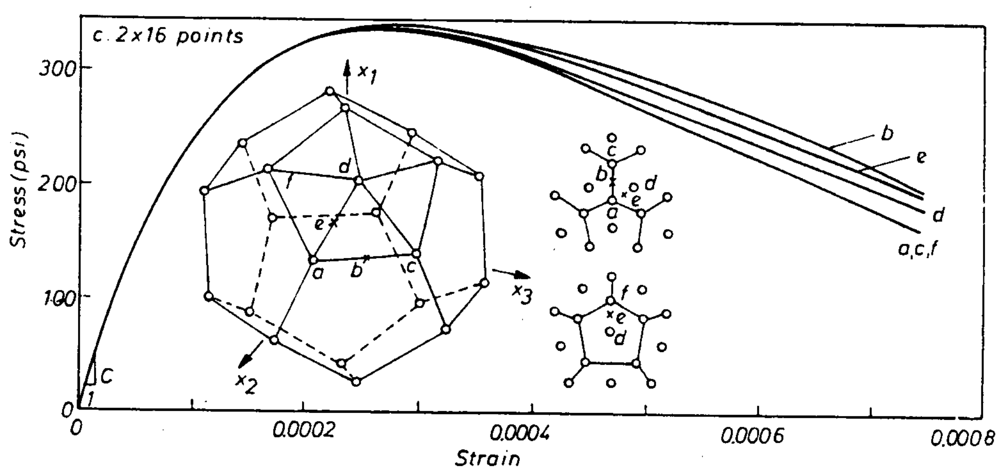
α	x_1^α	x_2^α	x_3^α	C_α
1	1	0	0	0.00795844204678
2	0.745355992500	0	0.666666666667	0.00795844204678
3	0.745355992500	−0.577350269190	−0.333333333333	0.00795844204678
4	0.745355992500	0.577350269190	−0.333333333333	0.00795844204678
5	0.333333333333	0.577350269190	0.745355992500	0.00795844204678
6	0.333333333333	−0.577350269190	0.745355992500	0.00795844204678
7	0.333333333333	−0.934172358963	0.127322003750	0.00795844204678
8	0.333333333333	−0.356822089773	−0.872677996250	0.00795844204678
9	0.333333333333	0.356822089773	−0.872677996250	0.00795844204678
10	0.333333333333	0.934172358963	0.127322003750	0.00795844204678
11	0.794654472292	−0.525731112119	0.303530999103	0.0105155242892
12	0.794654472292	0	−0.607061998207	0.0105155242892
13	0.794654472292	0.525731112119	0.303530999103	0.0105155242892
14	0.187592474085	0	0.982246946377	0.0105155242892
15	0.187592474085	−0.850650808352	−0.491123473188	0.0105155242892
16	0.187592474085	0.850650808352	−0.491123473188	0.0105155242892
17	0.934172358963	0	0.356822089773	0.0100119364272
18	0.934172358963	−0.309016994375	−0.178411044887	0.0100119364272
19	0.934172358963	0.309016994375	−0.178411044887	0.0100119364272
20	0.577350269190	0.309016994375	0.755761314076	0.0100119364272
21	0.577350269190	−0.309016994375	0.755761314076	0.0100119364272
22	0.577350269190	−0.809016994375	−0.110264089708	0.0100119364272
23	0.577350269190	−0.5	−0.645497224368	0.0100119364272
24	0.577350269190	0.5	−0.645497224368	0.0100119364272
25	0.577350269190	0.809016994375	−0.110264099708	0.0100119364272
26	0.356822089773	−0.809016994375	0.467086179481	0.0100119364272
27	0.356822089773	0	−0.934172358963	0.0100119364272
28	0.356822089773	0.809016994375	0.467086179481	0.0100119364272
29	0	0.5	0.866025403784	0.0100119364272
30	0	−1	0	0.0100119364272
31	0	0.5	−0.866025403784	0.0100119364272
32	0.947273580412	−0.277496978165	0.160212955043	0.00690477957966
33	0.812864676392	−0.277496978165	0.512100034157	0.00690477957966
34	0.595386501297	−0.582240127941	0.553634669695	0.00690477957966
35	0.595386501297	−0.770581752342	0.227417407053	0.00690477957966
36	0.812864676392	−0.582240127941	−0.015730584514	0.00690477957966
37	0.492438766306	−0.753742692223	−0.435173546254	0.00690477957966
38	0.274960591212	−0.942084316623	−0.192025554687	0.00690477957966
39	−0.076926487903	−0.942084316623	−0.326434458707	0.00690477957966
40	−0.076926487903	−0.753742692223	−0.652651721349	0.00690477957966
41	0.274960591212	−0.637341166847	−0.719856173359	0.00690477957966
42	0.947273580412	0	−0.320425910085	0.00690477957966
43	0.812864676392	−0.304743149777	−0.496369449643	0.00690477957966
44	0.595386501297	−0.188341624401	−0.781052076747	0.00690477957966
45	0.595386501297	0.188341624401	−0.781052076747	0.00690477957966
46	0.812864676392	0.304743149777	−0.496369449643	0.00690477957966
47	0.492438766306	0.753742692223	−0.435173546254	0.00690477957966
48	0.274960591212	0.637341166847	−0.719856173359	0.00690477957966
49	−0.076926487903	0.753742692223	−0.652651721349	0.00690477957966
50	−0.076926487903	0.942084316623	−0.326434458707	0.00690477957966
51	0.274960591212	0.942084316623	−0.192025554687	0.00690477957966
52	0.947273580412	0.277496978165	0.160212955043	0.00690477957966
53	0.812864676392	0.582240127941	−0.015730584514	0.00690477957966
54	0.595386501297	0.770581752342	0.227417407053	0.00690477957966
55	0.595386501297	0.582240127941	0.553634669695	0.00690477957966
56	0.812864676392	0.277496978165	0.512100034157	0.00690477957966
57	0.492438766306	0	0.870347092509	0.00690477957966
58	0.274960591212	0.304743149777	0.911881728046	0.00690477957966
59	−0.076926487903	0.188341624401	0.979086180056	0.00690477957966
60	−0.076926487903	−0.188341624401	0.979086180056	0.00690477957966
61	0.274960591212	−0.304743149777	0.911881728046	0.00690477957966

