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# WHY CONTINUUM DAMAGE IS NONLOCAL: JUSTIFICATION BY QUASIPERIODIC MICROCRACK ARRAY

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Abstract. - Strain-softening damage due to distributed cracking is modeled by an elastic continuum with a quasiperiodic array of cracks of regular spacing but varying sizes. As a model for the initial stage, the cracks are penny-shaped and small compared to their spacing, and as a model for the terminal stage the uncracked ligaments between the cracks are circular and small compared to their spacing. The strain due to cracks and the compliance per crack are calculated. The cracked material is homogenized in such a manner that the macroscopic continuum strains satisfy exactly the condition of compatibility with the actual strains due to cracks, and the macroscopic continuum stress satisfies exactly the condition of work equivalence with the actual stresses in the cracked material. The results show that, contrary to the existing theories, the damage variable used in continuum damage mechanics should be nonlocal, while the elastic part of the response should be local. In particular, the nonlocal continuum damage should be considered as a function of the spatial average of the cracking strain rather than its local value. The size of the averaging region is determined by the crack spacing.

## Introduction

Introduction of nonlocal continuum concepts into the analysis of strain-softening structures [1] has recently met with considerable success. It has eliminated problems with spurious mesh sensitivity and incorrect convergence and has assured that refinements of the finite element mesh cannot lead to spurious localization of energy dissipation into a softening zone of a vanishing volume [1-10]. However, a physical justification of the nonlocal approach is still lacking. The objective of the present brief study is to show such a justification for a certain case of strain-softening that is caused by distributed cracking.

## Array of Small Penny-Shaped Cracks

We consider an infinite elastic continuum containing an array of small circular (penny-shaped) cracks normal to axis x of cartesian coordinates x, y, z (Fig. 1a). The cracks lie in parallel planes $x = x_i = i\ell$ ($i = \ldots -2, 1, 0, 1, 2, \ \ )$ and their centers lie at the nodes of a spatial cubic lattice whose nodal spacing for each of the lattice directions x, y and z is $\ell$ To bring to light the nonlocal aspects, the crack array cannot be perfectly periodic. We make the array quasiperiodic by assuming that the radius $a_i$ of the cracks on the planes $x = x_i$ slightly

