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Equilibrium Path Bifurcation Due to Strain-Softening Localization in Ellipsoidal Region

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A preceding study of the loss of stability of a homogeneous strain state in infinite homogeneous solid due to localization of strain into an ellipsoidal region is complemented by determining the condition of bifurcation of equilibrium path due to ellipsoidal localization mode. The bifurcation occurs when the tangential moduli matrix becomes singular, which coincides with Hill's classical bifurcation condition for localization into an infinite layer. The bifurcation is normally of Shanley type, occurring in absence of neutral equilibrium while the controlled displacements at infinity increase. During the loading process with displacement increase controlled at infinity, this type of bifurcation precedes the loss of stability of equilibrium due to an ellipsoidal localization mode, except when the tangential moduli change suddenly (which happens, e.g., when the slope of the stress-strain diagram is discontinuous, or when temperature is increased).

Introduction

Strain softening due to distributed cracking may cause strain-localization instabilities. For uniaxial behavior and for bending, the localization instabilities were analyzed in 1974 by Bažant (1976). In multidimensional situations, the simplest localization instabilities are the localization into a planar band or into an ellipsoid, which were analyzed by Rudnicki and Rice (1975), Rice (1976), and Rudnicki (1977). These pioneering studies were limited to localizations in an infinite space and to von Mises or Drucker-Prager plasticity. Recently, Bažant (1988d) extended these classical works to ellipsoidal localization in a material with an arbitrary constitutive law and, for the cases of localizations into planar bands as well as circular or spherical regions, to finite bodies. He also went beyond the conditions of critical state and obtained the conditions of stability. A detailed study of the effects of various material parameters as well as body size on the critical states has been presented by Bažant and Lin (1989). For a more detailed literature review, see Bažant (1988d) and Bažant and Lin (1989).

The interest in the solutions of localization in ellipsoidal regions stems from the fact that, in contrast to the solutions for infinite bands, they can be used as approximate solutions of localization in finite bodies. If the body is finite, localization into an infinite band cannot represent and exact solution because, for example, the fixed boundary conditions cannot be satisfied at the location where the localization band intersects the boundary.

In particular, we will focus on localization into ellipsoidal regions for which analytical solutions can be found. Except for the special case of cylindrical and spherical localization regions, these solutions are available only for an infinite solid, and generally cannot satisfy the boundary conditions for a finite body. However, in contrast to the infinite localization band, they can at least satisfy the boundary conditions approximately, provided the body is sufficiently large compared to the size of the localization ellipsoid. This is due to the fact that the stresses, strains, and displacements in the analytical solution for the ellipsoidal localization region in an infinite solid decay rapidly with the distance from the ellipsoid, thus becoming negligible at a certain sufficient distance from the ellipsoid. If the boundary lies beyond that distance, the solution is nearly vanishing at the boundary and can, therefore, be used as an approximate solution for a finite body.

The previous studies of strain localization into ellipsoidal regions have been restricted to instabilities, which occur at constant load and represent a state of neutral equilibrium or limit of stability. Localizations, however, can also occur in a stable fashion as a bifurcation of equilibrium response path at increasing load, similarly to Shanley-type instability of plastic column. The objective of this paper is to extend the previous general stability analysis of Bažant (1988b) and Bažant and Lin (1989) to bifurcations of Shanley type, at which the ellipsoidal localization region is stable and the loading increases during bifurcation.

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Eshelby's Solution

We consider an ellipsoidal hole (Fig. 1(b)) in an infinite homogeneous elastic space characterized by the elastic moduli

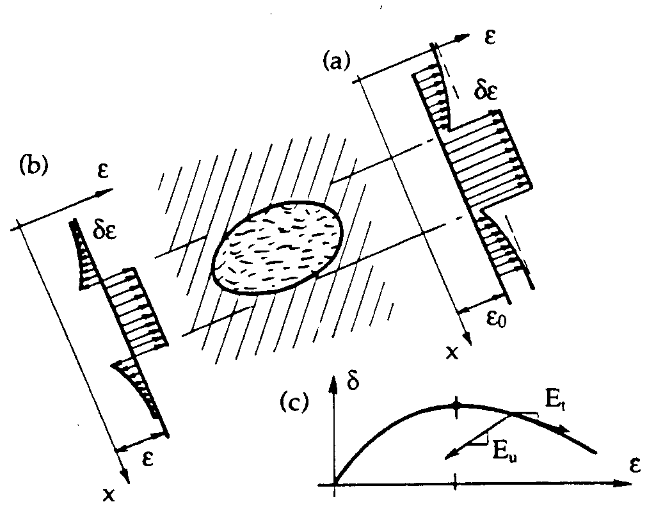

Fig. 1 Strain increment distribution for localization during loading into an ellipsoidal region, and stress-strain diagram ($E_t - D_{uu}$)