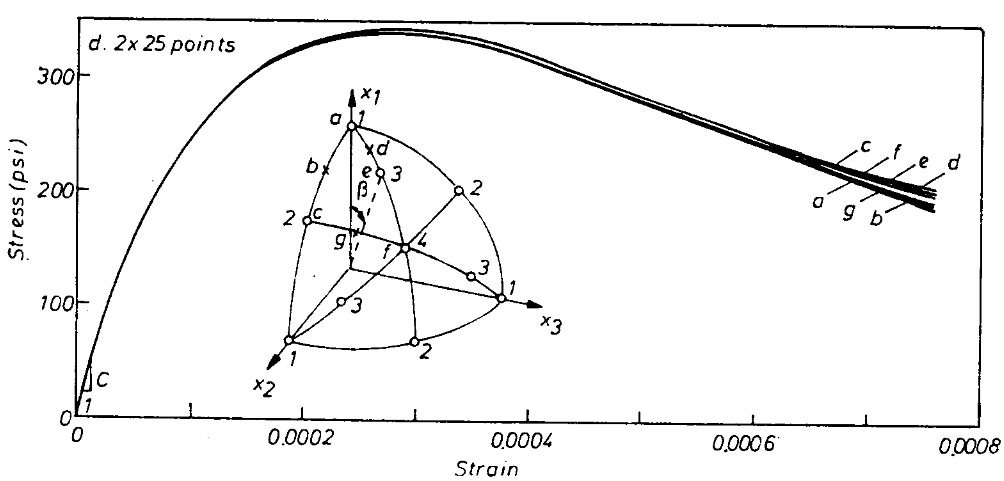
We solve the stress responses for uniaxial stresses applied at various directions (marked as a, b, c, ...) with regard to the set of integration points. For each direction of the applied uniaxial stress, the response curve (Fig. 2) is calculated in small increments of the axial normal strain. Thus, the stress state at the end of each strain increment is a function of not only the incremental stiffnesses on planes of all orientations, but also of the stress states in all previous increments.

