Sampling analysis of concrete structures for creep and shrinkage with correlated random material parameters

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The latin hypercube sampling method, which represents the most efficient way to determine the statistics of the creep and shrinkage response of structures, has previously been developed and used under the assumption that the random parameters of the creep and shrinkage prediction model are mutually independent. In reality they are correlated. On the basis of existing data, this paper establishes, by means of the method of maximum likelihood, the joint multivariate probability distribution of the random parameters involved, tests the hypothesis of mutual dependence of parameters on the basis of the χ^2-distribution, and generalizes the latin hypercube sampling method to the case of correlated multinormal random parameters. The generalization is accomplished by an orthogonal matrix transformation of the random parameters based on the eigenvectors of the inverse of the covariance matrix. This yields a set of new random parameters which are uncorrelated (independent) and can be subjected to the ordinary latin hypercube sampling, with samples of equal probabilities. Numerical examples of statistical prediction of creep and shrinkage effects in structures confirm the practical feasibility of the method and reveal a good agreement with the scatter observed in some previous experiments.

INTRODUCTION

To estimate the uncertainty of the creep and shrinkage effects in structures, the uncertainties of the parameters which influence creep and shrinkage must be determined first. Although there are many such parameters, eight of them appear to be sufficient[3,15]. They include the mix composition parameters, the strength of concrete, the environmental humidity, and several uncertainty factors in the shrinkage prediction model.

To determine the stochastic response characteristics of complex and large structures with a large number of random parameters, a host of methods have been studied[1–3,15]. Among them, the latin hypercube sampling method (a special case of stratified random sampling), which was developed by McKay and co-workers[16–19,26], is probably most efficient and general. With this method, a relatively small number of computer runs of deterministic structural creep and shrinkage analyses for chosen samples of the parameter values suffices to obtain the statistics of the response. This method is more efficient than the simple random sampling and is applicable even if the number of parameters is quite large. This method has been applied to concrete problems in Refs 3 and 13.

In all of the previous studies with this method, the random parameters were assumed to be mutually independent. Also, McKay gave his detailed proof of the superiority of latin hypercube sampling only for the case of independent random variables.

In reality, however, many of the random parameters involved are interrelated with each other. But a dependent multivariate analysis would make the probabilistic problem very complicated, and obviously through the assumption of independence the form of the joint distribution and probability density could be exceedingly simplified. On the other hand, sometimes it is impossible to establish a functional dependence among these parameters because of the lack of experimental evidence. These are the reasons why the previous investigators were willing to assume random variables to be mutually independent. A further reason is that the assumption of independence is on the safe side if the degree of correlation is unknown.

The present paper has a twofold purpose: (1) To establish and test a joint probability distribution for the eight random parameters of the creep and shrinkage prediction model, using the available experiment data, and (2) to generalize the latin hypercube sampling method to the case of correlated multinormal random parameters.

We will follow the common practice assuming the joint multivariate probability distribution to be normal. Then we will determine the correlation coefficients of the distribution by the method of maximum likelihood. Further, we will test the hypothesis of correlation to make sure that the variables in the multivariate normal distribution obtained cannot be completely independent. Finally, after extending the latin hypercube sampling method to correlated random parameters, we will present numerical examples of various applications to structures and comparison with experimental data.

It should be kept in mind that, even with the present generalization, other stochastic aspects of creep will still be treated in a simplified form. The environmental humidity is in reality a random process in time. Likewise, the creep and shrinkage strains are processes with random increments in time. These aspects have been treated elsewhere[10,11,27]

JOINT PROBABILITY DISTRIBUTION

We consider a creep and shrinkage prediction model involving a set of N parameters x_i $(i=1,2,\ldots,N)$ which may be considered as random. For the BP Model[4–6] we have eight random parameters $(N=8)$. These parameters may be assembled into a vector (column matrix) $\mathbf{x}$ and are defined as[3,15,27]:

$$\mathbf{x}^T=(x_1,x_2,\ldots,x_k)=(\psi_1,\psi_2,\psi_3,h,f_c',w/c,g/c,c) \quad (1)$$

Here T denotes a transpose, parameters ψ_i reflect the uncertainties of the creep and shrinkage model, ψ_1 is for shrinkage, ψ_2 for basic creep, and ψ_3 for drying creep; h is the relative environmental humidity, f_c' is the standard cylinder strength of concrete at 28 days, w/c is the water–cement ratio, g/c is the gravel–cement ratio (both ratios by weights), and c is the cement content of the concrete (in kg/m^3). The components of vector $\mathbf{x}$ are continuous one-dimensional random variables which are, in general, correlated. We assume these variables to be characterized by

$$f(x_1,\ldots,x_k)=(2\pi)^{-k/2}|\mathbf{C}|^{-1/2}\exp[-0.5(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})] \quad (2)$$

in which $\boldsymbol{\mu}$ is the mean vector of $\mathbf{x}$; $\mathbf{C}$ is the covariance matrix of the multinormal population of x_i values which is real, symmetric and positive definite. This k-dimensional joint probability density function is characterized by k mean values $\bar{\mu}_i$, k variances $V(x_i)$, and $k(k-1)/2$ correlation coefficients C_{ij}. In the special case that all the off-diagonal correlation coefficients are zero, i.e. matrix $\mathbf{C}$ degenerates to a diagonal matrix ($C_{ij}=0$ for $i\neq j$), the random variables x_i are mutually uncorrelated, or independent. For the case of multinormal distribution, the situations of uncorrelated and independent variables are equivalent.

The mean and variance of each random parameter x_i can be estimated by using the outcomes of one-dimensional sampling tests. For example, we can get the sample mean and variance for ψ_1, ψ_2 and ψ_3 by comparing the sampling test results with theoretical results, without taking other random parameters into account. The correlation coefficients, however, have to be estimated by multidimensional sampling tests.

The data in Table 1, which summarizes the basic experimental data on creep and shrinkage from Ref. 4, may be regarded as the outcomes of multidimensional sampling tests to estimate the correlation coefficient matrix of the sample. In this table, the definition of ψ_i is:

$$\psi_i=(\text{test result})/(\text{theoretical result}) \quad (3)$$

The test results may be represented by the measured curve (which should be hand-smoothed for reasons explained in Ref. 4). The theoretical results are obtained from some creep and shrinkage prediction models such as the BP Model[4–6]. ψ_i represent the time-averaged mean values estimated at discrete times, which are suitably chosen as uniformly spaced in the log-time scale. Usually one or two time points per decade in the log-time scale suffice. The definition in equation (3) differs from that in Refs 3–6 and 15; it has some advantages such as simplicity, as well as disadvantages. It is particularly suitable for comparing specimens of different types of concrete, e.g. different strengths. Compared to basing the statistics on the errors, defined as the differences (rather than ratios) between the measured and predicted values, one disadvantage of the definition in equation (3) is that it overemphasizes the errors when the theoretical values are at the small value side of the range.

Before starting the statistical analysis, it is worth pointing out that there are two different kinds of random deviations for variables h, f_c', w/c, g/c and c. The first one results from human choice; for example, in one test we may choose $w/c=0.50$, $g/c=3.00$ and want $f_c'=5000$ psi, whilst in another test we may take $w/c=0.4$, g/c as 3.4 and want $f_c'=4000$ psi. The test data in Table 1 include this kind of random deviations. The second is the uncertainty or random variability due to imperfect (subjective) human control. It can be regarded as a deviation in quality. For example, we choose w/c to be 0.5, but when the aggregate is too wet, the real w/c may be 0.55 because of the water contained in the aggregate. Obviously the first kind of deviation leads to a wider dispersion than the second. As a result, the variance of the first kind of deviations will be larger than second. Hence it would be meaningless to estimate the variances corresponding to deviations in quality based on Table 1.