matrix $\mathbf{D}_u$. We imagine to fit and glue into this hole an ellipsoidal plug made of the same material. To fit, the plug must first be deformed by a uniform strain, ϵ^0, called the eigenstrain (note that an ellipsoid can be transformed to any other ellipsoid by uniform strain). The eigenstrain is then imagined unfrozen. This causes the plug with ellipsoidal hole to undergo strain increment, ϵ^e, in order to establish equilibrium with the surrounding infinite space. According to the famous Eshelby's (1957) theorem, the strain ϵ^e in the plug is uniform and is expressed as

$$\epsilon_{ij}^e = S_{ijkm}\epsilon_{km}^0 \tag{1}$$

in which S_{ijkm} are the components of a fourth-rank tensor which depend only on the aspect ratios a_1/a_2, a_3/a_2 of the principal axes of the ellipsoid, as well as on the elastic moduli; the latin lower case subscripts refer to cartesian coordinates x_i (i = 1, 2, 3), and repeated subscripts imply summation over 1, 2, 3. Eshelby's coefficients S_{ijkm} are generally calculated as elliptic integrals (Mura, 1982). Always, $S_{ijkm} = S_{jikm} = S_{ijmk}$, but in general $S_{ijkm} \neq S_{kmij}$. For arbitrary anisotropic material properties, the expressions for coefficients S_{kmij} were derived by Kinoshita and Mura (1971) and Lin and Mura (1973). For convenience we rewrite equation 1 in a matrix form:

$$\epsilon^e = \mathbf{Q}_u\epsilon^0 \tag{2}$$

in which $\mathbf{Q}_u$ is a (6 × 6) square matrix formed from coefficients S_{ijkm} (see Bažant, 1988d), and ϵ^0 and ϵ^e are (6 × 1) column matrices of the eigenstrains and the equilibrium strains in the ellipsoidal region.

According to Hooke's law, the stress in the ellipsoidal plug, σ^e, which is uniform, may be expressed as $\sigma^e = \mathbf{D}_u(\epsilon^e - \epsilon^0)$. From equation 2, $\epsilon^0 = \mathbf{Q}_u^{-1}\epsilon^e$, and so

$$\sigma^e = \mathbf{D}_u(\mathbf{1} - \mathbf{Q}_u^{-1})\epsilon^e \tag{3}$$

in which $\mathbf{1}$ = unit (6 × 6) matrix.

Review of Ellipsoidal Localization Instability

For the readers' convenience, let us first briefly outline a simplified version of the analysis from Bažant (1988d). We consider an infinite strain-softening solid (without any hole) which is initially in an initial equilibrium state of uniform initial strain, ϵ, and uniform initial stress, σ, balanced by loads applied at infinity. We seek the condition under which the initial state loses stability in a mode in which the strain localizes into an ellipsoidal region (Fig. 1(*a*)) without changing the prescribed stresses (or the prescribed displacements) at infinity. If these variations can happen while maintaining equilibrium, we have a state of neutral equilibrium which represents the limit of stability, i.e., the critical state.

Due to localization, the strain and stress in the infinite continuum outside the ellipsoidal region becomes nonuniform; while according to Eshelby's theorem, the stress and strain inside the ellipsoidal region will remain uniform (Fig. 1(*b*)). The strains can become discontinuous across the ellipsoidal interface. On the other hand, the normal and shear stress components acting on this surface must remain continuous in order to maintain equilibrium.

It must now be recognized that a material in a strain-softening state can deform not only according to the matrix of tangential moduli for further loading (i.e., strain-softening), which is indefinite, but also according to the matrix of elastic moduli for unloading, which is positive definite. Because the stresses inside the ellipsoidal region decrease if there is strain softening at further loading, and because the stress components on the ellipsoidal interface must be continuous across the interface, the stresses immediately outside the ellipsoidal region must decrease if the stresses or displacements at infinity are fixed. So, immediately outside the ellipsoidal region, there can be unloading, i.e., the material can behave elastically. In fact, there must be unloading at least somewhere outside the ellipsoidal region, for if there were none, the strain variations would be non-negative everywhere along any straight line through the ellipsoid, and positive inside the ellipsoid; their integral from $-\infty$ to ∞ along such a line would be nonzero and thus incompatible with the boundary condition of constant displacements at infinity. The same must be true for the boundary condition of constant stresses at all infinitely remote points because integration of strains along any closed infinitely remote contour indicates that this boundary condition implies constant displacement at infinity. (Later, we will see that in the bifurcation problem, the situation is different: There need not be any unloading since bifurcation can occur while the displacements at infinity increase.)