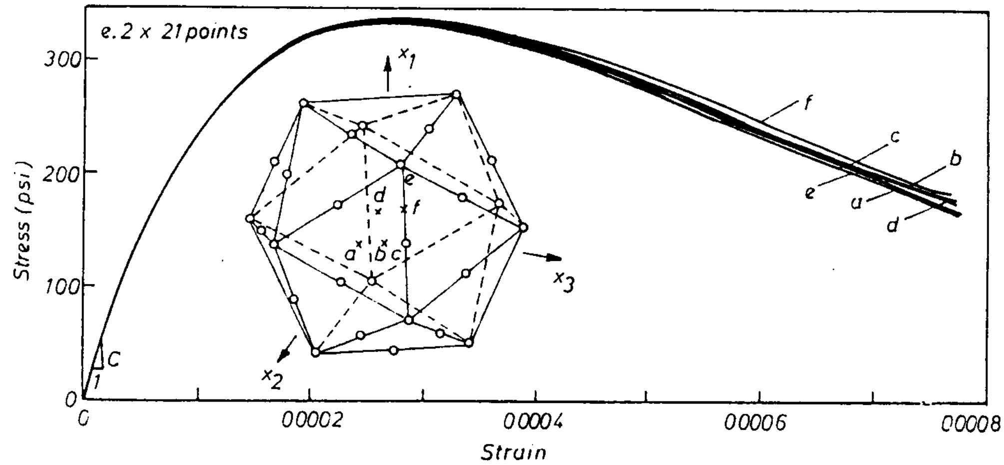
This type of problem appears to be particularly sensitive to the error of the integration formula, as we can see from the spread of the response curves in Fig. 2. For the 20-point, 26-point, and 32-point formulas, the errors indicated by the spread are unacceptable (cf. Figs. 2 a, b, c). Note that the 50-point formula and the 56-point formula perform far better, giving a much smaller spread of response curves (cf. Figs. 2 d, g) than does the 32-point formula, although all these formulas are of the same degree. The 42-point formula in Fig. 2e performs also much

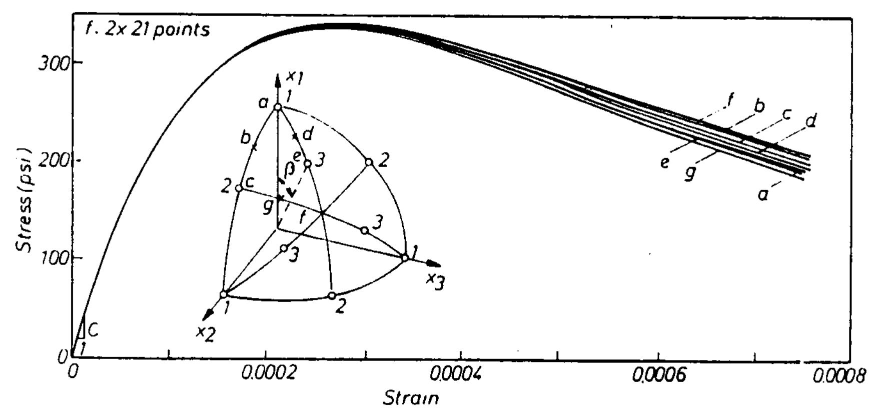

a. 2x10 points
Stress (psi)
Strain
0
100
200
300
400
0.0002
0.0004
0.0006
0.0008
d
b
a,c,f
e
C
1