The correlation coefficient, however, is a different probabilistic measure than variance. It is mainly a measure of the extent of linear dependence between two random variables. So, it appears we may employ a set of test data as a sample to estimate the correlation coefficient provided that it can cover the relative random variability of the variables, as Table 1 does. It may be noted that in the case of nonlinear complete dependence between two variables, the correlation coefficient fails to be a good measure. There are other measures of correlation, such as the contingency coefficient[20], which can be used in the case of nonlinear dependence. However, that is beyond the scope of this paper.

Although what we are really concerned with in practice are the deviations in quality, no pertinent test data have been collected up to now. The data should not only reflect deviations in multidimensional correlations between pairs of random variables but also represent accurately the random variability in quality. Only such kind of data can yield both the variances and the correlation coefficients. Therefore, this paper will focus only on estimation of the correlation coefficient, and leave determination of the variances of h, f_c', w/c, g/c and c to one-dimensional sampling tests.

Another point to mention is that the validity of our estimators and the test of hypothesis which follows is affected by the nature of randomness of the sampling. In practice, randomness means that the sampling units should be drawn independently of one another from a homogeneous population. The units must not have common characteristics which might induce dependence among their vectors. Even though some of the random parameters in the existing experimental data are designed systematically by the investigators, e.g., h, w/c and so forth, still every individual specimen can reasonably be considered to have been tested independently. So we can suppose that the requirements of independence and homogeneity have approximately been met by the sampling units in our sample.

Table 1. Experimental data on creep and shrinkage

No.	n_i	ψ_1	ψ_2	ψ_3	h	f'_c	w/c	g/c	c
1	1.00	0.978			50.0	41.37	0.71	2.70	362.0
2	8.00	0.781			55.0	36.99	0.51	3.37	350.0
3	1.00	0.775			35.0	36.99	0.49	3.07	350.0
4	1.00	0.788			50.0	36.99	0.49	3.07	350.0
5	1.00	0.927			75.0	36.99	0.49	3.07	350.0
6	1.00	0.966			50.0	17.24	0.59	3.67	
7	1.00	0.985			70.0	17.24	0.59	3.67	
8	6.00	0.890			50.0		0.50	2.40	472.0
9	1.00	0.857			50.0	50.47	0.43	2.62	404.0
10	1.00	1.013			60.0	42.75	0.43	2.62	404.0
11	6.00	1.025			20.0	45.16	0.46	2.07	450.0
12	11.0	1.018			50.0	45.16	0.46	2.07	450.0
13	3.00	1.255			75.0	45.16	0.46	2.07	450.0
14	1.00	0.992			55.0	39.10	0.50	2.48	400.0
15	1.00	1.300			65.0	40.60	0.38		350.0
16	1.00	0.646			65.0	28.00	0.55		334.0
17	1.00	0.450			65.0	34.10	0.55	2.27	336.0
18	5.00		0.822			36.28	0.49	3.07	350.0
19	4.00		1.117			22.26	0.58	7.10	
20	5.00		1.192			14.34	0.56	4.42	264.7
21	5.00		0.772			20.13	0.50	10.4	
22	5.00		1.284			34.30	0.56	7.14	
23	4.00		1.200			40.00	0.42	2.59	
24	5.00		0.761			44.10	0.38	2.80	
25	1.00		1.006			35.68	0.41	3.15	
26	1.00		1.054			49.40	0.49	2.98	418.0
27	1.00		0.880			43.40	0.43	2.62	404.0
28	1.00		0.893			30.00	0.85	3.81	253.0
29	1.00		0.880			35.90	0.85	3.81	253.0
30	1.00		1.363			45.20	0.46	2.07	450.0
31	1.00		0.890			42.80	0.43	2.62	404.0
32	4.00		1.087			33.40	0.70	3.06	
33	5.00			0.785	50.0	36.30	0.49	3.07	350.0
34	1.00			0.902	50.0	36.30	0.49	3.07	350.0
35	1.00			0.946	75.0	36.30	0.49	3.07	350.0
36	1.00			0.861	99.0	36.30	0.49	3.07	350.0
37	1.00			1.038	50.0	17.20	0.59	3.67	
38	1.00			1.046	75.0	17.20	0.59	3.67	
39	1.00			1.047	99.0	17.20	0.59	3.67	
40	1.00			0.989	65.0	39.04	0.56	4.00	275.0
41	6.00			0.813	50.0	41.40	0.71	2.70	362.0
42	1.00			1.046	65.0	40.60	0.38		350.0
43	3.00			1.061	65.0	42.70	0.55		334.0
44	1.00			1.053	50.0	49.40	0.49	2.98	418.0
45	1.00			0.879	60.0	45.90	0.43	2.62	404.0
46	3.00			1.248	20.0	45.16	0.46	2.07	451.2
47	3.00			1.272	50.0	45.16	0.46	2.07	451.2
48	3.00			1.269	70.0	45.16	0.46	2.07	451.2
49	1.00			0.890	60.0	26.08	0.61	4.47	270.0
50	1.00			0.830	60.0	32.40	0.50	3.56	350.0

Table 1—continued

No.	n_i	ψ_1	ψ_2	ψ_3	h	f'_c	w/c	g/c	c
51	1.00			0.831	60.0	42.60	0.35	3.49	400.0
52	1.00			1.049	60.0	42.20	0.45	3.55	360.0
53	1.00			0.862	60.0	38.10	0.58	3.04	400.0
54	1.00			1.111	60.0	37.40	0.40	3.17	400.0
55	1.00			1.004	60.0	40.96	0.45	3.17	400.0
56	1.00			0.822	60.0	41.60	0.43	2.82	450.0
57	1.00			0.952	95.0	39.80	0.48	3.71	350.0
58	1.00			0.833	95.0	45.30	0.49	3.59	350.0
59	1.00			0.781	95.0	45.80	0.47	3.98	362.0
60	1.00			1.090	95.0	36.20	0.52	3.71	350.0
61	2.00			0.923	50.0	35.90	0.85	3.81	253.0
62	1.00			0.913	50.0	43.40	0.43	2.62	404.0
63	1.00			1.047	98.0		0.40		
64	1.00			1.036	90.5		0.40		
65	1.00			0.943	80.5		0.40		
66	1.00			0.919	71.0		0.40		
67	1.00			0.933	53.0		0.40		
68	1.00			1.003	42.5		0.40		
69	1.00			1.053	25.2		0.40		
70	1.00			1.113	0.01		0.40		
71	1.00		0.805			51.60	0.47	2.66	
72	1.00		0.794			51.60	0.47	2.66	
73	1.00		0.768			51.60	0.47	2.66	
74	1.00		0.737			51.60	0.47	2.66	
75	1.00		1.105			45.00	0.40	3.22	377.0
76	1.00		0.980			45.00	0.40	3.22	377.0
77	1.00		0.746			43.40	0.43	2.62	404.0
78	1.00		0.850			43.40	0.43	2.62	404.0
79	1.00		0.849			33.12	0.56	2.63	
80	1.00		0.915			33.12	0.56	2.63	
81	1.00		0.883			33.12	0.56	2.63	
82	1.00		0.897			33.12	0.56	2.63	
83	1.00		0.998			28.80	0.45	4.00	
84	1.00		1.030			28.80	0.45	4.00	
85	1.00		1.028			28.80	0.45	4.00	
86	1.00		1.039			28.80	0.45	4.00	
87	1.00		1.073			28.80	0.45	4.00	
88	1.00		1.109			28.80	0.45	4.00	
89	1.00		0.911			28.80	0.45	4.00	
90	1.00		1.269			17.60	0.70	3.50	
91	1.00		1.231			17.60	0.70	3.50	
92	1.00		1.416			17.60	0.70	3.50	
93	1.00		1.250			17.60	0.70	3.50	
94	1.00		0.703			45.20	0.43	2.62	
95	1.00		0.813			45.20	0.43	2.62	404.0
96	3.00		1.559			23.28	0.53	3.84	314.6
97	3.00		1.328			23.28	0.50	3.84	314.6
98	1.00		0.680			39.00	0.60		320.0
99	1.00		0.996			39.00	0.60		320.0
100	1.00		0.488			50.00	0.60		320.0