According to equation (3), the stress variations inside the ellipsoidal region that correspond to Eshelby's solution for the outside are $\delta\sigma^e = \mathbf{D}_u(1 - \mathbf{Q}_u^{-1})\delta\epsilon^e$ in which $\mathbf{D}_u$ represents the matrix of the elastic moduli for unloading from the current state characterized by initial strain ϵ. This matrix is positive definite, and it is also isotropic if the material is isotropic. The vector of surface tractions that must be transmitted from the ellipsoidal region to the outside in order to provide the correct boundary conditions for the Eshelby's solution for the outside is $\delta\mathbf{p}^e = -\delta\sigma^e\mathbf{n}$, where $\mathbf{n}$ is the unit vector of the normals to the ellipsoidal surface, pointed outward from the ellipsoid. Therefore,

$$\delta\mathbf{p}^e = -\mathbf{D}_u(\mathbf{1} - \mathbf{Q}_u^{-1})\delta\epsilon^e\mathbf{n}. \tag{4}$$

At the same time, the stress variations inside the ellipsoidal region may be expressed as $\delta\sigma^e = \mathbf{D}_t\delta\epsilon^e$, in which $\mathbf{D}_t$ is the (6 × 6) matrix of tangential moduli for further loading from the current state of strain ϵ. This matrix is not positive definite if the initial state is in the strain-softening range, and, in general, is anisotropic (as a manifestation of stress-induced anisotropy). Based on the stress variations inside the ellipsoidal region, the vector of surface tractions acting on the surface of the ellipsoid must be $(\mathbf{D}_t\delta\epsilon^e)\ \mathbf{n}$, and the vector of surface tractions applied from the ellipsoidal region on the outside must be

$$\delta\mathbf{p}^e = -\mathbf{D}_t\delta\epsilon^e\mathbf{n}. \tag{5}$$

Substituting equation (5) into equation (4), one obtains for the strain variations $\delta\epsilon^e$ in the ellipsoidal localization region the condition:

$$\delta\mathbf{X}\ \mathbf{n} = \mathbf{0},\ \text{with}\ \delta\mathbf{X} = \mathbf{Z}\ \delta\epsilon^e, \tag{6}$$

in which

$$\mathbf{Z} = \mathbf{D}_t - \mathbf{D}_u(\mathbf{1} - \mathbf{Q}_u^{-1}), \tag{7}$$

where $\mathbf{Z}$ is a (6 × 6) matrix depending only on the loading and unloading stiffness matrices of the material in its current state.

Vector $\delta\mathbf{X}$ is the same for all the points of the ellipsoidal surface because the strain $\delta\epsilon^e$ in the ellipsoidal region is homogeneous, both according to Eshelby's solution for the outside and the assumed strain-softening deformation inside. Equation (6) represents a system of three homogeneous linear algebraic equations for the three components of $\mathbf{n}$. These equations must be satisfied not just for one vector, $\mathbf{n}$, but for infinitely many vectors of all the normals, $\mathbf{n}$, of the ellipsoidal surface. This is possible if and only if $\delta\mathbf{X} = \mathbf{0}$. Therefore.

$$\mathbf{Z}\delta\epsilon^e = \mathbf{0}. \tag{8}$$

This is a system of six homogeneous linear algebraic equations for the six components of column matrix $\delta\epsilon^e$. Thus, a localization instability is possible if and only if the determinant of this equation system vanishes, i.e.,

$$\det \mathbf{Z} = 0. \tag{9}$$

This condition, representing a condition of neutral equilibrium, was obtained in Bažant (1988d) by analyzing the positive definiteness of the second-order work in the infinite solid, which showed that the states in the loading process preceding the attainment of this condition are stable, and the subsequent states are unstable. The special case of equation (9) for Drucker-Prager plasticity was obtained by Rudnicki (1977).

Using the thermodynamic stability condition from Bažant (1988a, b), Bažant (1988d) further showed that the initial state of uniform strain ϵ is unstable if matrix $\mathbf{Z}$ is not positive definite, and is stable (with regard to the ellipsoidal localization mode) if matrix $\mathbf{Z}$ is positive definite.

