Response of Aging Linear Systems to Ergodic Random Input

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Abstract: Response of an aging linear system exposed from a certain age t_0 to ergodic random input is analyzed. It is shown that the response, while nonstationary with respect to time and age, is stationary and ergodic with regard to the birth time τ (the time when the system was built). Consequently, the instantaneous statistical characteristics of all possible response realizations at a chosen age, t, may be determined as the characteristics of the response at age t (and at a fixed exposure age, t_0) as the birth time τ is varied, i.e., as the input history is shifted in time against the instant when the system was built. Based on this new idea, the spectral method is generalized for aging systems, using a frequency response function and a spectral density of response that depend on both the current age t and the age t_0 when the exposure begins. The relation between the spectral densities of input and response is algebraic, similar to the case of stationary response of nonaging systems. For the special case of nonstationary response of nonaging systems, the proposed new method is simpler than the existing methods. The new method can be applied, e.g., to shrinkage stresses in an aging linearly viscoelastic structure (a concrete structure) exposed to relative humidity fluctuations of weather. A simple illustrative example is solved in a closed form. Another possible application is earthquake motion of a structure undergoing progressive damage, provided the problem is approximated as linear.

Introduction

When the properties of a linear system depend on its age, then a stationary random input produces a nonstationary random response. A practically important example is the shrinkage stress produced in a concrete structure by random humidity fluctuations of weather. Although it is acceptable to treat a concrete structure as linearly viscoelastic, the age dependence of concrete properties must be taken into account. Another problem that can be approximately treated as an aging linear system is the dynamic motion of a structure subjected to earthquake and undergoing progressive damage (i.e., decay of stiffness and damping), provided the damage is simplified as a function of time rather than of the energy dissipated during hysteretic cycles.

For stationary response of a nonaging system, the statistics of response at a certain point may be obtained as the statistics of the time history of response at that point. However, this approach is impossible for an aging structure. In the present study, based on a 1982 report (4) that was summarized in recent conference proceedings (5), the physical meaning of random variation in the presence of aging will be defined, an effective new mathematical approach to the nonstationarity of response will be formulated, and the use of power response spectra (9,12,13,14,17,18,25) will be generalized for this purpose, assuming the input to be ergodic. Although certain practical applications of the present new method have already been published (6,7,8,22), a simple illustrative example will be presented. Finally, similarities to and differences from previous studies of nonstationary response (5,9–11,15,16,21,23,25) will be pointed out.

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Statistical Concept and Ergodicity

Let θ denote time (the actual time), measured, e.g., from the foundation of Rome, and τ denote the birth time, i.e., the instant when the system was built (e.g., when the concrete was cast). Further, let $t = \theta - \tau$ = age of the system (concrete); θ_0 = the exposure time, i.e., the time at which the exposure to the random input $f(\theta)$ began, e.g., the time when the concrete structure was exposed to weather or to loading; and $t_0 = \theta_0 - \tau$ = the exposure age = the age of the system (concrete) at time θ_0 (see Fig. 1); always $t \geq 0$, $t_0 \geq 0$, $\theta_0 \geq \tau$.

The properties of a linear aging system may be completely characterized by its impulse response function $\phi(\mathbf{x}, t, t_0, \xi)$ where $\xi = \theta - \theta' = t$

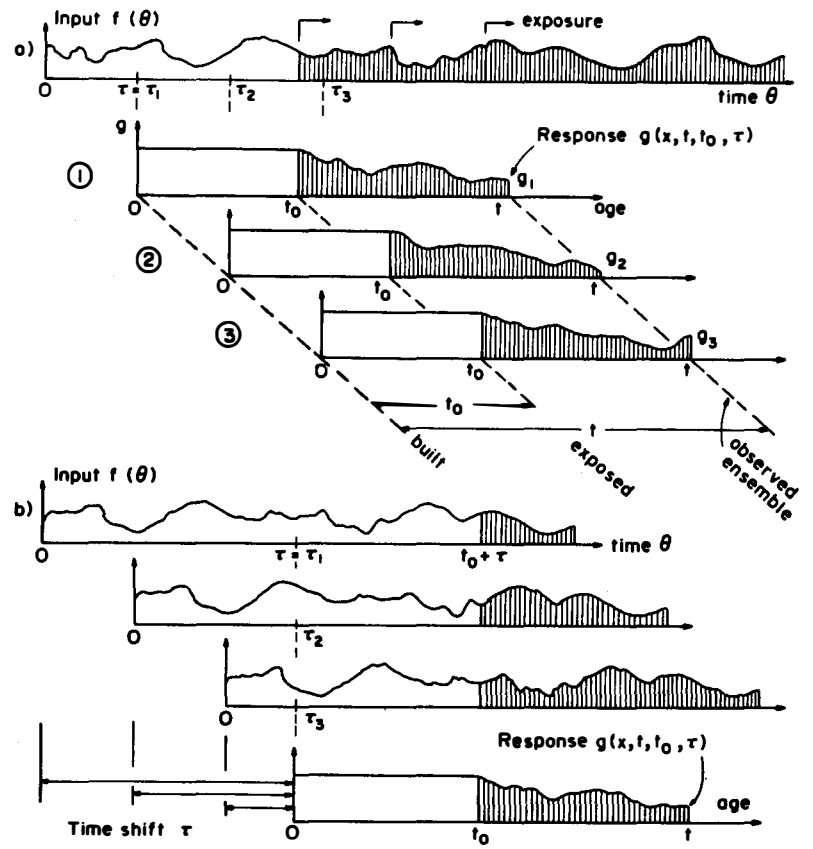

FIG. 1.—(*a*) Example of Response Realization; (*b*) Shift of Birth Time, τ

$- t'$ = time lag of current time $\theta = \tau + t$ behind the time, θ' (or age t'), at which a unit impulse $\delta(\theta - \theta')$ (Dirac delta function) was applied, and $\mathbf{x}$ = location coordinate vector of response variable $g(\mathbf{x}, t, t_0, \tau)$, representing, e.g., the stress or deflection at point $\mathbf{x}$ and at age t for a system built at time τ and exposed at age t_0. According to the principle of superposition, the response to an arbitrary integrable input history $f(\theta)$ may then be expressed as

$$g(\mathbf{x}, t, t_0, \tau) = \int_0^{t-t_0} \phi(\mathbf{x}, t, t_0, \xi) f(\tau + t - \xi) d\xi \quad (1)$$

Eq. 1 describes, in general, only the response that is caused by the random input $f(\theta)$. This means that our problem must be formulated in such a manner that $g(\ldots) = 0$ when $f(\theta) = 0$. In general, of course, a nonzero response may be also caused by the initial condition at $t = t_0$ or by an additional deterministic load. However, exploiting the principle of superposition, we can always reduce the problem to the form described by Eq. 1 (see Eqs. 77–86). A nonzero deterministic response due to deterministic nonzero initial conditions and to possible additional deterministic loads may be solved separately and then superimposed; it does not affect the statistics of the response deviations from the mean.

Note that the restriction that $g(\ldots) = 0$ when $f(\theta) = 0$ does not imply the response to be stationary in absence of aging. Initial nonstationarity of the response, caused by the initial load application, is covered by our formulation (see Eqs. 77–86 and Fig. 2).