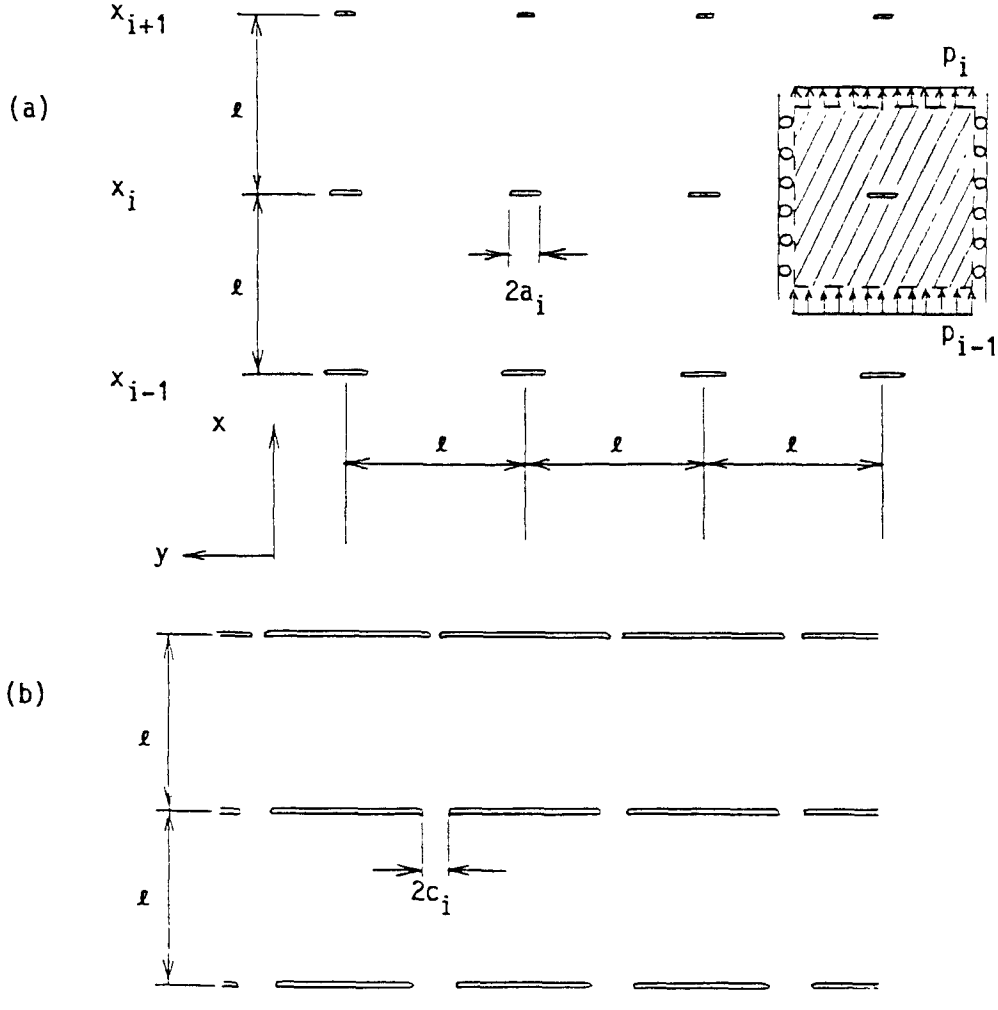


Fig 1 - Quasiperiodic Crack Array in Infinite Elastic Continuum (a) Small Penny-Shaped Cracks, (b) Small Circular Ligaments

varies from plane to plane. All the cracks are assumed to be small compared to $\ell$, i.e. $a_i \ll \ell$ for all i.

At $x \to -\infty$ we assume a fixed boundary, and at $x \to \infty$ a stress-free boundary. Small uniform normal distributed loads $p_i$ are applied on the planes $x = x_i + \ell/2$. As the boundary conditions at $y \to \pm\infty$, $z \to \pm\infty$, we assume that the boundary points are sliding on the planes parallel to x. These boundaries develop additional lateral normal stresses $\sigma'_z$ and $\sigma'_y$, which produce on the cracks stress intensity factors of zero values, and therefore they have no effect on the cracks. Consequently, these additional lateral stresses $\sigma'_y$ and $\sigma'_z$, which are superimposed on the stress field due loads $p_i$, are uniform in each layer $-\ell/2 < x < x_i + \ell/2$.

Due to the loads in the x-direction, there are additional nonuniform stresses (of all components) near each crack. If $a_i/\ell$ is sufficiently small, the normal stress $\sigma_x$ on each side of the plane $x_i + \ell/2$ is nearly uniform. Because of periodicity, each cell cross-hatched in Fig. 1a behaves as if its lateral boundaries were sliding. The average normal stress $\bar{\sigma}_x = \sigma_i$ on any plane x = const. is exactly the same for all x within the layer $x_i - \ell/2 < x < x_i + \ell/2$. Across the interfaces $x = x_i + \ell/2$ of these layers, $\sigma_x$ varies discontinuously if $p_i$ is nonzero. Note that the present boundary conditions and loading are introduced in such a manner that in absence of the cracks each layer is in a state of uniaxial strain, whose value varies from one layer to the next.

If the cracks are sufficiently small compared to $\ell$ (i.e., $a_i \ll \ell$), they do not interact and the formation of one crack does not release any appreciable amount of strain energy from the outside of the layer in which this crack is located. Therefore, the stress intensity factor $K_i$ is the same as for a single crack in an infinite elastic solid subjected to remote stress $\sigma_i$, which is [11, 12]:

$$K_i = 2\sigma_i \, (a_i/\pi)^{1/2} \tag{1}$$

Accordingly, the energy release rate per crack circumference $2\pi a_i$ is [13, 14]:

$$\frac{\partial W_i}{\partial a_i} = 2\pi a_i \frac{K_i^2}{E'} = \frac{8}{E'} \sigma_i^2 a_i^2 \tag{2}$$

where $E' = E / (1 - \nu^2)$, E = Young's modulus, and $\nu$ = Poisson's ratio. By integration, the total energy release per crack is:

$$W_i = \frac{8}{3E'} a_i^3 \sigma_i^2 \tag{3}$$

By Castigliano's theorem, the displacement $v_i$ due to the cracks lying in the plane $x = x_i$ is $v_i = \partial W_i / \partial P_i$ where $P_i$ = force resultant per crack. Since $P_i = \ell^2 \sigma_i$, we have

$$v_i = \frac{1}{\ell^2} \frac{\partial W_i}{\partial \sigma_i} = \frac{16\ a_i^3}{3E'\ell^2} \sigma_i \tag{4}$$

