Matrix Force-Displacement Relations in Aging Viscoelasticity

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Introduction

The age-adjusted effective modulus method (Bažant 1972), also called Trost-Bažant method (Neville, et al. 1983), has emerged as an effective and widely used tool for approximate calculation of long-time stress and deformation changes in concrete structures under permanent loads (Bažant and Najjar 1973; Bažant, et al. 1975; Bažant 1982; Lazić and Lazić 1984, 1985; Neville, et al. 1983; Chiorino, et al. 1984; Bažant 1986; ACI 1982; CEB-FIP 1978). This method, which represents a generalization of Trost's (1967a,b) method of relaxation coefficient, is mathematically based on Bažant's (1972) theorem, which states that if the strain history may be described linearly in terms of the compliance function, then the stress history may be expressed linearly in terms of the associated relaxation function, with simple expressions for the coefficients of this linear dependence.

The solution according to the age-adjusted effective modulus method is exact only for homogeneous structures, i.e., structures in which the material properties are the same at all points. By experience, it was discovered that good approximate solutions are often obtained even when

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the structure is nonhomogeneous, i.e., its creep properties differ from one part to another, as is typical of reinforced or composite structures. Sometimes the error may be more appreciable compared to the exact linearly viscoelastic solution; this can occur for certain compliance functions in some types of composite cross sections. For such problems, Lazić and Lazić (1984, 1985) showed that for composite cross sections of beams, error reduction can be achieved by a generalization of Bažant's original age-adjusted effective modulus method. In this generalization, the relationship of bending moment and normal force histories to the curvature and the neutral axis strain histories is considered according to an exact viscoelastic solution.

Matrix Relation between Force and Displacement Histories

It will now be shown that the age-adjusted effective modulus method can be generalized still further to any aging linearly viscoelastic system that is characterized by a matrix Volterra integral equation. This equation relates a column matrix $\mathbf{u}(t)$ of certain deformation variables of the structure to a column matrix $\mathbf{f}(t)$ of the associated force variables. The components of $\mathbf{u}$ may be the strain components or displacements, rotations, curvatures, etc., at a point of the structure, while the associated forces may be the strain components at a point of the structure or the applied or internal forces, bending moments, etc. The most general linear relationship between the histories $\mathbf{u}(t)$ and $\mathbf{f}(t)$ in time t may be written in the form of the following matrix Volterra integral equations

$$\mathbf{u}(t) = \int_{t_0}^{t} \mathbf{J}(t,t')d\mathbf{f}(t') = \underline{J}\mathbf{f}(t) \quad (1)$$

$$\mathbf{f}(t) = \int_{t_0}^{t} \mathbf{R}(t,t')d\mathbf{u}(t') = \underline{R}\mathbf{u}(t) \quad (2)$$

in which t_0 is the age when the stress and deformation first appear, the integrals are Stieltjes integrals, and $\mathbf{J}$ and $\mathbf{R}$ are the matrix compliance function and the matrix relaxation function of the structure, respectively. $\mathbf{J}(t,t')$ is defined as the components of $\mathbf{u}$ at age t caused by unit components of forces $\mathbf{f}$ applied at age t' ($t' < t$). $\mathbf{R}(t,t')$ is defined as the components of force column matrix $\mathbf{f}$ at age t caused by unit components of deformation column matrix $\mathbf{u}$ introduced at age t'. $\underline{J}$ and $\underline{R}$ represent the matrix creep operator and the matrix relaxation operator. These operators are obviously inverse to each other, i.e., $\underline{J}^{-1} = \underline{R}$. They represent Volterra's integral operators, which can be manipulated according to the rules of algebra except that they are not commutative (Mandel 1958; Bažant 1975). The functions $\mathbf{J}$ and $\mathbf{R}$ can, in principle, be determined either directly, by measurement on the structure, or indirectly, by calculation on the basis of material creep properties.