The input $f(\theta)$, e.g., the environmental humidity, is assumed to be a stationary random process. There are two possible practically meaningful concepts for defining the statistics of the response:

1. Ensemble Averaging (or Instantaneous Averaging).—We may imagine an ensemble of a great number, N, of identical systems (structures), $n = 1, 2, \ldots, N$, exposed to different possible realizations $f_n(\theta)$ of the random input $f(\theta)$ with the given statistical properties, while the age t_0 at the start of exposure is the same for all these structures [Fig. 1(*a*)]. We calculate the response value g_n (e.g., the stress at point $\mathbf{x}$ and age t) for each of the systems $n = 1, 2, \ldots, N$. Then we consider the ensemble of all the response values $g_1, g_2, \ldots, g_N$ at age t [Fig. 1(*a*)]. The

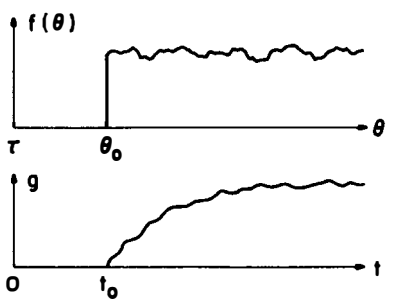

FIG. 2.—Nonstationarity of Response Due to Initial Condition $g = 0$ at $t = t_0$

statistical characteristics of the response at age t are then those of this ensemble of values. Obviously, different ensembles $(g_1, g_2, \ldots, g_N)$ are obtained for different ages t and exposures t_0 (as well as for different $\mathbf{x}$), and so the statistical characteristics will depend on t and t_0 (as well as $\mathbf{x}$).

2. Birth-Time Averaging.—Alternatively, we may consider only one structure, subjected to one possible realization $f(\theta)$ of the random input. In many practical design problems, the time τ at which the structure is built (the birth time) is not known in advance, and the design must be acceptable for many possible values of τ. For example, in the design of a nuclear concrete containment, the engineer may plan concrete to be poured on May 25, 1988, but cannot rule out that the permit to pour concrete might not be issued until January 4, 1992 or December 10, 2037. It is nevertheless wise for him to make sure that the design is acceptable for any possible time τ. So it is reasonable to choose some realization of input $f(\theta)$, and calculate for it the response values $g(\mathbf{x}, t, t_0, \tau)$ at desired values $\mathbf{x}$, t, t_0 for all possible values of τ, and then (for each $\mathbf{x}$, t, t_0) take the statistics of all these response values for all τ (for the same $\mathbf{x}$, t, and t_0). So, in this approach averaging of response needs to be carried out over the birth time τ rather than over an ensemble of all possible realizations. The variation of τ may be imagined as shifting of the input history $f(\theta)$ against the instant when the system is built [Fig. 1(*b*)].

Clearly, each of these two statistical concepts has practical meaning of its own, justified independently of the other. Thus, neither of these two concepts can be declared to be more fundamental than the other. However, the ensemble averaging (instantaneous averaging) concept is much more tedious to implement
require the engineer to carry out the structural analysis N-times, compared to a single structural analysis for the time-averaging concept. Fortunately, however, great simplification is possible if the input process $f(\theta)$ may be assumed to be ergodic, which is sufficient for nearly all practical applications (the exceptions, see p. 88 in Ref. 9, seem to be quite artificial).

A stationary random process, e.g., $f(\theta)$, is said to be ergodic if the statistical characteristics of all possible realizations $f_n(\theta)$ at time θ are the same as the statistical characteristics of the time history $f(\theta)$ for any one realization. We will consider here only weak ergodicity, in which case only the first- and second-moment statistical characteristics are assumed to be the same, i.e.

$$E_\theta f_n(\theta) = E_n f_n(\theta) \quad (2)$$

$$E_\theta[f_n(\theta) f_n(\theta + \lambda)] = E_n[f_n(\theta) f_n(\theta + \lambda)] \quad (3)$$

in which $$E_\theta f_n(\theta) = \bar{f}_n(\theta) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_n(\theta) d\theta \quad (4)$$

$$E_n f_n(\theta) = \lim_{N\to\infty} \frac{1}{N} {}_{n=1}^{N}$$

$$E_\theta[f_n(\theta) f_n(\theta + \lambda)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_n(\theta) f_n(\theta + \lambda) d\theta \quad (6)$$

$$E_n[f_n(\theta)f_n(\theta+\lambda)] = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_n(\theta)f_n(\theta+\lambda) = R_f(\lambda) \quad (7)$$

E denotes the operator of expectation and the subscript of E indicates the variable over which the averaging is carried out. Thus, E_θ or E_n indicates averaging over time θ (at fixed n) or over the ensemble $n = 1, 2, \ldots, N$ (at fixed θ), respectively. Eq. 7 defines the autocorrelation function of $f_n(\theta)$, which is denoted as $R_f(\lambda)$ and is obtained by ensemble averaging, and Eq. 6 defines a similar function obtained by birth-time averaging. The following theorem may now be proposed:

If the input random process is ergodic, then the response as a function of the birth time is also ergodic.

Proof.—The theorem means that the ensemble averaging of response is equivalent to the time averaging of response over the birth time τ at fixed current age t and fixed age t_0 at the start of exposure. So we need to prove that

$$E_\tau g_n(\mathbf{x},t,t_0,\tau) = E_n g_n(\mathbf{x},t,t_0,\tau) \quad (8)$$

and $E_\tau[g_n(\mathbf{x},t,t_0,\tau)g_n(\mathbf{x},t,t_0,\tau+\lambda)]$

$$= E_n[g_n(\mathbf{x},t,t_0,\tau)g_n(\mathbf{x},t,t_0,\tau+\lambda)] \quad (9)$$

in which the left-hand side of Eq. 9 defines for an aging system the autocorrelation function of response $R_g(\mathbf{x},t,t_0,\lambda)$. On the left-hand sides of these equations n is an arbitrary constant integer while the averaging is carried out over the birth times τ, and on the right-hand sides τ is an arbitrary constant while the averaging is carried out over the realizations $n = 1, 2, \ldots, N$. By substituting Eq. 1 for g_n and interchanging the order of the integrals over ξ and τ and of the summation over n, Eq. 8 may be proven as follows:

$$E_\tau g_n(\mathbf{x},t,t_0,\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g_n(\mathbf{x},t,t_0,\tau)d\tau \quad (10a)$$

$$E_\tau g_n(\mathbf{x},t,t_0,\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\int_0^{t-t_0}\phi(\mathbf{x},t,t_0,\xi)f_n(\tau+t-\xi)d\xi\right]d\tau \quad (10b)$$

$$E_\tau g_n(\mathbf{x},t,t_0,\tau) = \int_0^{t-t_0}\phi(\mathbf{x},t,t_0,\xi)\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_n(\tau+t-\xi)d\tau\right]d\xi \quad (10c)$$

$$E_\tau g_n(\mathbf{x},t,t_0,\tau) = \int_0^{t-t_0}\phi(\mathbf{x},t,t_0,\xi)\left[\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_n(\tau+t-\xi)\right]d\xi \quad (10d)$$

$$E_\tau g_n(\mathbf{x},t,t_0,\tau) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_0^{t-t_0}\phi(\mathbf{x},t,t_0,\xi)f_n(\tau+t-\xi)d\xi \quad (10e)$$

$$E_\tau g_n(\mathbf{x},t,t_0,\tau) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} g_n(\mathbf{x},t,t_0,\tau) = E_n g_n(\mathbf{x},t,t_0,\tau) \quad (10f)$$