This represents the difference of displacements between the planes $x = x_i - \ell/2$ and $x = x_i + \ell/2$. The average normal strain due to cracks in the interval $(x_i - \ell/2,\ x_i + \ell/2)$ may be defined as.

$$\bar{\gamma}_i = \frac{v_i}{\ell} = \left(\frac{16\ a_i}{3E'\ \ell^2}\right)^3 \sigma_i \tag{5}$$

We may call it the cracking strain. Furthermore, assuming the cracks to be propagating, we must have $K_i = K_c$ = critical stress intensity factor of the material. Thus, from Eq. 1,

$$a_i = \frac{\pi\ K_c^2}{4\ \sigma_i^2} \tag{6}$$

Substituting this into Eq. 5, we obtain:

$$\sigma_i = \left(\frac{\pi^3\ K_c^6}{12E'\ell^3\bar{\gamma}_i}\right)^{1/5} = \frac{\bar{\gamma}_i}{C(\bar{\gamma}_i,\ \ell)} \tag{7}$$

in which function C represents the overall secant compliance due to cracks and is defined as·

$$C(\bar{\gamma}_i,\ \ell) = \frac{16}{3E'} \left(\frac{a_i(\bar{\gamma}_i)}{\ell}\right)^3 = \left(\frac{12\ E'\ \ell^3\ \bar{\gamma}_i^{\,6}}{\pi^3\ K_c^6}\right)^{1/5} \tag{8}$$

Eq. 7 yields a decreasing stress $\sigma_i$ at increasing strain $\bar{\gamma}_i$, i e. strain softening.

Note that compliance C depends on length $\ell$, which may be regarded as the characteristic length of the cracked material. Dependence of the compliance or stiffness on some characteristic length is a typical property of nonlocal materials.

## Homogenization by Macroscopic Continuum Approximation

The conditions of macroscopic equivalence of the actual cracked material and the homogenizing continuum have to be stated integrally for the basic, periodically repeated cell, in this case the layer $(x_i - \ell/2,$

$x_i + \ell/2$). The basic conditions we must impose are the compatibility of deformations over this cell and the equivalence of work. For our example, we can satisfy them exactly.

Compatibility of the macroscopic cracking strains $\gamma(x)$ with the displacement $v_i$ due to cracks requires that

$$v_i = \int_{x_i-\ell/2}^{x_i+\ell/2} \gamma(x)\ dx \tag{9}$$

which implies that

$$\bar{\gamma}_i = \frac{1}{\ell} \int_{x_i-\ell/2}^{x_i+\ell/2} \gamma(x)\ dx = \langle\gamma(x_i)\rangle \tag{10}$$

The pointed brackets < > denote the averaging operator, and the superimposed bar is a label for the averaged (nonlocal) quantities.

Equivalence of work done by the stresses in the layer ($x_i - \ell/2$, $x_i + \ell/2$) requires that

$$\sigma_i \delta v_i = \int_{x_i-\ell/2}^{x_i+\ell/2} \sigma(x)\ \delta\gamma(x)\ dx \tag{11}$$

where $\sigma(x)$ is the macroscopic continuum stress; and $\delta v_i$ and $\delta\gamma(x)$ are any variations of the displacement due to cracks and the macroscopic cracking strains which are kinematically compatible, i.e. satisfy Eq. 9. Substituting for $v_i$ from this equation, we obtain from Eq. 11:

$$\sigma_i \int_{x_i-\ell/2}^{x_i+\ell/2} \delta\gamma(x)\ dx = \int_{x_i-\ell/2}^{x_i+\ell/2} \sigma(x)\ \delta\gamma(x)\ dx \tag{12}$$