Table 1—continued

No.	n_i	ψ_1	ψ_2	ψ_3	h	f'_c	w/c	g/c	c
101	1.00		0.624			50.00	0.60		320.0
102	1.00		0.864			50.00	0.60		320.0
103	1.00		0.809			50.00	0.60		320.0
104	1.00		0.971			42.17	0.46	2.25	450.0
105	1.00		0.965			42.17	0.46	2.25	450.0
106	2.00		1.131			43.60	0.40	3.83	343.0
107	2.00		1.177			43.60	0.40	3.83	343.0
108	2.00		1.030			43.60	0.40	3.83	343.0
109	4.00		1.203			40.00	0.42	2.95	
110	4.00		1.272			40.00	0.42	2.95	
111	5.00		1.391			40.00	0.42	2.95	
112	5.00		1.105			40.00	0.42	2.95	
113	6.00		1.053			45.40	0.38	2.61	419.0
114	6.00		1.044			45.40	0.38	2.61	419.0
115	6.00		0.951			45.40	0.38	2.61	419.0
116	2.00			1.240	3.00	43.60	0.40	3.83	343.0
117	2.00			1.446	28.0	43.60	0.40	3.83	343.0
118	2.00			1.347	50.0	43.60	0.40	3.83	343.0
119	1.00			0.937	1.00	33.12	0.56	2.63	380.0
120	1.00			0.783	3.00	33.12	0.56	2.63	380.0
121	1.00			1.025	25.0	33.12	0.56	2.63	380.0
122	1.00			1.054	70.0	33.12	0.56	2.63	380.0
123	1.00			0.943	1.00	51.70	0.47	3.86	332.0
124	1.00			0.961	1.00	51.70	0.47	3.86	332.0
125	1.00			0.833	3.00	51.70	0.47	3.86	332.0
126	1.00			0.846	4.00	51.70	0.47	3.86	332.0
127	1.00			0.819	50.0	51.70	0.47	3.86	332.0
128	1.00			0.946	50.0	33.60	0.60	2.39	350.0
129	1.00			1.044	50.0	33.60	0.60	2.39	350.0
130	1.00			1.223	50.0	33.60	0.60	2.39	350.0
131	1.00			1.211	50.0	33.60	0.60	2.39	350.0
132	1.00			0.921	0.10	37.90	0.50	2.80	400.0
133	1.00			0.744	1.00	37.90	0.50	2.80	400.0
134	1.00			0.725	3.00	37.90	0.50	2.80	400.0
135	1.00			0.993	35.0	37.90	0.50	2.80	400.0
136	1.00			0.937	50.0	37.90	0.50	2.80	400.0
137	1.00	0.983			60.0	42.75	0.43	2.62	404.0

h in %, f'_c in megapascals, c in kg/m^3, T in degrees Celsius.

Now let us write Table 1 as a data matrix:

$$\mathbf{X}=\begin{bmatrix} x_{11} & \cdots & \cdots & x_{1k} \\ \cdots & & \cdots & \cdots \\ x_{N1} & \cdots & \cdots & x_{NK} \end{bmatrix}=\begin{bmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_i^T \\ \vdots \\ \mathbf{x}_N^T \end{bmatrix} \tag{4}$$

Each row stands for one specimen in the creep and shrinkage test and is considered to be a sampling unit drawn from one and the same multinormal population, $\mathbf{x}_i$ is the column vectors in data matrix $\mathbf{X}$. According to the method of maximum likelihood, equation (4) represents one set of realizations of an 8-dimensional random variable governed by the multinormal density function (2) with mean vector $\boldsymbol{\mu}$ and nonsingular covariance matrix $\mathbf{C}$. So the likelihood of the observations (4) is

$$L(\boldsymbol{\mu},\mathbf{C})=[(2\pi)^k|\mathbf{C}|]^{-N/2}\exp\left[-\frac{1}{2}\sum_{i=1}^{N}(\mathbf{x}_i-\boldsymbol{\mu})^T\mathbf{C}^{-1}(\mathbf{x}_i-\boldsymbol{\mu})\right] \tag{5}$$

The maximum-likelihood estimators of the mean vector and the covariance matrix are

$$\hat{\boldsymbol{\mu}}=\bar{\mathbf{x}}=\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i \tag{6}$$

$$\hat{\mathbf{C}}=\frac{1}{N}\sum_{i=1}^{N}(\mathbf{x}_i-\hat{\boldsymbol{\mu}})(\mathbf{x}_i-\hat{\boldsymbol{\mu}})^T \tag{7}$$

The unbiased estimate of the population covariance matrix is

$$\mathbf{S}=\frac{1}{N-1}\sum_{i=1}^{N}(\mathbf{x}_i-\hat{\boldsymbol{\mu}})(\mathbf{x}_i-\hat{\boldsymbol{\mu}})^T \tag{8}$$

$\mathbf{S}$ is often referred to as the sample covariance matrix. The matrix of correlation coefficients is

$$\mathbf{R}=\mathbf{DCD} \tag{9}$$

where $\mathbf{D}$ is a $k\times k$ diagonal matrix whose diagonal elements are $1/\sqrt{c_{ii}}$; coefficients c_{ii} are the diagonal elements of matrix (7). The correlation coefficients are the most important parameters in analyzing correlations among random variables.

Before calculating the correlation coefficients, there are certain questions to discuss. In the data table, there are some test data where each of the eight random components coincides with that for another specimen. In such a case we simply consider the specimen to be a repeated sampling unit. The numbers in column 2 of Table 1 indicate the repeat times of every sampling unit. Because of these repetitive sampling units, the size of our sample is not the total number of the rows of Table 1. Rather, it is the sum of all the numbers, n_i, in column 2 of Table 1. Hence, the estimates (6), (7) become

$$\hat{\boldsymbol{\mu}}=\bar{\mathbf{x}}=\frac{1}{N}\sum_{i=1}^{m} n_i\mathbf{x}_i \tag{10}$$

$$\hat{\mathbf{C}}=\frac{1}{N}\sum_{i=1}^{m} n_i(\mathbf{x}_i-\hat{\boldsymbol{\mu}})(\mathbf{x}_i-\hat{\boldsymbol{\mu}})^T \tag{11}$$

where

$$N=\sum_{i=1}^{m} n_i \tag{12}$$

In Table 1 one can notice many blanks. There are two reasons for these blanks. One is that the data were not reported (e.g., the blanks in the columns g/c, c, f_c'). Another is that some random parameters are uncorrelated; for example, the first 17 rows in Table 1 correspond to the shrinkage experiments for which parameters ψ_2, ψ_3 are irrelevant since they characterize creep. The sample mean (10) can be calculated ignoring the blanks. For estimator $\hat{\mathbf{C}}$, however, we have to consider their effect. To eliminate any effect of these blanks on the estimator of $\hat{\mathbf{C}}$, the best way is to fill the blanks with sample mean $\bar{\mathbf{x}}$. From (11), we can see that this will cause the blanks to have no contributions to $\hat{\mathbf{C}}$, which is just what we want.