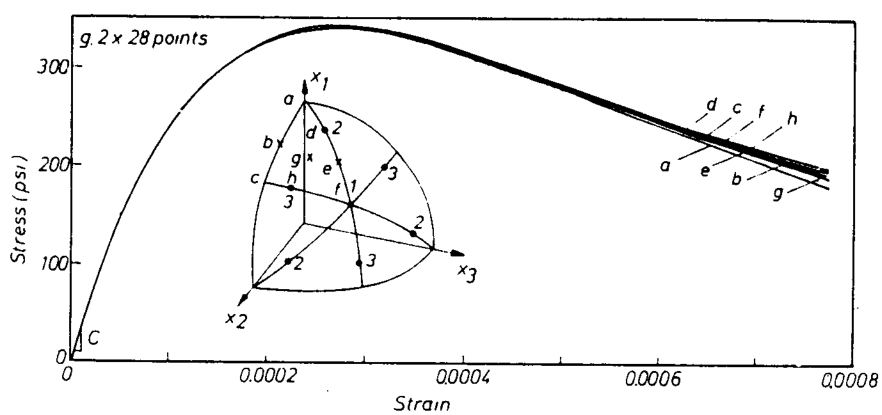

b. 2x13 points
Stress (psi)
Strain
0
100
200
300
0.0002
0.0004
0.0006
0.0008
a
e
c
b
d

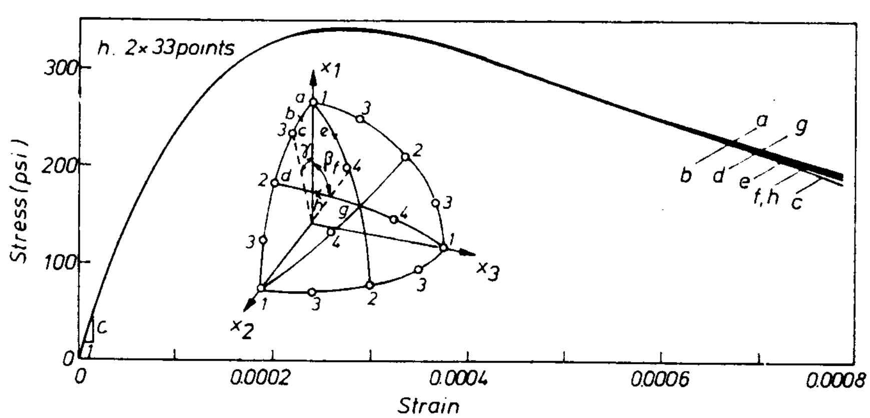

c. 2x16 points
Stress (psi)
Strain
0
100
200
300
0.0002
0.0004
0.0006
0.0008
b
e
d
a,c,f


d. 2x25 points
Stress (psi)
Strain
0
100
200
300
0.0002
0.0004
0.0006
0.0008
c
f
e
d
a
g
b


e. 2 x 21 points
Stress (psi)
Strain
0
100
200
300
0.0002
0.0004
0.0006
0.0008
a
b
c
d
e
f
x_1
x_2
x_3
C
1


f. 2x 21 points
Stress (psi)
Strain
0
100
200
300
0.0002
0.0004
0.0006
0.0008
a
b
c
d
e
f
g
x_1
x_2
x_3
β
C
1


g. 2 x 28 points
Stress (psi)
Strain
0
100
200
300
0.0002
0.0004
0.0006
0.0008
a
b
c
d
e
f
g
h
x_1
x_2
x_3
C


h. 2× 33 points
Stress (psi)
Strain
0
100
200
300
0.0002
0.0004
0.0006
0.0008
a
b
c
d
e
f,h
g
x_1
x_2
x_3
β
γ
C
1