Since $\sigma_i$ is constant within the layer, Eq. 12 may be rewritten as

$$\int_{x_i-\ell/2}^{x_i+\ell/2} [\sigma_i - \sigma(x)]\ \delta\gamma(x)\ dx = 0 \tag{13}$$

and because this must hold generally for all possible variations $\delta\gamma(x)$, we have

$$\sigma(x) = \sigma_i \text{ for } x_i - \ell/2 < x < x_i + \ell/2 \tag{14}$$

Note that this equation for stresses involves no averaging integral while Eq. 10 for strains does. For the type of loading we assumed (i.e. distributed loads $p_i$ on planes $x = x_i + \ell/2$), $\sigma(x)$ is according to Eq. 14 a piecewise constant function. If the loads were not concentrated in these planes but distributed also over x, $\sigma(x)$ as well as $\sigma(x_i)$ would vary continuously and $\sigma(x)$ would not exactly equal $\sigma_i(x)$ (representing the

average of stresses $\sigma_x$ over any plane x = const ). The reason we assumed the applied loads to be concentrated into discrete planes was to construct an example for which the continuum homogenization conditions can be satisfied exactly.

The presence of the spatial averaging integral in Eq. 10 is what ultimately impresses on the macroscopic continuum a nonlocal character, as we will see. At the same time, the absence of the averaging integral from Eq 14 causes the nonlocal character to be restricted only to the cracking strains. In this regard it may be noted that in the theory of heterogeneous materials [15] the work equivalence condition is usually stipulated in a slightly different form, namely $\ell\sigma_i\epsilon^* = \int \sigma(x)\epsilon^* dx$ over an interval of length $\ell$, $\epsilon^*$ being an arbitrary uniform strain (virtual strain). This condition would imply that $\sigma_i = \int \sigma(x)dx/\ell$. However, it would not guarantee work equivalence for the actual displacements $v_i$ and the actual cracking strains $\gamma(x)$ Thus Eq. 14 appears better justified for the present case.

According to Eqs. 10 and 14, the relation $\bar{\gamma}_i = C(\bar{\gamma}_i, \ell)\sigma_i$ (Eq 7) has the following generalization.

$$\langle\gamma(x)\rangle = C\{\langle\gamma(x)\rangle, \ell\}\ \sigma(x) \tag{15}$$

We see that the continuum cracking strain in this stress-strain relation is nonlocal, i.e., is processed through a spatial averaging operator. However, this is not true of stress $\sigma(x)$.

The total displacement u in an elastic body with cracks represents a sum of the displacement obtained for the same body under the same loads if there are no cracks, and the additional displacement v due to the creation of the cracks while the loads are kept constant. Thus, the total relative displacement $u_i$ of the macroscopic continuum between the planes $x = x_i - \ell/2$ and $x = x_i + \ell/2$ is

$$u_i = \ell e(x_i) + v_i \tag{16}$$

where e is the elastic normal strain in the x-direction which is produced by stresses $\sigma(x)$ in the same elastic continuum if there are no cracks, and $\gamma(x)$ represents the increase of this elastic strain due to introduction of cracks of radii $a_i$. Strains e(x) are obtained by stress analysis of the continuum with no cracks, which is local.

Substituting $v_i = \ell\bar{\gamma}_i$ and defining the macroscopic strain at $x = x_i$ as $\epsilon(x_i) = u_i/\ell$, we get from Eq. 16 $\epsilon(x_i) = e(x_i) + \bar{\gamma}_i$, and generalizing this to any x, we obtain:

$$\epsilon(x) = e(x) + \langle\gamma(x)\rangle \tag{17}$$

So the total stress-strain relation for the normal x-components of stress and strain in the macroscopic continuum must have the form

$$\epsilon(x) = e(x) + C[\ \langle\gamma(x)\rangle,\ \ell]\ \sigma(x) \tag{18}$$

The elastic strain is determined as the strain in the continuum with no cracks. For the type of boundary conditions and loading that we introduced, the continuum with no cracks is in a state of uniaxial strain, i.e. all the strain components except $\epsilon_x$ are zero. Thus

$$e(x) = C_0\ \sigma(x) \tag{19}$$

where $C_0$ is the elastic compliance in uniaxial strain,

$$C_0 = \frac{(1+\nu)(1-2\nu)}{(1-\nu)\ E} \tag{20}$$

To sum up, the stress-strain relation for the macroscopic continuum should not be fully nonlocal. The elastic strains should be local, while the (macroscopically smoothed) strains due to cracking should be nonlocal. This property of the nonlocal formulation, introduced on the basis of numerical experience and intuition in Refs. 3-8, has been found to be essential for achieving well behaving finite element solutions.