Using the method just described, we can obtain the following mean vector and correlation coefficient matrix of the sample:

In vector (13) the first three elements are the mean values of ψ_i; we can see that they are close to 1.0 (if the prediction model were perfect, they would have to be 1.0). In (14), the first three diagonal elements are the variances of ψ_i. They are much smaller than those indicated in Ref. 1. We can confirm that the BP formulae agree quite well with test data and can predict deformations of creep and shrinkage quite accurately in the time average sense.

In matrix (15), the correlation is said to be positive when $r_{ij}>0$ and negative when $r_{ij}<0$. A positive correlation means that the variables x_i and x_j increase simultaneously, while a negative correlation means that if x_j decreases then x_i increases. For example, the value $r_{65}=-0.45$ means that the strength f_c' decreases with an increasing ratio w/c.

We also see that none of the absolute values of the off-diagonal elements in matrix (15) is equal to 1 or even close to 1. This means that the variables in vector (1) are nonlinearly correlated. This is one reason why we adopt the latin hypercube sampling. This type of sampling approach can, in general, better capture the nonlinear correlation of random parameters.

TEST OF HYPOTHESIS

Now we have obtained the estimate for the correlation coefficient matrix. But it is possible that all the correlation coefficients of the population may be equal to zero. This can happen if we misjudge the properties of the normal population for only one sample (Table 1). If this happens, our following analysis, based upon the sample correlation coefficient matrix, would be in vain or even misleading. Therefore it is prudent to carry out a test of the hypothesis

$$\mathbf{R}=\mathbf{I} \tag{16}$$

where $\mathbf{I}$ is the unit matrix.

$$\hat{\boldsymbol{\mu}}^T=\bar{\mathbf{x}}^T=(0.944, 1.051, 1.0004, 0.498, 38.29, 0.491, 3.3, 379.78) \tag{13}$$

$$\hat{\mathbf{C}}=\begin{bmatrix} 0.0046 & 0.0 & 0.0 & -0.00024 & 0.086 & -0.00067 & -0.008 & 0.8 \\ 0.0 & 0.022 & 0.0 & 0.0 & -0.36 & 0.0004 & -0.0023 & -0.44 \\ 0.0 & 0.0 & 0.01 & -0.065 & 0.054 & -0.002 & -0.005 & 0.68 \\ -0.00024 & 0.0 & -0.065 & 260.5 & -19.7 & 0.062 & 0.79 & -47.7 \\ 0.086 & -0.36 & 0.054 & -19.7 & 73.8 & -0.36 & -6.41 & 152.3 \\ -0.00067 & 0.0004 & -0.002 & 0.062 & -0.36 & 0.009 & 0.023 & -1.83 \\ -0.008 & -0.0023 & -0.005 & 0.79 & -6.41 & 0.023 & 1.95 & -21.7 \\ 0.8 & -0.44 & 0.68 & -47.7 & 152.3 & -1.83 & -21.7 & 2095 \end{bmatrix} \tag{14}$$

$$\hat{\mathbf{R}}=\begin{bmatrix} 1.0 & 0.0 & 0.0 & -0.00022 & 0.15 & -0.11 & -0.085 & 0.26 \\ 0.0 & 1.0 & 0.0 & 0.0 & -0.28 & 0.028 & -0.011 & -0.065 \\ 0.0 & 0.0 & 1.0 & -0.04 & 0.062 & -0.2 & -0.035 & 0.15 \\ -0.00022 & 0.0 & -0.04 & 1.0 & -0.14 & 0.041 & 0.035 & -0.065 \\ 0.15 & -0.28 & 0.062 & -0.14 & 1.0 & -0.45 & -0.53 & 0.39 \\ -0.11 & 0.028 & -0.2 & 0.041 & -0.45 & 1.0 & 0.18 & -0.42 \\ -0.085 & -0.011 & -0.035 & 0.035 & -0.53 & 0.18 & 1.0 & -0.34 \\ 0.26 & -0.065 & 0.15 & -0.065 & 0.39 & -0.42 & -0.34 & 1.0 \end{bmatrix} \tag{15}$$

A statement equivalent to (16) is that $\mathbf{C}$ is a diagonal matrix. The opposite hypothesis is

$$\mathbf{R} \neq \mathbf{I} \tag{17}$$

This hypothesis means that in the correlation coefficient matrix at least one off-diagonal element is nonzero, that is to say, the test is one of complete independence as opposed to some dependence among the random variables because we have assumed the variates to be multinormal.

We use the generalized likelihood-ratio criterion[21] to test the hypothesis. The likelihood of the random sample is given by (5). When the estimates $\hat{\boldsymbol{\mu}}, \hat{\mathbf{R}}$ of the mean vector and the population correlation matrix are inserted into the likelihood, its maximized value becomes

$$\begin{aligned} L(\hat{\Omega}) &= L(\hat{\boldsymbol{\mu}}, \hat{\mathbf{C}}) \\ &= (2\pi)^{-Nk/2}|\hat{\mathbf{C}}|^{-N/2} \exp\left[-\frac{1}{2}\sum_{i=1}^{N}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^T\hat{\mathbf{C}}^{-1}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})\right] \\ &= (2\pi)^{-Nk/2}|\hat{\mathbf{C}}|^{-N/2}e^{-Nk/2} \end{aligned} \tag{18}$$

The parameter space Ω of the distribution is that region of the $k(k+3)/2$-dimensional Euclidean space in which the components of $\hat{\boldsymbol{\mu}}$ are finite and those of $\hat{\mathbf{C}}$ constitute a symmetric positive definite matrix. The subspace ω corresponding to the null hypothesis (16) is the $2k$-dimensional region of Ω for which

$$\left.\begin{array}{ll} -\infty < \hat{\mu}_i < \infty & \\ 0 < \hat{C}_{ii}^2 < \infty & \text{for all } i \\ \hat{C}_{ij} = 0 & \text{for } i \neq j \end{array}\right\} \tag{19}$$

The likelihood of the sample in the subspace has the same form as matrix (5). Only the correlation matrix of the sample is a diagonal matrix, i.e.,

$$L(\hat{\omega}) = L(\hat{\boldsymbol{\mu}}, \hat{\mathbf{C}}) = (2\pi)^{-Nk/2}\left(\prod_{i=1}^{k}\hat{C}_{ii}^2\right)^{-N/2} e^{-kN/2} \tag{20}$$

The test statistic is

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{|\hat{\mathbf{C}}|^{N/2}}{\left(\prod_{i=1}^{k}\hat{C}_{ii}^2\right)^{N/2}} = |\mathbf{R}|^{N/2} \tag{21}$$

The difference of the dimensionalities of Ω and ω is $k(k-1)/2$, and we can use the large-sample chi-squared distribution

$$\chi^2 = -\left(N - 1 - \frac{2k+5}{6}\right)\ln|\mathbf{R}| \tag{22}$$

to obtain a decision rule for accepting or rejecting the independence hypothesis at a specified significance level. For a test of level α we should accept the hypothesis (16) if

$$\chi^2 < \chi^2_{\alpha,\frac{1}{2}k(k-1)} \tag{23}$$

and accept the alternative (17) otherwise. We choose α to be the upper 100-percentage point of the chi-squared distribution with $k(k-1)/2$ degrees of freedom.