Similarly, Eq. 9 may be proven as follows:

$$E_\tau[g_n(\mathbf{x},t,t_0,\tau)g_n(\mathbf{x},t,t_0,\tau+\lambda)]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g_n(\mathbf{x},t,t_0,\tau)g_n(\mathbf{x},t,t_0,\tau+\lambda)d\tau \quad (11a)$$

$$E_\tau[g_n(\mathbf{x},t,t_0,\tau)g_n(\mathbf{x},t,t_0,\tau+\lambda)]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\int_0^{t-t_0}\phi(\mathbf{x},t,t_0,\xi)f_n(\tau+t-\xi)d\xi\right]$$

$$\left[\int_0^{t-t_0}\phi(\mathbf{x},t,t_0,\eta)f_n(\lambda+\tau+t-\eta)d\eta\right]d\tau \quad (11b)$$

$$E_\tau[g_n(\mathbf{x},t,t_0,\tau)g_n(\mathbf{x},t,t_0,\tau+\lambda)] = \int_{\xi=0}^{t-t_0}\int_{\eta=0}^{t-t_0}\phi(\mathbf{x},t,t_0,\xi)\phi(\mathbf{x},t,t_0,\eta)$$

$$\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_n(\tau+t-\xi)f_n(\lambda+\tau+t-\eta)d\tau\right]d\xi d\eta \quad (11c)$$

$$E_\tau[g_n(\mathbf{x},t,t_0,\tau)g_n(\mathbf{x},t,t_0,\tau+\lambda)] = \int_{\xi=0}^{t-t_0}\int_{\eta=0}^{t-t_0}\phi(\mathbf{x},t,t_0,\xi)\phi(\mathbf{x},t,t_0,\eta)$$

$$\left[\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_n(\tau+t-\xi)f_n(\lambda+\tau+t-\eta)\right]d\xi d\eta \quad (11d)$$

$$E_\tau[g_n(\mathbf{x},t,t_0,\tau)g_n(\mathbf{x},t,t_0,\tau+\lambda)]$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left[\int_0^{t-t_0}\phi(\mathbf{x},t,t_0,\xi)f_n(\tau+t-\xi)d\xi\right]$$

$$\left[\int_0^{t-t_0}\phi(\mathbf{x},t,t_0,\eta)f_n(\lambda+t+\tau-\eta)d\eta\right] \quad (11e)$$

$$E_\tau[g_n(\mathbf{x},t,t_0,\tau)g_n(\mathbf{x},t,t_0,\tau+\lambda)]$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} g_n(\mathbf{x},t,t_0,\tau)\,g_n(\mathbf{x},t,t_0,\tau+\lambda) \quad (11f)$$

$$E_\tau[g_n(\mathbf{x},t,t_0\ \tau)g_n(\mathbf{x},t,t_0,\tau+\lambda)] = E_n[g_n(\mathbf{x},t,t_0,\tau)g_0(\mathbf{x},t,t_0,\tau+\lambda)] \quad (11g)$$

It should be noted that, in contrast to stationary response of nonaging systems, averaging over the actual time, θ, or over age t at fixed τ, is not equivalent to averaging over the ensemble of all realizations, i.e., in general

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g_n(\mathbf{x},t,t_0,\tau)dt \neq \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} g(\mathbf{x},t,t_0,\tau) \quad (12)$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g_n(\mathbf{x},t,t_0,\tau)g_n(\mathbf{x},t+\lambda,t_0,\tau)dt$$

$$\neq \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} g_n(\mathbf{x},t,t_0,\tau)g_n(\mathbf{x},t+\lambda,t_0,\tau) \quad (13)$$

Such averaging would be of no practical use for aging systems.

In earthquake design, averaging over time or over the ensemble of realizations is often assumed to be equivalent to averaging over various locations. Such interpretations of ergodicity, however, may introduce serious errors. For example, the earthquake spectra at different locations are often assumed to be the same, whereas in reality they may be quite different.

Spectral Description

It is useful to allow the input and output processes to be complex-valued, in which case the reality is described by either the real part or the imaginary part. The properties of the system may be conveniently characterized by determining the response $g(\mathbf{x}, t, t_0, 0)$ to a periodic input $f(\theta) = e^{i\omega t}$ where ω = circular frequency (a real number) and i = imaginary unit. This is accomplished by solving the linear differential or integral equations that describe the linear system. The function $\Phi(\mathbf{x}, t, t_0, \omega) = g(\mathbf{x}, t, t_0, 0)e^{-i\omega t}$ is then called the frequency response function. This function completely characterizes the properties of the linear system. The responses to periodic inputs may then be expressed as follows:

For $f(\theta) = e^{i\omega t}$: $g(\mathbf{x}, t, t_0, \tau) = \Phi(\mathbf{x}, t, t_0, \omega)e^{i\omega t}$ ..... (14)

For $f(\theta) = e^{i\omega\theta} = e^{i\omega t}e^{i\omega\tau}$:

$$g(\mathbf{x}, t, t_0, \tau) = \Phi(\mathbf{x}, t, t_0, \omega)e^{i\omega t}e^{i\omega\tau} = \Phi(\mathbf{x}, t, t_0, \omega)e^{i\omega\theta} \quad \text{..... (15)}$$

Note that at t_0, the beginning of exposure, there is a sudden jump of input from 0 to the value $e^{i\omega\theta_0}$. The nonstationary response to this initial jump is included in the function $\Phi(\ldots)$ as defined above. Since either the real part or the imaginary part of $e^{i\omega\theta_0}$ is always nonzero, it is impossible for the complex-valued periodic input to start with a zero initial value, i.e., without a discontinuous jump.

A general random input history $f(\theta)$ may be characterized by the input spectrum (spectral decomposition of the input), which is defined as the Fourier transform of $f(\theta)$, i.e.

$$F(\omega) = \int_{-\infty}^{\infty} f(\theta)e^{-i\omega\theta}d\theta = e^{-i\omega t}\int_{-\infty}^{\infty} f(\theta)e^{-i\omega\tau}d\tau \quad \text{..... (16)}$$

The inverse relation (inverse Fourier transform) is

$$f(\theta) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega\theta}d\omega \quad \text{..... (17)}$$

Note that factor 2π is placed in Eq. 17 while in random vibration theory it is often placed in Eq. 16. The Fourier transform and its inverse exist if $f(\theta)$ is absolutely integrable (i.e., the integral of $|f(\theta)|$ from $-\infty$ to ∞ is finite) and function $f(\theta)$ is piece-wise smooth (24).

Due to linearity of the system, both the input and output in Eq. 15 may be multiplied by the same constant, say $F(\Omega)$ (with Ω as a real parameter). Furthermore, these inputs and outputs may be summed or integrated over parameter Ω. These operations yield the following input-response pairs:

For $f(\theta) = F(\Omega)e^{i\Omega\theta}$: $g(\mathbf{x}, t, t_0, \tau) = F(\Omega)\Phi(\mathbf{x}, t, t_0, \Omega)e^{i\Omega\theta}$ ..... (18)

For $f(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\Omega)e^{i\Omega\theta}d\Omega$:

$$g(\mathbf{x}, t, t_0, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\Omega)\Phi(\mathbf{x}, t, t_0, \Omega)e^{i\Omega\theta}d\Omega \quad \text{..... (19)}$$

It should be noted that our transition from Eq. 15 to Eq. 18 requires the problem to be formulated in such a manner that $g(\ldots) = 0$ when $f(\theta) = 0$, as we have stipulated below Eq. 1.