It is possible to take an alternative approach to homogenization in which the relative total displacement $u_i = \ell C_0\sigma_i + \int \gamma(x)dx$ between $x = x_i \pm \ell/2$ is obtained before homogenization. Instead of Eq. 9 we may now impose the compatibility condition in the form $u_i = \int\epsilon(x)dx$ over the layer. Writing the work equivalence in the form $\sigma_i\delta u_i = \int \sigma(x)\delta\epsilon(x)\ dx$ instead of Eq. 11, we again recover Eq. 14 by the same procedure, however from the foregoing expressions for $u_i$ we obtain $\langle\epsilon(x)\rangle = e(x) + C[\ \langle\gamma(x)\rangle, \ell]\sigma(x)$. The fact that $\langle\gamma(x)\rangle$ is nonlocal and $e(x)$, $\sigma(x)$ are local agrees with Eq. 18 but the appearance of nonlocal $\langle\epsilon(x)\rangle$ disagrees. It would make the use of this stress-strain relation more complicated, probably unnecessarily so It should nevertheless be kept in mind that this aspect of homogenization is not without ambiguity.

The idea of nonlocal continuum was originally introduced without any reference to strain softening [16-22] and all strains and stresses were considered as nonlocal, i.e. the continuum was fully nonlocal. When this

idea was first proposed [2] to deal with strain-softening damage such as cracking, the stress-strain relation was also assumed to be fully nonlocal:

$$\langle\epsilon(x)\rangle = \langle e(x)\rangle + C[\ \langle\epsilon(x)\rangle - \langle e(x)\rangle,\ \ell]\ \langle\sigma(x)\rangle \tag{21}$$

This stress-strain relation, whose finite element implementation leads to the so-called imbricate medium and an imbricated finite element system [2], was found to cause various problems in finite element programming, due to difficulties in the implementation of the boundary conditions and interface conditions as well as the existence of certain spurious zero-energy instability modes. Thus, even though these difficulties have been overcome by certain artificial devices [23], this original version of the nonlocal formulation for strain-softening is complicated and less than satisfactory from the theoretical viewpoint. From the comparison of Eqs. 18 and 21 we now understand why.

To recapitulate, the elastic strain e is the strain of the continuum with no cracks, and so it must be local. On the other hand the macroscopic cracking strain must be defined by averaging (smoothing) of the effect of the discrete cracks in order to satisfy the strain compatibility requirement, and so it must be nonlocal. The basic reason is that the displacement due to a crack (Eq. 4) is defined with a unique value only at a sufficient distance from the crack.

## Nonlocal Continuum Damage Mechanics

In continuum damage mechanics [24-27, 1], the stress-strain relation is written in the form:

$$\epsilon(x) = \frac{C_0}{1 - \Omega(x)}\ \sigma(x) \tag{22}$$

where $\Omega(x)$ is called damage and $C_0$ is the elastic compliance for the continuum with no damage ($\Omega = 0$), i.e. no cracks. Comparison of Eqs. 18 and 22 and substitution $e(x) = C_0\sigma(x)$ yields for the damage the expression:

$$\Omega(x) = 1 - \frac{C_0}{C[\ \langle\gamma(x)\rangle,\ \ell] + C_0} = f(\ \langle\gamma(x)\rangle\ ) \tag{23}$$

i.e. the damage is a function, f, of the average cracking strain rather than the local cracking strain.

From Eq. 23, it is clear that continuum damage $\Omega$ should be considered to be a nonlocal variable. Moreover, Eq. 23 shows that $\Omega$ should be a function of the average of the macroscopic cracking strain $\gamma(x)$ Such a definition was introduced for reasons of proper convergence at mesh refinements in Refs. 3, 4 and 6. Ref. 3 also considered an alternative in which $\Omega$ is obtained as the average of the local damage energy release rate Y (or local damage $\omega$) that depends on local $\gamma$. This is equivalent only approximately, not exactly. The same is true when $\Omega$ is considered to be a function of $\langle\epsilon(x)\rangle$ rather than $\langle\gamma(x)\rangle$, as done in Ref. 8 for other reasons.