From (14) we can calculate $|\mathbf{R}| = 0.3129$. Then we have

$$\chi^2 = 287.5 > \chi^2_{\alpha,\frac{1}{2}k(k-1)} \tag{24}$$

So we should accept hypothesis (17) for any test level. This means that our hypothesis about the correlations among random variables is correct.

EQUAL PROBABILITY SAMPLING METHOD

Now that the joint probability density function has been established, we can analyze the stochastic creep and shrinkage effects in structures. We want to calculate the probabilistic characteristics of structural responses, such as the mean values and variances of stresses and deflections, from the input joint probability distribution of the random parameters. Our goal is to obtain the most efficient estimators, with as small a number of computer runs as possible, that is, with as small a size of the sample as possible. The basic criterion for estimators means that the smaller the variance of the estimator, the better the estimator will be[9,23–25]. McKay has proven that for the latin hypercube sampling some estimators, such as the distribution functions and the mean values, have much smaller variances than those obtained by simple random sampling for the same number of computer runs, provided that the following conditions are met:

1. The random parameters are mutually independent.
2. Every sampling unit has equal probability content, that is, the method is an equal probability sampling method.
3. The structural responses are monotonic with respect to each component random parameter (i.e. of the input random vector).

These conditions can be met for independent parameters easily. So many investigators used the latin hypercube sampling to solve problems of independent variables in recent years. But for multidimensional dependent variables it is difficult to use an equal probability sampling method because sometimes it is impossible to develop a successful subspace partition procedure which assures every sampling unit to have the same probability content within a multidimensional population.

In the general case of an unequal probability sampling method, the mean and the variance of the structural response may be estimated as

$$\bar{Y} = \frac{1}{W_n}\sum_{j=1}^{n} Y_j P_{u_j} \tag{25}$$

$$\sigma^2 = \frac{1}{W_n}\sum_{j=1}^{n} P_{u_j}(Y_j - \bar{Y})^2 \tag{26}$$

where

$$W_n = \sum_{j=1}^{n} P_{u_j} \tag{27}$$

Y_j is the structural response (such as the maximum 50-year deflection) calculated in the jth computer run; W_n is the probability content of the entire sample; and P_{u_j} is defined as the probability content of the sampling unit u_j,

$$P_{u_j}=\int_{u_j} f(x_1,\ldots,x_k)\,dx_1\ldots dx_k \quad (28)$$

where u_j means the integration is to be made over the subspace corresponding to the sampling unit u_j. In the special case of independent random variables, one has $f(x_1,\ldots,x_k)=f(x_1)\ldots f(x_k)$ and then (28) becomes

$$P_{u_j}=\int_{u_{j_1}} f(x_1)\,dx\ldots\int_{u_{j_k}} f(x_k)\,dx \quad (29)$$

where $f(x_i)$ is the one-dimensional probability density function. An interval partition procedure can be easily developed to give every interval of x_i the same probability content. Then equations (25)–(27) simplify to the well-known forms:

$$\bar{Y}=\frac{1}{n}\sum_{j=1}^{n} Y_j \quad (30)$$

$$\sigma^2=\frac{1}{n}\sum_{j=1}^{n}(Y_j-\bar{Y})^2 \quad (31)$$

$$W_n=n^{-k+1} \quad (32)$$

McKay[16–19] has obtained analytical results on the efficiency of the mean estimator (30). In a similar manner, analysis of the general case can show that the variance of the estimator (25) obtained by latin hypercube sampling will also be better than that of the estimators obtained by simple random sampling, provided that conditions 2 and 3 hold. But if only condition 3 holds, the estimators of the latin hypercube sampling will be probably less efficient than those obtained by simple random sampling. Hence equal probability sampling is necessary, and a method to obtain such samples must be developed for dependent variables if the benefits of latin hypercube sampling should materialize.

Fortunately, if the joint probability distribution is multinormal, a linear orthogonal transformation of random variables can be employed to obtain an equal probability procedure. In fact, the normal density is proportional to e to the power of $(\mathbf{x}_i-\hat{\boldsymbol{\mu}})^T\mathbf{C}^{-1}(\mathbf{x}_i-\hat{\boldsymbol{\mu}})$, which is a quadratic form in which $\mathbf{C}^{-1}$ is a symmetric, positive definite and real square matrix. So there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}^T\mathbf{C}^{-1}\mathbf{P}=\mathbf{I} \quad (33)$$

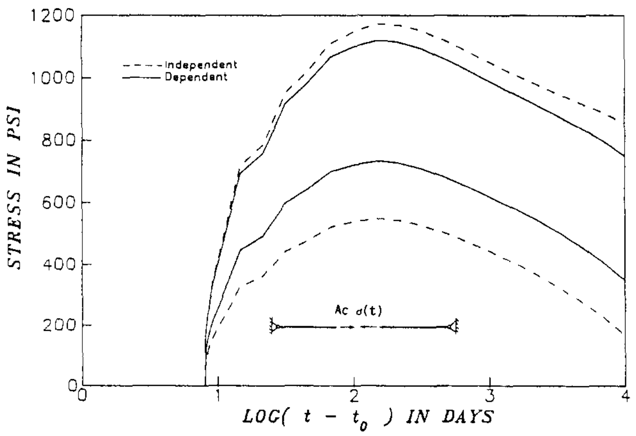

Fig. 1. Shrinkage stress in a restrained bar

since, for orthogonal matrix, $\mathbf{P}^T=\mathbf{P}^{-1}$. Matrix $\mathbf{P}$ consists of all the eigenvectors of matrix $\mathbf{C}^{-1}$. The linear transformation from random vector $\mathbf{X}$ to random vector $\mathbf{Z}$ may be written as $\mathbf{Z}=\mathbf{P}^{-1}(\mathbf{X}-\boldsymbol{\mu})$, and the inverse transformation is $\mathbf{X}=\mathbf{PZ}+\boldsymbol{\mu}$. Since $x_i=\sum_{j=1}^{k}P_{ij}Z_j+\mu_i$, we see that $\partial x_i/\partial z_j=P_{ij}$. Hence, the Jacobian of the transformation is given by $J=|\mathbf{P}|$. But the relation $|\mathbf{I}|=1=|\mathbf{P}^{-1}\mathbf{C}^{-1}\mathbf{P}|=|\mathbf{P}^2||\mathbf{C}^{-1}|$ implies that $|\mathbf{P}|=1\sqrt{|\mathbf{C}^{-1}|}$, since $|\mathbf{C}^{-1}|>0$ ($\mathbf{C}^{-1}$ is positive definite). Thus it follows that

$$\begin{aligned}P_{u_j}&=\int_{U_j}[(2\pi)^k|\mathbf{C}|]^{-1/2}\\&\quad\times\exp[-\tfrac{1}{2}(\mathbf{X}-\boldsymbol{\mu})^T\mathbf{C}^{-1}(\mathbf{X}-\boldsymbol{\mu})]\,dx_1\ldots dx_k\\&=\int_{S_j}(2\pi)^{-k/2}\exp[-\tfrac{1}{2}\mathbf{Z}^T\mathbf{P}^T\mathbf{C}^{-1}\mathbf{PZ}]\,dz_1\ldots dz_k\\&=\int_{S_j}(2\pi)^{-k/2}\exp\left[-\frac{1}{2}\sum_{i=1}^{k}z_i^2\right]dz_1\ldots dz_k\\&=\int_{S_{j1}}\frac{1}{\sqrt{2\pi}}e^{-z_1^2/2}\,dz_1\ldots\int_{S_{jk}}\frac{1}{\sqrt{2\pi}}e^{-z_k^2/2}\,dz_k\end{aligned} \quad (34)$$

$$U_j=U_{j1}\ldots U_{jk} \quad (35)$$

$$S_j=S_{j1}\ldots S_{jk} \quad (36)$$

Here S_j is the subspace corresponding to the jth sampling unit $\mathbf{Z}_j$ in the standard normal space; S_{ji} is the interval of the ith component of $\mathbf{Z}_j$; U_j is the subspace corresponding to the jth sampling unit $\mathbf{X}_j$ in the initial sample space; and U_{ji} is the interval of the ith component of $\mathbf{X}_j$.