Let us now calculate the Fourier transform of the general response (Eq. 19), defined as

$$G(\mathbf{x}, t, t_0, \omega) = \int_{-\infty}^{\infty} g(\mathbf{x}, t, t_0, \tau)e^{-i\omega\theta}d\theta$$

$$= -e^{i\omega t}\int_{-\infty}^{\infty} g(\mathbf{x}, t, t_0, \tau)e^{-i\omega\tau}d\tau \quad \text{..... (20)}$$

Substituting Eq. 19, we have

$$G(\mathbf{x}, t, t_0, \omega) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\Omega)\Phi(\mathbf{x}, t, t_0, \Omega)e^{i\Omega\theta}d\Omega\right]e^{-i\omega\theta}d\theta$$

$$= \int_{-\infty}^{\infty} F(\Omega)\Phi(\mathbf{x}, t, t_0, \Omega)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\Omega-\omega)\theta}d\theta d\Omega \quad \text{..... (21)}$$

Noting that $\frac{1}{\pi}\int_{-\infty}^{\infty} e^{i(\Omega-\omega)\theta}d\theta = \delta(\Omega - \omega)$ ..... (22)

in which δ denotes Dirac delta function (see Ref. 24, p. 294), and evaluating the integral over Ω in Eq. 21, we thus obtain the result

$$G(\mathbf{x}, t, t_0, \omega) = F(\omega)\Phi(\mathbf{x}, t, t_0, \omega) \quad \text{..... (23)}$$

We see that the spectra of input and response are related *algebraically*. This is the main advantage of the spectral approach. The response may then be obtained as the inverse Fourier transform of $G(\mathbf{x}, t, t_0, \omega)$, i.e.

$$g(\mathbf{x}, t, t_0, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\mathbf{x}, t, t_0, \omega)e^{i\omega(t+\tau)}d\omega \quad \text{..... (24)}$$

Substituting Eqs. 23 and 16 into Eq. 24, we obtain

$$g(\mathbf{x}, t, t_0, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\mathbf{x}, t, t_0, \omega)e^{i\omega(t+\tau)}\int_{-\infty}^{\infty} f(\eta)e^{-i\omega\eta}d\eta d\omega \quad \text{..... (25)}$$

The second integral is $e^{-i\omega\zeta}$ if $f(\eta)$ is a unit impulse at time ζ, i.e., $f(\eta) = \delta(\eta - \zeta)$ where δ denotes Dirac delta function. We may denote as $\xi = \theta - \zeta$ the time lag after the impulse, and the response g to the impulse $f(\eta)$ is then the impulse response function $\phi(\mathbf{x}, t, t_0, \xi)$. Thus, noting that $e^{i\omega(t+\tau)}e^{-i\omega\zeta} = e^{i\omega\theta}e^{-i\omega\zeta} = e^{i\omega\xi}$, we find Eq. 25 to yield

$$\phi(\mathbf{x},t,t_0,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\mathbf{x},t,t_0,\omega)e^{i\omega\xi}d\omega \quad (26)$$

This means that the impulse response function is the inverse Fourier transform of the frequency response function, same as for stationary response of nonaging systems. Conversely, the frequency response function is the Fourier transform of the impulse response function:

$$\Phi(\mathbf{x},t,t_0,\omega) = \int_{-\infty}^{\infty} \phi(\mathbf{x},t,t_0,\xi)e^{-i\omega\xi}d\xi \quad (27)$$

in which we extend the definition of the impulse response function as follows:

$$\phi(\mathbf{x},t,t_0,\xi) = 0 \quad \text{for} \quad \xi < 0 \quad \text{and for} \quad \xi > t - t_0 \quad (28)$$

This expresses the fact that the future ($\xi < 0$) can have no effect on the present, and that the system cannot feel any input prior to the age t_0 at which exposure begins. It is also assumed that function ϕ is such that its Fourier transform exists, which is true for viscoelastic structures.

Note that the response to input $e^{i\omega t}$ could be, alternatively, also written as $e^{i\omega(t-t_0)}\,\Psi(\mathbf{x},t-t_0,t_0,\omega)$ where Ψ could be regarded as the frequency response function. This would be, however, strictly a matter of choice and would be physically equivalent because $\Psi(\mathbf{x},t,t_0,\omega) = e^{-i\omega t_0}\,\Psi(\mathbf{x},t - t_0,t_0,\omega)$, i.e., function Ψ differs from function Φ only by a constant multiplier $e^{-i\omega t_0}$, same for all times and all locations. As will be seen later (Eqs. 41 and 47), the response statistics depend only on $|\Phi|$ or $|\Psi|$, and not on their real and imaginary parts separately. The uses of Φ and Ψ are equivalent because $|\Phi| = |\Psi|$, due to the fact that $|e^{-i\omega t_0}| = 1$. All the present mathematical formulation could be easily converted to this alternate description by replacing everywhere $e^{i\omega t}$ with $e^{i\omega(t-t_0)}$ and $G(\omega)$ with $G'(\ldots) = F(\omega)\Psi(\ldots)$. Our use of $\Phi(\mathbf{x},t,t_0,\omega)$ conforms with the established practice of writing the compliance function of aging viscoelastic material as $J(t,t_0)$. The alternative notation $J(t,t-t_0)$ was used in some older papers but has later been abandoned.

Spectral Density of Response

The second-moment statistical properties of a stationary random process, such as $f(\theta)$, may be characterized by its autocorrelation function, which is defined as

$$R_f(\lambda) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\theta)f(\theta+\lambda)d\theta = E_\theta[f(\theta)f(\theta+\lambda)] \quad (29)$$

and has already appeared in Eq. 3. As is well-known (12,17), the spectral density of a stationary random process, i.e., of input $f(\theta)$, then is

$$S_f(\omega) = \int_{-\infty}^{\infty} R_f(\lambda)e^{-i\omega\lambda}d\lambda \quad (30)$$

By inversion

$$R_f(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)e^{i\omega\lambda}d\omega \quad (31)$$

Consider now the autocorrelation function of response at fixed age t (and at fixed t_0 and $\mathbf{x}$) as τ is varied, i.e., as the input history is shifted against the instant the system is built. The definition of this function, which already appeared in Eq. 9, is

$$R_g(\mathbf{x},t,t_0,\lambda) = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T} g(\mathbf{x},t,t_0,\tau)g(\mathbf{x},t,t_0,\tau+\lambda)d\tau$$

$$= E[g(\mathbf{x},t,t_0,\tau)g(\mathbf{x},t,t_0,\tau+\lambda)] \quad (32)$$

Now, according to the Wiener-Khintchine relation (12) the spectral density of the response is given by the Fourier transform

$$S_g(\mathbf{x},t,t_0,\omega) = \int_{-\infty}^{\infty} R_g(\mathbf{x},t,t_0,\lambda)e^{-i\omega\lambda}d\lambda \quad (33)$$

which has the inverse

$$R_g(\mathbf{x},t,t_0,\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_g(\mathbf{x},t,t_0,\omega)e^{i\omega\lambda}d\omega \quad (34)$$

Eq. 33 is crucial. It is well-known (12,13) that the Wiener-Khintchine relation does not apply to nonstationary processes. Our response is nonstationary as a function of age t. The device that allows us to apply the Wiener-Khintchine relation is to freeze the age and consider how the response at fixed t depends on the shift τ of the stationary input against the instant the system is built. As a function of τ, the response must be stationary.