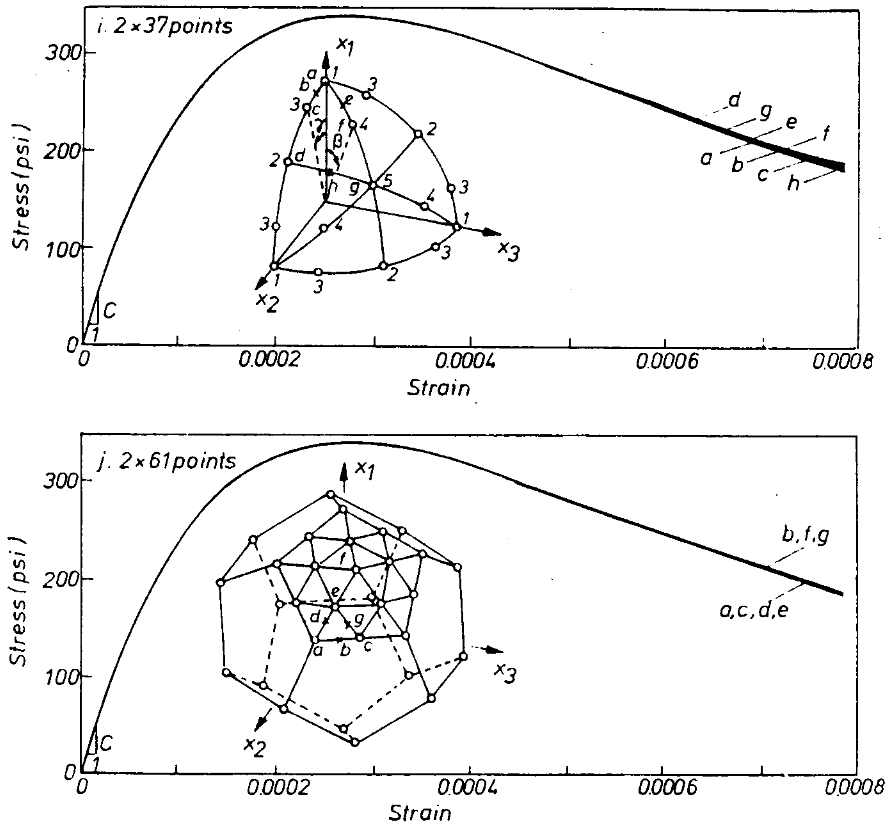


Fig. 2

better than the 32-point formula (Fig. 2b) although they are of the same degree. On the other hand, there is a surprisingly small difference in the spread of the response curves between the 66-point and 74-point formulas (cf. Figs. 2 h, i), despite the fact that they are of different degrees.

From these applications, described in full detail in Ref. 4, we must therefore conclude that optimality in the sense of Condition I (i.e., the degree of the formula attainable with a certain number of points) is an insufficient indicator of the actual error. Other criteria, e.g., such as that in Condition III, may be equally relevant.

Stroud [13, p. 301] derived also a fully symmetric 11th degree formula with 2×28 points (arranged as shown in Fig. 1j). Based on the Condition III test (Fig. 2g), this formula appears to have about the same error as McLaren's fully symmetric 11th degree formula which has fewer points (2×25). Thus, Stroud's formula is generally less efficient. However, it is more efficient for integrating functions possessing both central and plane symmetries, such as the plane stress state. In that case, Stroud's 2×28 formula (Fig. 1j) reduces to only 14 integration points, whereas McLaren's 2×25 point formula (Fig. 1e) reduces to 16 points. This is because McLaren's formula has many points on the symmetry planes (Fig. 1e), while Stroud's formula has none (Fig. 1j). The present new 2×21 point formula (Fig. 1f), which is of a lesser degree (9th) but has only a slightly larger error based on the Condition III test, reduces also to 14 points for the case of central and plane symmetries. Therefore, Stroud's 2×28 point formula (Fig. 1j) is the best known formula for this type of symmetry (plane stress state).