Comparing equation (34) with (29), we see that equation (34) is equivalent to the case of independence and every component of $\mathbf{Z}$ is a random variable in the standard normal space. Hence we can first easily introduce the ordinary equal probability latin hypercube sampling with vector $\mathbf{Z}$. After a set of simple interval partitions $\mathbf{S}=(S_1,\ldots,S_j,\ldots,S_n)$ has been determined, the linear transformation can be used to transform $\mathbf{S}$ back to the initial sample space, i.e. $\mathbf{S}\to\mathbf{U}=(U_1,\ldots,U_j,\ldots,U_n)$ where n is the size of the sample (e.g., the number of all computer runs). This guarantees every U_j to possess the same probability content. So the estimators obtained in this way are assured to be more efficient than those of the simple random sampling.

In the special case that every component of vector $\mathbf{X}$ is independent of the others, $\mathbf{C}$ will degenerate into a diagonal matrix, and so will matrix $\mathbf{P}$. Hence the above method will remain valid for this special case.

NUMERICAL EXAMPLES

Let us now illustrate the equal probability latin hypercube sampling combined with our orthogonal transformation method by analyzing some practical structural problems with multinormal correlated random variables. The first two examples which we solve are sketched in Figs 1–2; they are the same as those solved by the equal probability latin hypercube sampling with independent random variables in a previous paper[3]. Since that paper explained in detail the method of structural creep analysis, we do not need to explain it here again. Further examples to be solved are sketched in Figs 3–12, which also compare the

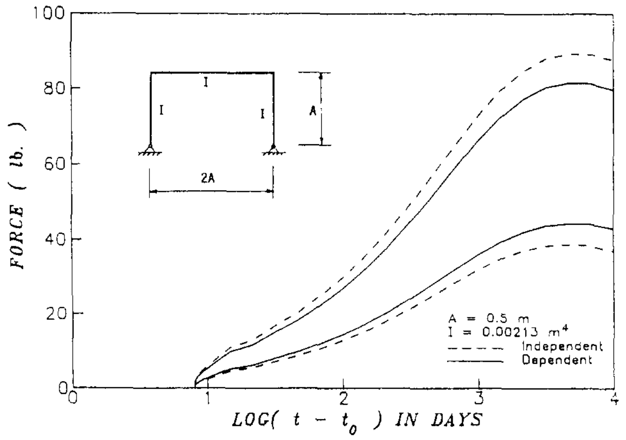

Fig. 2. Shrinkage force in a frame

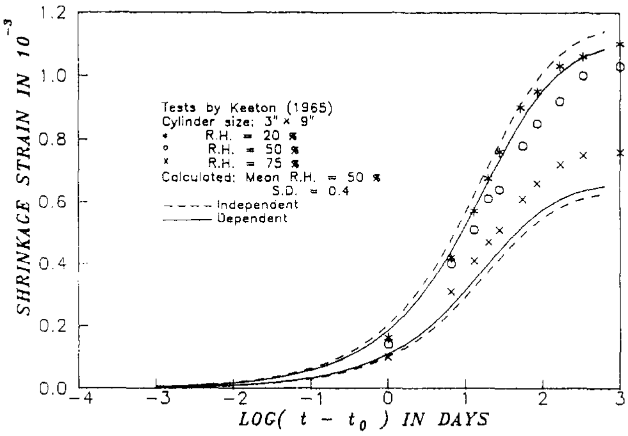

Fig. 3. Line of mean ± standard deviation compared with the measurements of shrinkage strain

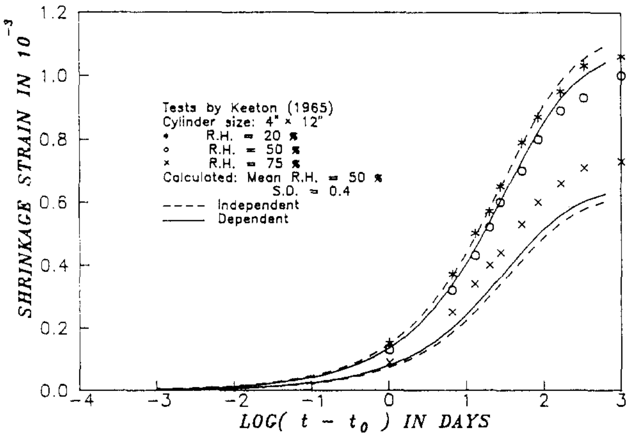

Fig. 4. Line of mean ± standard deviation compared with the measurements of shrinkage strain

curves calculated by the present method with the data points from the tests of L'Hermite *et al.*[14], Keeton[12], and Mossiossian and Gamble[22].

In Figs 1–12, the solid curves show the responses of the structure or the specimen calculated by the present method, and the dashed curves show for comparison the responses calculated previously[4] by the latin hypercube sampling with independent random parameters. The top and bottom curves represent $\bar{Y}(t)+\sigma(t)$ and $\bar{Y}(t)-\sigma(t)$, respectively; $\bar{Y}(t)$ is the mean response at age t estimated by equation (25) and $\sigma(t)$ is the standard deviation of the response at age t estimated by equation (26).

The probabilistic analysis of these examples with dependent random parameters uses the correlation coefficient matrix of multinormal distribution as given by equation (15). The expected values and the coefficients of variation of the 8 random parameters involved (equation (1)), as well as other data and parameters for the BP model[4,7,8] used, are listed in Tables 2 and 3. The mean values for ψ_i are in these tables simply taken as 1, because from (13) we can see that they are rather close to 1. The coefficients of variation for ψ_i are calculated

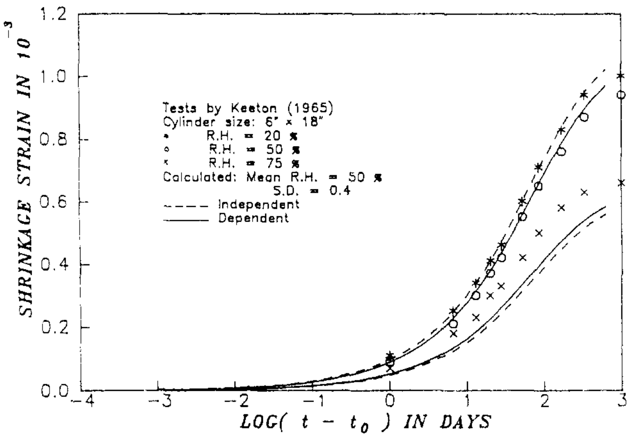

Fig. 5. Line of mean ± standard deviation compared with the measurements of shrinkage strain