Using the extended definition of the frequency response function according to Eq. 28, Eq. 1 describing the response on the basis of the principle of superposition may be rewritten as

$$g(\mathbf{x},t,t_0,\tau) = \int_{-\infty}^{\infty} \phi(\mathbf{x},t,t_0,\xi)f(\tau+t-\xi)d\xi \quad (35)$$

Consequently,

$$g(\mathbf{x},t,t_0,\tau)g(\mathbf{x},t,t_0,\tau+\lambda)$$

$$= \int_{-\infty}^{\infty}\phi(\mathbf{x},t,t_0,\xi)f(\tau+t-\xi)d\xi\int_{-\infty}^{\infty}\phi(\mathbf{x},t,t_0,\eta)f(\tau+\lambda+t-\eta)d\eta \quad (36)$$

Assuming now each of these integrals to be convergent, we have

$$E_\tau[g(\mathbf{x},t,t_0,\tau)\,g(\mathbf{x},t,t_0,\tau+\lambda)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_\tau[f(\tau+t-\xi)f(\tau+\lambda+t-\eta)]\phi(\mathbf{x},t,t_0,\xi)\phi(\mathbf{x},t,t_0,\eta)d\xi d\eta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_\kappa[f(\kappa)f(\kappa+\xi-\eta+\lambda)]\phi(\mathbf{x},t,t_0,\xi)\phi(\mathbf{x},t,t_0,\eta)d\xi d\eta \quad (37)$$

in which $\kappa = \tau + t - \xi$. In Eq. 37 we recognize the autocorrelation functions of the input and the response, and so we have

$$R_g(\mathbf{x}, t, t_0, \lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_f(\xi - \eta + \lambda)\phi(\mathbf{x}, t, t_0, \xi)\phi(\mathbf{x}, t, t_0, \eta)\, d\xi\, d\eta \quad \ldots\ldots (38)$$

The spectral density of response may now be calculated, according to Eq. 33, as follows:

$$S_g(\mathbf{x}, t, t_0, \omega) = \int_{-\infty}^{\infty} R_g(\mathbf{x}, t, t_0, \lambda) e^{-i\omega\lambda} d\lambda$$

$$S_g(\mathbf{x}, t, t_0, \omega)$$

$$= \int_{-\infty}^{\infty} e^{-i\omega\lambda} d\lambda$$

$$S_g(\mathbf{x}, t, t_0, \omega) = \int_{-\infty}^{\infty} \phi(\mathbf{x}, t, t_0, \eta) e^{-i\omega\eta} d\eta$$

$$\int_{-\infty}^{\infty} R_f(\xi - \eta + \lambda) e^{-i}$$

The last integral is to be evaluated before the preceding ones, and so the integration over λ in the last integral is made at constant ξ and η. Setting $\xi - \eta + \lambda = \zeta$ and $d\lambda = d\zeta$, we recognize the last integral to be $\int_{-\infty}^{\infty} R_f(\zeta) e^{-i\omega\zeta} d\zeta$,
Eq. 27, the first integral in Eq. 39 is $\Phi(\mathbf{x}, t, t_0, \omega)$, and the second one is $\Phi(\mathbf{x}, t, t_0, -\omega)$. So we have

$$S_g(\mathbf{x}, t, t_0, \omega) = \Phi(\mathbf{x}, t, t_0, \omega)\, \Phi(\mathbf{x}, t, t_0, -\omega)\, S_f(\omega) \quad \ldots\ldots (40)$$

Furthermore, if $\Phi(\ldots\ \omega) = \Phi_1 + i\Phi_2$, in which Φ_1 and Φ_2 are the real and imaginary parts of Φ, then $\Phi(\ldots\ -\omega) = \Phi_1 - i\Phi_2$, i.e., $\Phi(\ldots\ -\omega) = \Phi^*(\ldots\ \omega)$ = complex conjugate of $\Phi(\ldots\ \omega)$. Also, $\Phi(\ldots\ \omega)\, \Phi(\ldots\ -\omega) = \Phi_1^2 + \Phi_2^2 = |$

$$S_g(\mathbf{x}, t, t_0, \omega) = |\Phi$$

This is the same as the classical result for stationary response of nonaging systems (12) except for the presence of arguments t and t_0. In fact, the entire procedure of derivation is a generalization of that used for nonaging response (12).

It is important that the relation between the spectral densities of input and response of an aging system is algebraic, whereas the relation between the autocorrelation functions of input and response (Eq. 38) is given by a double integral. Thus, we see that this principal advantage of the spectral approach is retained in the case of aging systems.

According to Eq. 41, the spectral density of response at age t depends on the entire history of system properties from t_0 to t if the calculation of function Φ depends on it. This is so, e.g., for the stress caused by drying and creep in a concrete structure.

Mean and Variance of Response

Calculate now the mean response. Using the impulse response function, the response may be expressed as

$$g(\mathbf{x}, t, t_0, \tau) = \int_{-\infty}^{\infty} \phi(\mathbf{x}, t, t_0, \xi) f(\tau + t - \xi)\, d\xi \quad \ldots\ldots (42)$$

and thus the mean of the response is

$$\bar{g}(\mathbf{x}, t, t_0) = E_\tau[g(\mathbf{x}, t, t_0, \tau)] = \int_{-\infty}^{\infty} \phi(\mathbf{x}, t, t_0, \xi) E_\tau[f(\tau + t - \xi)]\, d\xi$$

$$= \bar{f} \int_{-\infty}^{\infty} \phi(\mathbf{x}, t, t_0, \xi)\, d\xi \quad \ldots\ldots (43)$$

Here $\bar{f} = E_\theta[f(\theta)] = E_\xi[f(\tau + t - \xi)]$ = constant, and the averaging is carried out over τ at constant ξ and t. In the foregoing calculation, the order of the operations of expectation, E_τ, and of integration is switched since both operations are linear. A well-behaved integral is, of course, assumed. Since the last integral in Eq. 43 is identical to that in Eq. 27 for $\omega = 0$, Eq. 43 may be written as

$$\bar{g}(\mathbf{x}, t, t_0) = \bar{f}[\Phi(\mathbf{x}, t, t_0, \omega)]_{\omega=0} \quad \ldots\ldots (44)$$

This equation implies that if input $f(\theta)$ has a zero mean, so has the response at all times.

Consider now the mean of the squared random process. For a stationary input we have, as usual

$$E_\theta\{[f(\theta)]^2\} = \{E_\theta[f(\theta) f(\theta + \lambda)]\}_{\lambda=0} = R_f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)\, d\omega \quad \ldots\ldots (45)$$

and for the response we have

$$E_\tau\{[g(\mathbf{x}, t, t_0, \tau)]^2\} = \{E_\tau[g(\mathbf{x}, t, t_0, \tau) g(\mathbf{x}, t, t_0, \tau + \lambda)]\}_{\lambda=0}$$

$$= R_g(\mathbf{x}, t, t_0, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_g(\mathbf{x}, t, t_0, \omega)\, d\omega \quad \ldots\ldots (46)$$

We see that the mean square response at $\mathbf{x}$, t, and t_0 is proportional to the area under the spectral density curve for the same $\mathbf{x}$, t and t_0. Consequently, if we split the input and the response into the mean and the deviation from the mean, i.e., $f(\theta) = \bar{f} + \tilde{f}(\theta)$ and $g(\mathbf{x}, t, t_0, \tau) = \bar{g}(\mathbf{x}, t, t_0) + \tilde{g}(\mathbf{x}, t, t_0, \tau)$, then the mean input alone produces response $\bar{g}(\mathbf{x}, t, t_0)$, and the input deviation from its mean, $\tilde{f}(t + \tau)$, applied as the sole input, produces response $\tilde{g}(\mathbf{x}, t, t_0, \tau)$. The variance s_{0g}^2 of the response is defined as the mean of the square of $\tilde{g}(\mathbf{x}, t, t_0, \tau)$, and according to Eq. 46 it may be expressed as

$$s_{0g}^2(\mathbf{x}, t, t_0) = \mathrm{var}_\tau[\tilde{g}(\mathbf{x}, t, t_0, \tau)] = E_\tau\{[\tilde{g}(\mathbf{x}, t, t_0, \tau)]^2\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_g(\mathbf{x}, t, t_0, \omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\Phi(\mathbf{x}, t, t$$

in which the spectral densities of $\tilde{f}(\theta)$ and $\tilde{g}(\ldots)$ may be expressed as

$$S_{\tilde{f}}(\omega) = \int_{-\infty}^{\infty} R_{\tilde{f}}(\lambda) e^{-i\omega\lambda} d$$

$$S_{\tilde{g}}(\mathbf{x}, t, t_0, \omega) = \int_{-\infty}^{\infty} R_{\tilde{g}}(\mathbf{x}, t, t_0, \lambda)\, e^{-i\omega\lambda} d$$