The definition of damage in Eq.23, along with the stress-strain relation in Eq. 18 in which $\epsilon$, $\epsilon^e$ and $\sigma$ are local, was found to yield excellent results in various finite element applications. The solutions were found to converge properly on mesh refinement [3, 4-6] and the numerical implementation proved to be quite easy even in large finite element programs [5, 7, 8]. Moreover, the nonlocal version of such programs provided faster convergence than the local version. As one application, the problem of cave-in induced by compressive strain softening at the sides of a subway tunnel excavated in a cement-grouted soil was solved using meshes with up to 3248 degrees of freedom [7].

If $\gamma(x)$ varies so slowly that the change of $\gamma$ over distance $\ell$ would be negligible, then of course $\langle\gamma(x)\rangle$ can be replaced by $\gamma(x)$, and the damage is then local. However, since strain softening causes strain localization, such a slow variation of $\gamma(x)$ cannot be assumed to occur for all x. This is why a local treatment of strain-softening damage is always inadequate.

## Other Crack Systems and Generalizations

Many other types of crack systems can be analysed similarly. As one further example consider the terminal stage of damage in which the planes $x = x_i$ are fully cracked except for small circular ligaments of diameters $2c_i$ (Fig. 1b). The centers of these ligaments coincide again with the nodes of a cubic lattice of step $\ell$, and the ligaments are assumed to be very small compared to $\ell$, i.e. $c_i \ll \ell$ for all i. This problem was analysed for a different purpose in Ref. 28. It was shown that (for $K_i$ =

$K_c$) the curve $v_i$ ($\sigma_i$) attains a maximum of $v_i$ after which it exhibits a snapback, whose final stage is described by the equation

$$\sigma_i = \frac{E'^3}{4\pi K_c^2 \ell^2} v_i^3 \tag{24}$$

Again $v_i$ is the relative displacement between the planes $x = x_i - \ell/2$ and $x = x_i + \ell/2$.

The average normal strain due to cracks in the interval ($x_i - \ell/2$, $x_i + \ell/2$) may again be defined as $\bar{\gamma}_i = v_i/\ell$. Eq 24 may then be written as

$$\sigma_i = \frac{\bar{\gamma}_i}{C(\bar{\gamma}_i, \ell)} \tag{25}$$

in which

$$C(\bar{\gamma}_i, \ell) = \frac{E'^3}{4\pi K_c^2} \ell \bar{\gamma}_i^2 \tag{26}$$

To satisfy the requirement of compatibility of displacement $v_i$ with the macroscopic cracking strain $\gamma(x)$, the value of $\bar{\gamma}_i$ may be replaced by $\langle\gamma(x)\rangle$, and further analysis is the same as before (Eqs 9-20), with the same conclusions

The detailed form of the compliance function $C(\bar{\gamma}_i, \ell)$, as defined by Eq. 8 or 26 is for our conclusions about the nonlocal aspects obviously unimportant. What is important is that C depends on the characteristic length $\ell$, and even more that it depends on the average cracking strain $\bar{\gamma}_i$ which is macroscopically equivalent to $\langle\gamma(x)\rangle$ rather than $\gamma(x)$. Various other types of crack arrays should therefore lead to similar conclusions.

The preceding analysis neglected the microscopic heterogeneity of the material, such as the differences in elastic moduli between the aggregate and the matrix found in concrete. This heterogeneity no doubt has a large influence but it alone might not necessitate a nonlocal treatment According to numerical experience, the nonlocal approach is required only if strain softening takes place, and strain softening is the consequence of microcracking, void growth or other damage

The crack spacing in real materials is randomly irregular. To model this, one might consider a certain statistical distribution of lengths $\ell$. Accordingly, one could introduce a certain weighting function in the averaging integral (operator < >) in Eq. 18. This has already been done in Refs 3-8 for reason of numerical efficiency. The analysis should also be

generalized to three dimensions, although this would bring about much greater complexity.

The number of cracks increases during the progress of damage, and this reduces their average spacing. This might require the characteristic length to be considered as a variable. The question would then arise as to what is the proper domain over which the averages should be taken. Also, it needs to be recognized that there are other sources of softening damage than cracking, e.g., void nucleation and growth in metals, or interface slips with softening. Further extensions would be needed for such problems.

## Conclusion

1. Based on the analysis of an array of small penny-shaped cracks it appears that the damage variable in continuum damage mechanics should be treated as nonlocal while the elastic part of the response should be local.

2. The nonlocal damage should be formulated as a function of the spatial average of the cracking strain over a zone whose size, representing the characteristic length of the macroscopic continuum, coincides with the crack spacing.

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