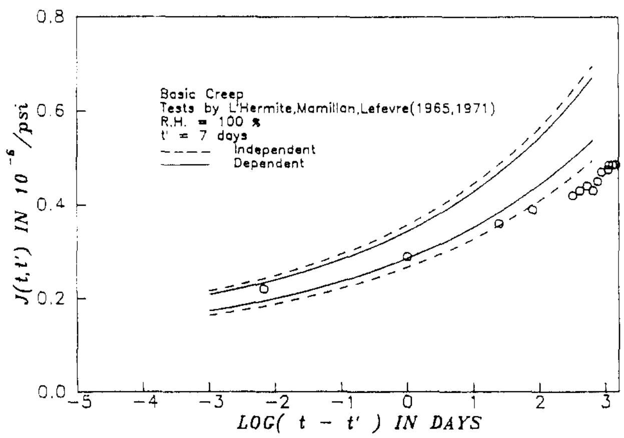

Fig. 6. Line of mean ± standard deviation compared with the measurements of basic creep

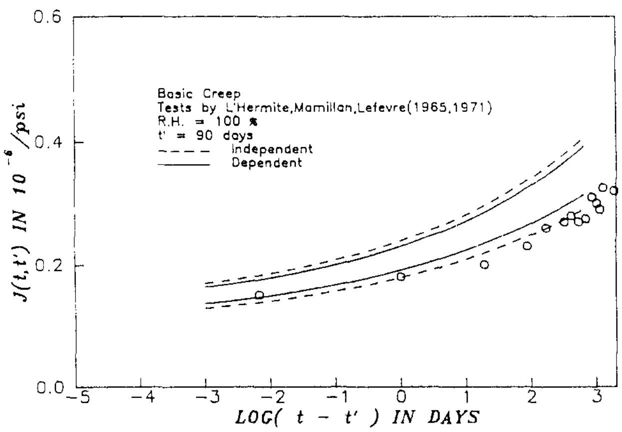

Fig. 7. Line of mean ± standard deviation compared with the measurements of basic creep

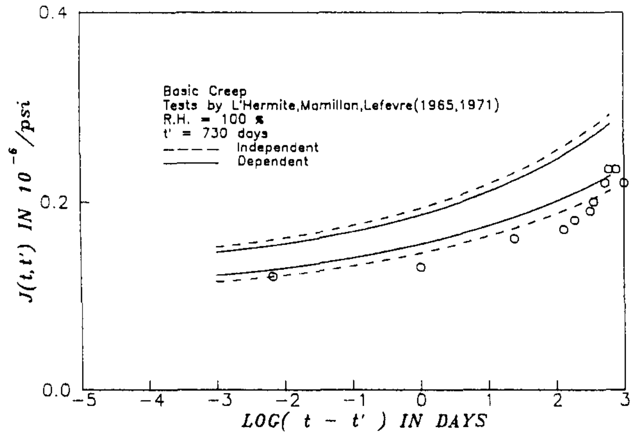

Fig. 8. Line of mean ± standard deviation compared with the measurements of basic creep

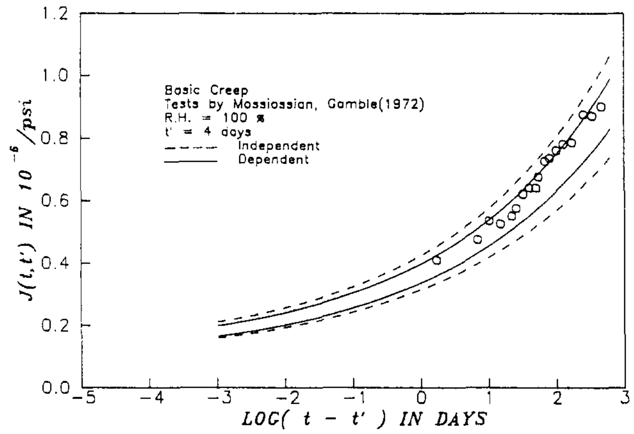

Fig. 9. Line of mean ± standard deviation compared with the measurements of basic creep

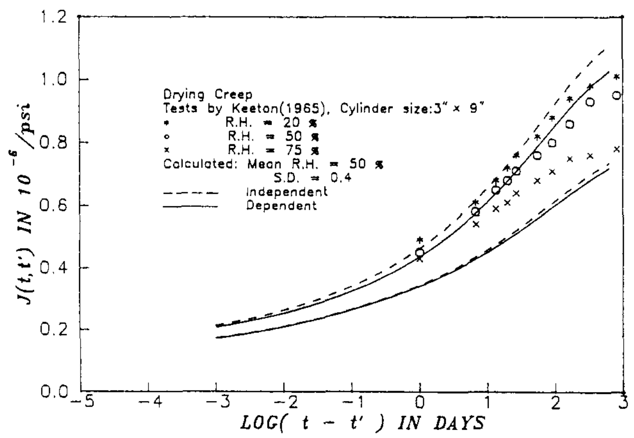

Fig. 10. Line of mean ± standard deviation compared with the measurements of drying creep

according to our results in equations (13) and (14). The coefficients of variation for h, f_c', w/c, g/c and c are not selected on the basis of precise statistical analysis, but merely on the basis of experience and intuitive judgement. The number of computer runs, equal to the number of sampling intervals, is $n=16$ for all the cases.

Figures 1–12 indicate a good agreement of the data with the curves obtained by our latin hypercube sampling method with dependent random parameters. The standard deviations obtained with the present method are generally smaller, that is to say, the present method gives more accurate results. On the other hand, the results of the previous calculations with independent random parameters[3], which are simpler, are generally on the safe side.

By comparing the predicted curves with the test data in Figs 3–12, we can also see that the present method realistically simulates the random variability observed in experiments.

COMMENTS ON A SIMPLIFIED ALTERNATIVE APPROACH TO CORRELATION

Among some of the parameters of the creep and shrinkage prediction model, there exist some approximate deterministic relations. For example, the strength, $f_c'=x_5$, is known to be approximately related to parameter $w/c=x_6$, i.e.

$$x_5 \simeq f(x_6) \tag{37}$$

where $f(x_6)=[(w/c)-0.5]3300$ psi. In another current investigation at Northwestern University by Bažant and Jung-Koo Kim, this aspect is being handled by replacing equation (37) with

$$f_c'=x_5 f(x_6) \tag{38}$$

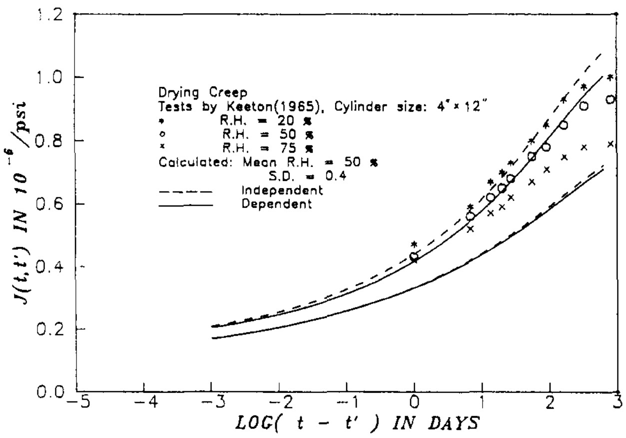

Fig. 11. Line of mean ± standard deviation compared with the measurements of drying creep