Here $R_{\tilde{f}}(\lambda)$ and $R_{\tilde{g}}(\mathbf{x}, t, t_0, \lambda)$ are the autocorrelation functions for the deviations $\tilde{f}(\theta)$ and $\tilde{g}(\ldots)$ of input and response from their means. Note that for a single frequency input, i.e., for $S_{\tilde{f}}(\omega) = 2\pi s_{0\tilde{f}}^2 \delta(\omega - \omega_0)$ = Dirac delta function, Eq. 47 reduces to

$$s_{0\tilde{g}}(\mathbf{x}, t, t_0) = |\Phi(\mathbf{x}, t, t_0,$$

Example: Random Shrinkage Stress in Aging Viscoelastic Material

For the sake of simple illustration, we now try to calculate the statistical characteristics of the shrinkage stress $\sigma = g(\mathbf{x}, t, t_0, \tau) = g$ in a concrete structure that is idealized as perfectly thin so that all concrete feels the environmental humidity h instantly (this means that the delay due to diffusion of moisture through concrete is neglected; in Refs. 7 and 8 this was not neglected, but an exact closed-form solution was then impossible and a numerical approximation had to be used). Stress σ may represent, e.g., the maximum stress in a statically indeterminate frame, or in a cylindrical shell or roof dome. The structure is assumed to be homogeneous in its material properties. The reinforcement stiffness is neglected, and no cracking is assumed to take place, which normally means the structure must be prestressed.

The ergodic random input consists of the environmental relative humidity denoted as $h = 1 + f(\theta)$, to which the structure is exposed at age t_0. The initial pore relative humidity of concrete for $t \le t_0$ is $h = 1$ (saturation), and so $f(\theta) = 0$ at t_0, and the input $f(\theta)$ represents the drop of pore relative humidity. Concrete is assumed, for the sake of simplicity, to be an aging Maxwell solid for which the stress-strain relation is

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E_c(t)} + \frac{\sigma}{\eta(t)} + \dot{\epsilon}_{sh} \quad \ldots\ldots \quad (51)$$

in which superimposed dots denote time derivatives; $E_c(t)$ = Young's elastic modulus of concrete and $\eta(t)$ = viscosity, which both depend on concrete age t, and $\dot{\epsilon}_{sh}$ = shrinkage strain rate, which may be approximately expressed as $\dot{\epsilon}_{sh} = \kappa \dot{f}(\theta)$ where κ = constant = shrinkage coefficient. Furthermore, we may assume $\eta(t) = E_c(t)/\gamma$ where γ = constant and $1/\gamma$ represents the relaxation time of concrete, a material property. Although in general concrete must be described by a Maxwell chain model with a broad relaxation spectrum, a single Maxwell unit (assumed in Eq. 51) provides an adequate description within a limited time range, in particular for response delays and concrete ages roughly between $0.3/\gamma$ and $3/\gamma$.

Denote as K_x the shrinkage stress at location x of the structure caused

by a unit homogeneous shrinkage strain $\epsilon_{sh} = 1$ if the structure is perfectly elastic and has a unit elastic modulus, $E_c = 1$. According to McHenry's analogy (2), the response stress $\sigma = g(\mathbf{x}, t, t_0, \tau)$ satisfies Eq. 51 in which $\dot{\epsilon} = 0$ and $\dot{\epsilon}_{sh}$ is replaced by $K_x \dot{\epsilon}_{sh} = k_x \dot{f}(\theta)$ where $k_x = \kappa K_x$. This yields the differential equation

$$\dot{g} + \gamma g = -k_x E_c(t) \dot{f} \quad \ldots\ldots \quad (52)$$

The initial condition at age $t = t_0$ when exposure to the environment begins is $g = 0$. Immediately after time t_0, i.e., at $t = t_0^+$, there is a jump of stress to the value $g = g(\mathbf{x}, t_0^+, t_0, \tau) = -k_x E_c(t_0) f(t_0 + \tau)$. As may be verified by substitution in Eq. 52, the solution for $t > t_0$ may be represented as

$$g(x, t, t_0, \tau) = -k_x R(t, t_0) f(t_0 + \tau) - \int_{t'=t}^{t}$$

in which $R(t, t') = E_c(t') e^{-\gamma(t-}$

$R(t, t')$ is the relaxation function, representing the stress g at age t caused by a unit constant strain imposed at age t', i.e., the response to a Heaviside step function input. If $f(\theta)$ is continuous and continuously differentiable, one may set $df(\theta) = [df(\tau + t')/dt']dt'$, and Eq. 53 may then be transformed by integration by parts:

$$g(x, t, t_0, \tau) = -k_x E_c(t) f(t + \tau) - \int_{t'=t}^{t}$$

$$\text{in which} \quad L(t, t') = -\frac{\partial R(t, t')}{\partial t'} \quad \ldots\ldots \quad (56)$$

$L(t, t')$ is called in aging viscoelasticity the memory function. Eq. 55 may further be written as

$$g(x, t, t_0, \tau) = \int_{t_0}^{t} \phi(x, t, t_0, \xi) f(t + \tau - \xi) d\xi$$

$$= \int_{-\infty}^{\infty} \phi(x, t, t_0, \xi) f(t + \tau - \xi) d\xi \quad \ldots\ldots \quad (57)$$

in which $\xi = t - t'$ = time lag, and

$$\text{for} \quad t' \le t_0 \quad (\xi \ge t - t_0) \quad \text{or} \quad t' > t (\xi < 0): \quad \phi(t, t_0, \xi) = 0 \quad \ldots\ldots \quad (58)$$

$$\text{for} \quad t_0 < t' \le t \quad (0 < \xi < t - t_0): \quad \phi(x, t, t_0, \xi) = -k_x \frac{\partial R(t, t - \xi)}{\partial \xi}$$

$$- k_x E_c(t - \xi) \delta(\xi) = k_x [E_c(t_0) \gamma e^{-\gamma\xi}$$

$\phi(x, t, t_0, \xi)$ is the impulse response function, representing the stress g at age t caused by a unit strain impulse (Dirac delta function) at age t', and $\delta(\xi)$ is the Dirac delta function.