Theorem

If the deformation history for $t \geq t_0$ is expressed linearly in terms of the matrix compliance function, i.e.

$$\mathbf{u}(t) = \mathbf{a} + \mathbf{J}(t, t_0)\mathbf{b} \quad (3)$$

in which $\mathbf{a}$ and $\mathbf{b}$ are constant column matrices, then the corresponding force history for $t \geq t_0$ can be expressed linearly in terms of the matrix relaxation function and has the form:

$$\mathbf{f}(t) = \mathbf{b} + \mathbf{R}(t, t_0)\mathbf{a} \quad (4)$$

Proof.—Since Eq. 3 is valid only for $t \geq t_0$, it may be rewritten as:

$$\mathbf{u}(t) = H(t - t_0)\mathbf{a} + \underline{J}H(t - t_0)\mathbf{b} \quad (5)$$

where H denotes heaviside step function. Multiplying all terms of this equation by operator $\underline{R}$ from the left, we have

$$\underline{R}\mathbf{u}(t) = \underline{R}H(t - t_0)\mathbf{a} + \underline{R}\,\underline{J}H(t - t_0)\mathbf{b} \quad (6)$$

By definition, $\underline{R}H(t - t_0) = R(t, t_0)$ and $\underline{R}\,\underline{J} = \underline{I}$ where $\underline{I}$ = identity operator. Also $\underline{I}\mathbf{b} = \mathbf{b}$. Eq. 6 thus becomes Eq. 4.

Matrix Generalization of Age-Adjusted Effective Modulus Method

Similar to the age-adjusted effective modulus method, Eq. 4 can be reformulated in terms of the increments of $\mathbf{u}$ and $\mathbf{f}$ from the instant of first loading, t_0, to the current time, t. Denoting the initial elastic compliances and stiffnesses of the structure as $\mathbf{J}(t_0, t_0) = \mathbf{J}_0$, $\mathbf{R}(t_0, t_0) = \mathbf{R}_0$, we have:

$$\mathbf{u}_0 = \mathbf{a} + \mathbf{J}_0\mathbf{b} \quad (7a)$$

$$\mathbf{f}_0 = \mathbf{b} + \mathbf{R}_0\mathbf{a} \quad (7b)$$

$$\Delta\mathbf{u} = (\mathbf{J} - \mathbf{J}_0)\mathbf{b} \quad (8a)$$

$$\Delta\mathbf{f} = (\mathbf{R} - \mathbf{R}_0)\mathbf{a} \quad (8b)$$

From this we get:

$$\mathbf{b} = (\mathbf{J} - \mathbf{J}_0)^{-1}\Delta\mathbf{u} \quad (9a)$$

$$\mathbf{a} = (\mathbf{R} - \mathbf{R}_0)^{-1}\Delta\mathbf{f} \quad (9b)$$

and from Eqs. 3 and 4 we obtain:

$$\mathbf{a} = \mathbf{u}_0 - \mathbf{J}_0(\mathbf{J} - \mathbf{J}_0)^{-1}\Delta\mathbf{u} \quad (10a)$$

$$\mathbf{b} = \mathbf{f}_0 - \mathbf{R}_0(\mathbf{R} - \mathbf{R}_0)^{-1}\Delta\mathbf{f} \quad (10b)$$

Substitution into Eq. 8 finally yields the following algebraic matrix relations between the column matrices of incremental forces and displacements:

$$\Delta\mathbf{f} = (\mathbf{R}_0 - \mathbf{R})[\mathbf{J}_0(\mathbf{J} - \mathbf{J}_0)^{-1}\Delta\mathbf{u} - \mathbf{u}_0] \quad (11)$$

$$\Delta\mathbf{u} = (\mathbf{J} - \mathbf{J}_0)[\mathbf{R}_0(\mathbf{R}_0 - \mathbf{R})^{-1}\Delta\mathbf{f} + \mathbf{f}_0] \quad (12)$$

These equations represent generalizations of the stress-strain relation of the age-adjusted effective modulus method (ACI 1982).