Note: For reader's convenience, according to ref. 13 (p. 301) McLaren's formula with $N = 2 \times 25$ is as follows. — The points in the first octant (Fig. 1e) are (1, 0, 0), (0, 1, 0), (0, 0, 1) with weights 9216/725760; $(c_1, c_1, 0)$, $(c_1, 0, c_1)$, $(0, c_1, c_1)$ with weights 16384/725760; (c_2, c_2, c_2) with weight 15309/725760; and (c_3, c_3, c_4), (c_3, c_4, c_3) (c_4, c_3, c_3) with weight 14641/725760, in which $c_1^2 = 1/2$, $c_2^2 = 1/3$, $c_3^2 = 1/11$, $c_4^2 = 9/11$. For Stroud's formula [13, p. 301] with $N = 2 \times 28$, the points in the first octant (Fig. 1j) are (c_1, c_1, c_1) with weight 9/560; (c_2, c_3, c_3), (c_3, c_2, c_3), (c_3, c_3, c_2) with weights $(122 + 9\sqrt{3})/6720$; (c_4, c_5, c_5), (c_5, c_4, c_5), (c_5, c_5, c_4) with weights $(122 - 9\sqrt{3})/6720$, in which $c_1^2 = 1/3$, $c_2^2 = (15 + 8\sqrt{3})/33$, $c_3^2 = (9 - 4\sqrt{3})/33$, $c_4^2 = (15 - 8\sqrt{3})/33$ and $c_5^2 = (9 + 4\sqrt{3})/33$. The points and weights for the other octants are symmetric.

References

1 Abramowitz, M.; Stegun, I. A. (ed.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables National Bureau of Standards, Appl. Math. Ser. 1974, p. 894.

2 Albrecht, J.; Collatz, L., Zur numerischen Auswertung mehrdimensionaler Integrale, Z. angew. Math. Mech. (ZAMM) 38 (1958) 1/3, 1—15.

3 Bažant, Z. P.; Oh, B. H., Microplane model for fracture analysis of concrete structures, Proc., Symp. on The Interaction of Non-Nuclear Munitions with Structures, held at U.S. Air Force Academy, Colo., May 1983, pp. 49—55.

4 Bažant, Z. P.; Oh, B. H., Microplane model for progressive fracture of concrete and rock, J. Engineering Mech. ASCE (Am. Soc. of Civil Engineers) 111 (1985), 559—582; see also Bažant, Z. P.; Gambarova, P. G., Crack shear in concrete: Crack band microplane model, J. Structural Engineering ASCE 110 (1984), 2015—2035.

5 COURANT, R.; HILBERT, D., Methods of Mathematical Physics. Vol. II: Partial Differential Equations, Interscience Publishers 1962, p. 287.
6 Encyclopedic Dictionary of Mathematics. Vol. II, ed. by SHOHICHI LYANAGA and YUKIYOSI KAWADA, M.I.T. Press, 1980, p. 1105.
7 FINDEN, C., Spherical Integration, Dissertation, Cambridge Univ. 1961.
8 KEATS, P.; DIAZ, J. C., Fully symmetric integration formulas for the surface of the sphere in S dimensions, SIAM J. Numerical Analysis 20. (1983) 2, 406—419.
9 LEBEDEV, V. I., Values of the base-points and weights of Gauss-Markov quadrature formulae for a sphere from the 9th to the 17th order of accuracy, invariant under the octahedron group with inversion, Zh. Vychisl. Mat. i. Mat. Fiz. 16 (1976), 293—306.
10 LEBEDEV, V. I., Quadratures on a sphere (in Russian), Zh. Vychisl. Mat. i Mat. Fiz. 16 (1976), 293—306 [English transl. in USSR Computational Math. & Math. Phys. 16 (1976) 2, 10—24].
11 MCLAREN, A. D., Optimal numerical integration on a sphere, Math. Comput. 17 (1963), 361—383.
12 SOBOLEV, S. L., On the number of nodes of cubature formulas on a sphere (in Russian), Doklady Akad. Nauk SSSR 146 (1962), 770—773.
13 STROUD, A. H., Approximate Calculation of Multiple Integrals, Prentice Hall, Englewood Cliffs, New Jersey, 1971, pp 296—302.

Received November 12, 1984, in original form January 3, 1984

Addresses: Dr. ZDENĚK P. BAŽANT, Professor of Civil Engineering and Director, Center for Concrete and Geomaterials, The Technological Institute, Northwestern University, Evanston, Illinois 60201, U.S.A.; Dr. BYUNG H. OH, Visiting Scholar Center for Concrete and Geomaterials, Northwestern University, Evanston, Illinois 60201, U.S.A. Presently Asst. Prof. of Civil Engineering, Seoul National University, Seoul, Korea.