Let the age-dependence of the elastic modulus be

$$E_c(t) = E_0 - E_1 e^{-\gamma t}$$

in which $E_0 > E_1 \geq 0$. Let us now calculate the response to a single periodic component of input, $f(\theta) = e^{i\omega\theta} = e^{i\omega\tau} e^{i\omega t}$.
sumed in the form $g(x,t,t_0,\tau) = \Phi(0,t,t_0,\omega)e^{i\omega}$
expressions for f and g in Eq. 52, $e^{i\omega\theta}$ cancels out and we obtain for Φ the differential equation

$$\dot{\Phi} + (\gamma + i\omega)\Phi = i\omega k_x(E_1 e^{-\gamma t}$$

The general solution of this differential equation is

$$\Phi = Ce^{-(\gamma+i\omega} \quad {}_{x\ 0} \quad \gamma + i\omega \quad + \quad {}_{x\ 1}e \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (62)$$

as may be verified by substitution in Eq. 61; C is an arbitrary constant to be found from the initial condition at age t_0 at which $(h - 1)$ suddenly jumps from 0 to $e^{i\omega t_0} e^{i\omega\tau}$, causing $g(\ldots)$ to jump instantly from 0 to the value $-k_x E(t_0) e^{i\omega t_0}$
is $\Phi(x,t_0,t_0,\omega) = -k_x E(t_0)$, from which C may be calculated. This leads for the present problem to the frequency response function:

$$\Phi(x,t,t_0,\omega) = k_x \left[E_1 e^{-\gamma t} \quad \gamma + i\omega \quad (\gamma e \quad + i\omega) \quad \ldots\ldots\ldots\ldots \quad (63)\right.$$

Strange though it might seem on a first look, it does not matter whether the input is continuous at t_0, beginning with a zero value, e.g., $f(\theta) = Re(e^{i\omega}$
jump from 0 to an arbitrary value, e.g., $F(\omega)\ e^{i\omega\theta}$.
clear from Eqs. 15 and 18, it may be worthwhile to illustrate it. Consider the input $f(\theta) = e^{i\omega\theta}$
viously. The response then should be $g(x,t,t_0,\tau) = \Phi(0,t,t_0,\omega)\ e^{i\omega\theta}F$
Indeed, substituting both functions into Eq. 52, we find that $F(\omega)$ cancels out and the same differential equation for Φ (Eq. 61) results. The initial condition for Φ also remains the same. Indeed, $h - 1$ jumps at t_0 instantly from 0 to the value $e^{i\omega t_0}\ e^{i\omega\tau}\ F(\omega)$, causing $g(\ldots)$ to jump instantly from 0 to the value $-k_x E(t_0)\ e^{i\omega t_0}$
$e^{i\omega\tau}F(\omega)$; this yields the initial condition $\Phi = -k_x E(t_0)$ at $t = t_0$. This is the same as before, and so is the differential equation for Φ. Therefore, the frequency response function $\Phi(\ldots)$ in Eq. 63 also remains the same; obviously, it embodies the nonstationarity due to the initial condition.

Eq. 63 may be also obtained by integration according to Eqs. 53–54 or Eqs. 55–56 (or Eqs. 57–59). Since $\lim E(t') = \text{const.} > 0$ for $t \to \infty$, the asymptotic frequency response function $\bar{\Phi}(x,\omega)$ for $t \to \infty$ must be independent of t and t_0. Indeed, Eq. 63 yields

$$\bar{\Phi}(x,\omega) = \lim_{t\to\infty} \Phi(x,t,t_0,\omega) = -k_x E_0 \frac{i\omega}{\gamma + i\omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (64)$$

The same function results by setting in Eq. 52 $E(t) = E_0$
$e^{i\omega t}$ and $g = \Phi e^{i\omega t}$.

For nonstationary transient response of a nonaging system ($E_1 = 0$), the frequency response function should depend only on $t - t_0$, and not on t or t_0 separately. Indeed, Eq. 63 reduces to

$$\Phi(x,t,t_0,\omega) = -\frac{k_x E_0}{\gamma + i\omega}(\gamma e^{-(\gamma+i\omega)} \quad + i\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (65)$$

As an illustration of statistical analysis, consider that the spectral density of input is

$$S_f(\omega) = a_1\delta(\omega - \omega_1) + a_2\delta(\omega - \omega_2) + a_3[H(\omega - \omega_3) - H(\omega - \omega_4)] \ldots\ldots \quad (66)$$

in which a_1, a_2, a_3 = given amplitudes; $\omega_1, \ldots, \omega_4$ = given frequencies; and $\omega_4 > \omega_3$. Typically ω_1 and ω_2 may represent the daily and annual fluctuations, and ω_3, ω_4 are the limits of a white noise band characterizing the weather. According to Eq. 41, the spectral density of the response at age t is

$$S_g(x,t,t_0,\omega) = |\Phi \qquad)(t$$

$$+ a_3[H(\omega - \omega_3) - H(\omega - \omega_4)]\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (67)$$

Evaluating $|\Phi$
following expression for the variance of response at age t:

$$s_{0g}^2(x,t,t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\left(\frac{k_x E_0}{\gamma^2 + \omega^2}\right)[\omega^2 + \gamma e^{-\gamma(t-}\right.$$

$$\left. - \omega\sin\omega(t - t_0))] - k_x E_1 e^{-\gamma t}\right\}^2 \quad \left\{\frac{k\ E}{\gamma\ + \omega} \qquad (\gamma\sin\omega(t - t_0)\right.$$

$$\left. + \omega\cos\omega(t - t_0))]\right\}^2 \{a_1\delta(\omega - \omega_1) + a_2\delta(\omega - \omega_2) + a_3[H(\omega - \omega_3)^{)(t}$$

This integral can be evaluated in a closed form.

A closed-form solution is also possible for a more general viscoelastic behavior described by an aging standard solid model (1) for which the relaxation function of the material has the form:

$$R(t,t') = E_0 - E_1 e^{-\gamma t'} \qquad)$$

in which E_0, E_1, C_0, C_1 and γ = given material constants. According to Eq. 54, the response to periodic input $f(\tau + t) = e^{i\omega}$

$$\sigma(x,t,t_0) = e^{i\omega t}\Phi(x,t,t_0,\omega) = e^{i\omega t}(a_1 + a_2 e^{-\gamma t} + a_3 e^{-i\omega t} + a_4 e^{-(\gamma+i\omega)t})k_x \quad (70)$$

in which $a_1 = E_0 + C_0\left(\frac{\omega(i\gamma + \omega)}{\gamma^2 + \omega^2} - 1\right)$ ..... (71*a*)

$$a_2 = (E_1 - C_1)\frac{\omega(i\gamma - \omega)}{\gamma^2 + \omega^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (71b)$$

$$a_3 = (C_0 - E_0)e^{i\omega t_0} + (C_1 - E_1)\frac{\omega(i\gamma - \omega)}{\gamma^2 + \omega^2}$$

$$+ [(C_1 - E_1)e^{-\gamma t_0}$$

$$a_4 = C_1 e^{i\omega t_0} - C_0 \frac{\omega(i\gamma + \omega)}{\gamma^2 + \omega^2} + (C_0 e^{\gamma t_0} - C_1) e^{-i\omega t_0} \quad (71d)$$

The special case $E_1 = C_1 = 0$ represents a nonaging system.

Various numerical applications to structures, with diagrams of response, are presented in Refs. 6–8.

Previous Work on Nonstationary Response and Alternative Approaches

The special case of nonstationary or transient response of a nonaging system to stationary input, the nonstationarity being due to the initial condition, has been studied by many investigators (9–11,15–19). The usual approach is to define transient autocorrelation functions with two independent variables, e.g.

$$R_g(t_1, t_2) = E[g(t_1) g(t_2)] \quad (72)$$

in which expectation E may be interpreted only as ensemble averaging at fixed t_1 and t_2, not as time averaging. The spectral density is then defined by the double Fourier transform

$$S_g(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} R_g(t_1, t_2) e^{i\omega_1 t_1} e^{-i\omega_2 t_2} dt_2 \right) dt_1 \quad (73)$$

For the relation to the spectral density of input, a relation similar to Eq. 41 applies (17):

$$S_g(\omega_1, \omega_2) = \Phi^*(\omega_1) \Phi(\omega_2) S_f(\omega_1, \omega_2) \quad (74)$$

As an alternative, nonstationary processes have been described by the evolutionary spectral density function (17,19)

$$\mathcal{S}_g(t, \omega) = \int_{-\infty}^{\infty} \mathcal{R}_g(t, \tau) e^{-i\omega\tau} d\tau \quad (75)$$

in which $\mathcal{R}_g(t, \tau) = E_\tau \left[g\left(t - \frac{\tau}{2}\right) g\left(t + \frac{\tau}{2}\right) \right] \quad (76)$

is the instantaneous time-averaging autocorrelation function.