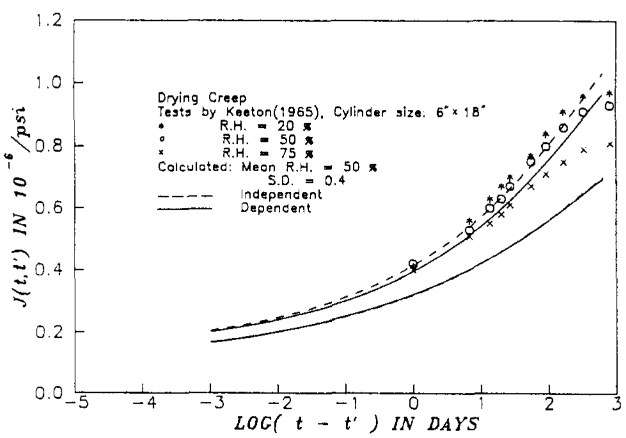

Fig. 12. Line of mean ± standard deviation compared with the measurements of drying creep

Table 2. Mean values and coefficients of variation used in Fig. 1–Fig. 12

Fig.		ψ_1	ψ_2	ψ_3	h	f'_c	w/c	g/c	c
1	m	1	1	1	0.65	45.2	0.46	2.07	450.0
	v	0.07	0.14	0.10	0.20	0.10	0.10	0.10	0.10
2	m	1	1	1	0.70	54.0	0.40	3.50	350.0
	v	0.07	0.14	0.10	0.20	0.10	0.10	0.10	0.10
3	m	1	1	1	0.50	45.2	0.46	2.07	450.0
	v	0.07	0.14	0.10	0.40	0.10	0.10	0.10	0.10
4	m	1	1	1	0.50	45.2	0.46	2.07	450.0
	v	0.07	0.14	0.10	0.40	0.10	0.10	0.10	0.10
5	m	1	1	1	0.50	45.2	0.46	2.07	450.0
	v	0.07	0.14	0.10	0.40	0.10	0.10	0.10	0.10
6	m	1	1	1	1.0	36.3	0.49	3.07	350.0
	v	0.07	0.14	0.10	0.10	0.10	0.10	0.10	0.10
7	m	1	1	1	1.0	36.3	0.49	3.07	350.0
	v	0.07	0.14	0.10	0.10	0.10	0.10	0.10	0.10
8	m	1	1	1	1.0	36.3	0.49	3.07	350.0
	v	0.07	0.14	0.10	0.10	0.10	0.10	0.10	0.10
9	m	1	1	1	1.0	49.4	0.49	2.98	418.0
	v	0.07	0.14	0.10	0.10	0.10	0.10	0.10	0.10
10	m	1	1	1	0.50	45.2	0.46	2.07	451.0
	v	0.07	0.14	0.10	0.40	0.10	0.10	0.10	0.10
11	m	1	1	1	0.50	45.2	0.46	2.07	451.0
	v	0.07	0.14	0.10	0.40	0.10	0.10	0.10	0.10
12	m	1	1	1	0.50	45.2	0.46	2.07	451.0
	v	0.07	0.14	0.10	0.40	0.10	0.10	0.10	0.10

m = mean value; v = coefficient of variation.

i.e., the strength is no longer considered as one of the primary random parameters but is treated as a function of other random parameters. Parameter x_5 models the uncertainty in function $f(x_6)$.

In using equation (38) it is assumed that all the parameters can be treated as independent random variables. The ordinary latin hypercube sampling is then applied to determine the statistics of the structural response.

This alternative approach has some disadvantages as well as advantages. The advantage is of course simplicity. The second advantage is that one can exploit a known,

Table 3. Parameters used in Fig. 1–Fig. 12

Fig.	T	A_1	K_s	s/c	t'	t_0	D
1	23	1.0	1.25	1.66		8	50
2	20	1.0	1.25	2.00		8	200
3	23	0.93	1.15	1.66		8	38.1
4	23	0.93	1.15	1.66		8	50.8
5	23	0.93	1.15	1.66		8	76.2
6	23	1.00	1.25	1.75	7		31.1
7	23	1.00	1.25	1.75	90		31.1
8	23	1.00	1.25	1.75	730		31.1
9	21	0.93	1.55	1.35	4		60.8
10	24	0.93	1.15	1.66	8		38.1
11	24	0.93	1.15	1.66	8		50.8
12	24	0.93	1.15	1.66	8		76.2

approximate functional relationship, which can be nonlinear, while the present method can capture only the linear aspects of the interdependence of random parameters. The third advantage is that equation (38) reflects the fact that, at least in the strict physical sense of cause and effect, the w/c ratio influences the strength but not *vice versa*, i.e. the strength cannot be said to influence the water–cement ratio of the concrete mix (the w/c ratio is chosen by the designer so as to obtain approximately the desired strength, which is not the same).

The disadvantage of equation (38) is that it introduces as a deterministic function $f(x_6)$ a relation whose form may be highly uncertain. One could certainly concoct a dozen other formulas which would agree with the existing data on the relation between f_c', w/c and c equally well. Moreover, f_c' is no doubt at least slightly dependent on other parameters, too, e.g., on h and g/c, which can of course be reflected in the correlation matrix for the present method. The structure of the error can also differ from that in equation (38). For instance, the error could be additive instead of multiplicative, i.e.

$$f_c' = f(x_6) + x_5 \tag{39}$$

or both,

$$f_c' = x_5 f(x_6) + x_9 \tag{40}$$

or such as

$$\log f_c' = x_5[\log f(x_6) + x_9] \tag{41}$$

or

$$\sqrt{f_c'} = x_5\sqrt{f(x_6)} + x_9 \tag{42}$$

To take into account a possible correlation of f_c' to other parameters, e.g., x_8, one could consider a more general relationship $f_c' = x_5 = f(x_6, x_8)$ where f is again a deterministic function, and introduce the randomness by replacing this relation with $f_c' = x_5 f(x_6, x_8)$. This formulation, however, would imply x_6 and x_8 to be mutually independent. In reality they may be correlated, as our solution shows, and so such an approach would be imperfect too.

To sum up, the present method circumvents the errors and uncertainties of a relationship, such as $f_c' = f(x_6)$, which is approximate and incomplete. Extensive statistical experiments would be needed to decide which approach is better justified.

CONCLUSIONS

1. The random parameters of concrete creep and shrinkage prediction models are not mutually independent as generally assumed in the previous investigations. The sample correlation coefficients can be estimated on the basis of the available data. They can characterize the linear aspect of the correlations among these random parameters. To describe the uncertainties of structural effects of creep and shrinkage more accurately, the random parameters involved should be treated as multidimensional correlated random variables.

2. Test of the hypothesis of correlation based on the χ^2-distribution confirms that the correlated parameters cannot be mutually independent.

3. An orthogonal matrix transformation of the random variables yields linear combinations of the random parameters which are mutually independent, under the hypothesis of multinormal distribution. This transformation makes it possible to determine samples of equal probabilities, which permits generalization of the latin hypercube method to multinormal correlated parameters. The estimators obtained with this method are assured to be better than those of simple random sampling.

4. The present method can reflect the general experimentally observed tendencies of response scatter quite well.

5. Numerical examples demonstrate that the method can be applied in practical design problems of concrete structures. Same as the original latin hypercube sampling method with uncorrelated parameters, the present method reduces the problem of determining the statistics of creep and shrinkage effects in structures to a number of deterministic computer calculations of the structural responses for various sets (samples) of model parameters.

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