As a special case, the column matrices $\mathbf{f}$ and $\mathbf{u}$ may be replaced by σ and $\epsilon - \epsilon''$ where ϵ'' = shrinkage strain, while the matrix functions $\mathbf{J}$ and $\mathbf{R}$ are replaced by the compliance and relaxation functions of the aging material, $J(t, t')$ and $R(t, t')$. Eq. 11 then yields the uniaxial stress-strain relation of the age-adjusted effective modulus method (Bažant 1972):

$$\Delta\sigma = E''(t, t_0)[\Delta\epsilon - \Delta\epsilon'' - \epsilon(t_0)\phi(t, t_0)] \quad (13)$$

$$\text{in which} \quad E''(t, t_0) = \frac{E(t_0) - R(t, t_0)}{\phi(t, t_0)}; \quad \phi(t, t_0) = E(t_0)J(t, t_0) - 1 \quad (14)$$

where $E(t_0) = R(t_0, t_0) = 1/J(t_0, t_0)$ = elastic modulus at age t_0; $\phi(t, t_0)$ = creep coefficient; $E''(t, t_0)$ = age-adjusted effective modulus; $\epsilon(t_0) = \sigma(t_0)/E(t_0)$ = initial elastic strain; $\Delta\sigma = \sigma(t) - \sigma(t_0)$; and $\Delta\epsilon = \epsilon(t) - \epsilon(t_0)$. A similar replacement in Eq. 12 yields the same result (Eqs. 13 and 14).

As a more general special case, one can further obtain an algebraic triaxial stress-strain relation for the case of general aging linearly viscoelastic material behavior for which Poisson ratio ν is not constant but variable, i.e., $\nu(t, t')$. In that case the basic aging viscoelastic stress-strain relations and their inverses may be written as:

$$\epsilon(t) = \underline{J}\sigma(t) \quad (15a)$$

$$\sigma(t) = \underline{R}\epsilon(t) \quad (15b)$$

where $\underline{J}$ and $\underline{R}$ are matrix operators, inverse to each other and expressed on the basis of the matrix compliance function $\mathbf{J}(t, t')$ and the matrix relaxation function $\mathbf{R}(t, t')$ of the aging material. The incremental algebraic matrix stress-strain relations are then obtained from Eqs. 5 and 6 by replacing the column matrices $\Delta\mathbf{f}$ and $\Delta\mathbf{u}$ with $\Delta\epsilon$ and $\Delta\sigma$. The results are:

$$\Delta\epsilon = (\mathbf{J} - \mathbf{J}_0)[\mathbf{R}_0(\mathbf{R}_0 - \mathbf{R})^{-1}\Delta\sigma + \Delta\sigma_0] \quad (16)$$

$$\Delta\sigma = (\mathbf{R}_0 - \mathbf{R})[\mathbf{J}_0(\mathbf{J} - \mathbf{J}_0)^{-1}\Delta\epsilon - \Delta\epsilon_0] \quad (17)$$

The expressions:

$$\mathbf{D} = (\mathbf{J} - \mathbf{J}_0)\mathbf{R}_0(\mathbf{R}_0 - \mathbf{R})^{-1} \quad (18a)$$

$$\mathbf{C} = (\mathbf{R}_0 - \mathbf{R})\mathbf{J}_0(\mathbf{J} - \mathbf{J}_0)^{-1} \quad (18b)$$

may be regarded as the age-adjusted elastic compliance matrix and the age-adjusted elastic stiffness matrix of the material, respectively.

Conclusion

To sum up, Bažant's (1972) theorem underlying the age-adjusted effective modulus method can be generalized to a matrix form that gives the long-time increments of forces in the structure as a linear function of the long-time deformation increments. This linear function is characterized by matrices expressed in terms of the compliance and relaxation functions of the structure.

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Appendix.—References

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