The present method may be also applied, as a special case, to nonstationary response of a nonaging system. One may then set $t_0 = 0$ and delete argument t_0 from all the functions. This method yields a simpler input-output relation (Eq. 41 without t_0) than the aforementioned previous methods (e.g., Eq. 74). Moreover, the previous methods blur the statistics over time and thus are incapable of providing the precise characteristics of the randomness of response at any particular time.

For the special case of nonstationary response whose statistical characteristics evolve in time very slowly, the concept of a locally stationary response (9,20) has been introduced. Here the expectations may be approximately determined by time averaging over short (finite) segments of the record. Ideally, the segment should be infinitely short, but then the data of a single record are insufficient to determine statistics. So a

certain bias is inevitably introduced due to finite duration of the segments. With this concept, the response is simply calculated as a time-modulated stationary response of a nonaging system (9,11,16,20).

A random process for which the cause of nonstationarity is other than the initial condition has been investigated in the theory of fading communication media and time-varying filters (23). The mathematical treatment is however quite different and, unlike the present formulation, is inapplicable to aging systems. Signal fading is caused by the existence of signal paths that have different lengths from the transmitter to the receiver, due to scattered reflections, and by Doppler effects, due to random movements of the signal transmitting medium. In those problems (23), the frequency response function depends, aside from frequency ω, also on time t, just like in this study, but otherwise the mathematical formulation is different, based on special features such as a narrow (Gaussian) distribution of the path travel times. The main difference is that the fading (variation of the system properties in time) is predominantly periodic, and so the objective of the theory of fading communication media still is to find the overall time-independent statistical characteristics of the entire time history, just like for nonaging system properties. By contrast, the variation of the properties of an aging system [e.g., time dependence of $E_c(t)$ in our example] cannot be smeared out over time, and the principal requirement is to find the statistical characteristics for each particular time.

Reduction of Problems with Nonzero Response for $f(\theta) = 0$

As mentioned below Eq. 1, the principle of superposition must be used in this case to ensure that, for the stochastic problem, $g(\ldots) = 0$ when $f(\theta) = 0$. For the sake of illustration, consider the initial-boundary value problem defined by the differential equation

$$L[z(\mathbf{x}, t)] = c(\mathbf{x}, t) \quad (77)$$

with the following initial condition at $t = t_0$ and the boundary condition for $t > t_0$ at boundary points $\mathbf{x} = \mathbf{x}_b$:

$$A[z(\mathbf{x}, t_0)] = a(\mathbf{x}); \quad B[z(\mathbf{x}_b, t)] = f(\theta) + b(t) \quad (78)$$

in which L is a linear partial differential operator in $\mathbf{x}$ and t, homogeneous of degree 1 (i.e., such that $L[kz] = kL[z]$); B is an algebraic or differential operator in $\mathbf{x}$, homogeneous of degree 1; and A is an algebraic or differential operator in t, homogeneous of degree 1; $a(\mathbf{x})$ defines a deterministic initial condition, and $b(t)$, $c(\mathbf{x}, t)$ are additional deterministic loads (e.g., surface tractions and deterministic thermal strains). It is now easy to verify by substitution in Eqs. 77–78 that the solution may be obtained as

$$z(\mathbf{x}, t) = g(\mathbf{x}, t) + y(\mathbf{x}, t) \quad (79)$$

in which:

1. $g(\mathbf{x}, t)$ is a solution of the differential equation

$$L[g(\mathbf{x}, t)] = 0 \quad (80)$$

with the initial condition at $t = t_0$ and the boundary conditions for $t > t_0$:

$$A[g(\mathbf{x}, t_0)] = 0; \quad B[g(\mathbf{x}_b, t)] = f(\theta) \quad (81)$$

2. $y(\mathbf{x}, t)$ is a solution of the differential equation

$$L[y(\mathbf{x}, t)] = c(\mathbf{x}, t) \quad (82)$$

with the initial condition at $t = t_0$ and the boundary condition for $t > t_0$:

$$A[y(\mathbf{x}, t_0)] = a(\mathbf{x}); \quad B[z(\mathbf{x}_b, t)] = b(t) \quad (83)$$

The stochastic problem, defined by Eqs. 80–81 with random input $f(\theta)$, is obviously such that $g(\mathbf{x}, t) = 0$ if $f(\theta) = 0$, as required for the validity of Eq. 1. The solution of $y(\mathbf{x}, t)$ of Eqs. 80–81 is deterministic.

According to Eq. 79, the statistics of the complete response $z(\mathbf{x}, t)$ are the same as those of $g(\mathbf{x}, t)$, except for the mean, which is $\bar{z}(\mathbf{x}, t) = \bar{g}(\mathbf{x}, t) + y(\mathbf{x}, t)$. This is an important advantage of the present statistical concept, made possible by the fact that we use no averaging over the time t. All our averaging is either ensemble averaging at fixed t or, equivalently, averaging over the birth time τ, again at fixed t. If, by contrast, we used some of the previous methods, which are described by Eqs. 72–76 and do involve averaging over time t, then the decomposition in Eq. 79 would not reduce the stochastic problem to one for which a nonzero initial condition or possible additional deterministic loads have no effect on the statistics of the response (except the mean).

As a simpler illustration, let us now extend our previous example, considering that a nonzero stress σ_0 is produced by autogeneous shrinkage before the exposure to random environment begins. In analogy to Eq. 52, the dfferential equation for stress σ is

$$\dot{\sigma} + \gamma\sigma = -k_x E_c(t)\dot{f} \quad (84)$$

and the initial condition is $\sigma = \sigma_0$ at $t = t_0$. The solution may be obtained as

$$\sigma = g(x, t, t_0, \tau) + y(t) \quad (85)$$

in which $g(\ldots)$ is the same as before, i.e., is the solution of Eq. 52 with the initial condition $g = 0$ at $t = t_0$; and $y(t)$ is the solution of the differential equation

$$\dot{y} + \gamma y = 0 \quad (86)$$

with the initial condition of $y = \sigma_0$, i.e., $y(t) = \sigma_0 e^{-\gamma t}$. The mean response then is $\sigma = \bar{g}(\ldots) + \sigma_0 e^{-\gamma t}$ and the variance of σ is the same as that of $g(\ldots)$.

Conclusions

1. The statistical characteristics of the response of an aging linear system subjected to ergodic random input may be determined by considering a fixed age of the system and a fixed age at the start of exposure, and imagining the birth time to be varied, i.e., the input to be shifted in time against the instant when the system is built. The response, while nonstationary in time, becomes then a stationary ergodic random process as a function of the birth time, and response averaging over the birth time becomes equivalent to instantaneous ensemble averaging over all possible realizations.

2. The spectral method can be generalized for aging linear systems under ergodic input. The relation between the spectral densities of input and response is algebraic, similar to the case of stationary response, although the frequency response function needs to be obtained by solving integral or differential (rather than algebraic) equations in time.

3. In the special case of nonstationary response of nonaging systems under ergodic input, the present formulation appears to offer a simpler alternative to the existing formulations, taking advantage of ergodicity with respect to the birth time.

4. The problem must be formulated so that a zero response corresponds to a zero input. This can always be done according to the principle of superposition because, in contrast to previous formulations, no averaging is done over the actual time. Due to this fact, the additional response caused by additional deterministic loads and by a nonzero initial condition may be solved separately and then superimposed.

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Appendix.—References

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