The critical state condition in equation (8) ignores second-order nonlinear geometric effects. As is well known from various three-dimensional stability problems (Biot, 1965; Bažant 1971), the nonlinear geometric effects can have influence on three-dimensional instabilities only if they occur at stresses that are of the same order of magnitude as the tangential moduli. For a material with a stress-strain diagram that lacks a prolonged plastic plateau (Fig. 1(*c*)), this situation arises only if the instability occurs very near the peak of the stress-strain diagram. Since Eshelby's theorem is limited to small-strain elasticity, we must exclude this case from our analysis (as done by Rudnicki, 1977). Our solution will be applicable only to postpeak states at which the tangential modulus E_t (Fig. 1(*c*)) is of a higher order of magnitude than the stresses. Thus, if we find the bifurcation to occur at the peak stress, it means that according to finite strain theory it should occur near the peak stress.

Bifurcation Analysis

For inelastic structures, the condition of stable equilibrium does not necessarily indicate which equilibrium path will be followed by the structure under a given loading process. It can happen that, after a bifurcation of the equilibrium path, the states on all the equilibrium branches emanating from the bifurcation point are stable, yet only one of them can be followed by the structure. Which one is followed is decided by the condition of stable path (Bažant 1988a,b), which differs from the condition of stable equilibrium if the structure is inelastic. (An example of such behavior is Shanley's bifurcation at the tangent modulus load of an elastoplastic column.) We will now show that, for ellipsoidal localizations, the bifurcation of equilibrium path happens earlier and can take place during a succession of stable equilibrium states at increasing load, i.e., in absence of neutral equilibrium.

The loss of stability due to strain localization is considered to occur while the remote displacements, strains, and stresses are constant (Fig. 1(*b*)). Bifurcation of the equilibrium path, on the other hand, can occur while the remote displacements, strains, and stresses increase (Fig. 1(*c*)). In this manner, it can happen that the strains increase everywhere, but the increase inside the ellipsoidal region is larger than outside, so that the strain localizes simultaneously with the progress of loading; see Fig. 1(*a*). For the mode of instability (Fig. 1*b*), matrix $\mathbf{D}_t$ of the moduli for loading applies for the interior of the ellipsoid, and matrix $\mathbf{D}_u$ of the moduli for unloading applies for the exterior. But in the bifurcation mode (Fig. 1*a*), matrix $\mathbf{D}_t$ applies for both the interior and the exterior of the ellipsoid.

As shown in Bažant (1988d, equation 12), the second-order work for an arbitrary deviation from the initial uniform equilibrium state with loading inside and unloading outside the ellipsoid is $\delta^2 W = (1/2)\,\delta\epsilon^T \mathbf{Z}\,\delta\epsilon\, V$, where V = volume of the ellipsoidal region and $\delta\epsilon$ is the (6×1) column matrix of arbitrary homogeneous strain increments inside the ellipsoid. According to the derivation of equation (1), one can check that the second-order work done along the equilibrium path when loading takes place both inside and outside the ellipsoid is $\delta^2 W = (1/2)\,\delta\epsilon^T \mathbf{Z}_t\,\delta\epsilon\, V$ where $\mathbf{Z}_t$ is obtained from $\mathbf{Z}$ (equation (7) by replacing $\mathbf{D}_u$ with $\mathbf{D}_t$. Therefore, $\mathbf{Z}_t = \mathbf{D}_t(\mathbf{I} + \mathbf{Q}_t^{-1} - \mathbf{I})$ or

$$\mathbf{Z}_t = \mathbf{D}_t\mathbf{Q}_t^{-1}, \tag{10}$$

where $\mathbf{Q}_t$ is the matrix of Eshelby's coefficients based on $\mathbf{D}_t$.

Assuming $\mathbf{D}_t$ to vary during the loading process continuously, the first bifurcation of the equilibrium path is obtained when $\delta^2 W = 1/2\,\delta\epsilon^T \mathbf{Z}_t\,\delta\epsilon\, V = 0$ for some nonzero vector $\delta\boldsymbol{\epsilon}$. This case occurs when the matrix equation

$$\mathbf{Z}_t\delta\epsilon = \mathbf{0} \tag{11}$$

admits a nonzero solution $\delta\boldsymbol{\epsilon}$. This means that matrix $\mathbf{Z}_t$ given by equation (10) is singular.

Matrix $\mathbf{Q}_t$ relates the equilibrium strain of an elastic plug of moduli, $\mathbf{D}_t$, in an infinite space of moduli, $\mathbf{D}_t$, to the eigenstrain, $\delta\epsilon^0$, by which the plug must be deformed to fit it into an ellipsoidal hole in the infinite space; $\delta\epsilon = \mathbf{Q}_t\delta\epsilon^0$. From this physical meaning it is clear that if $\mathbf{D}_t$ is positive definite, then a finite $\delta\epsilon$ can be produced only by a finite $\delta\epsilon^0$, and if $\mathbf{D}_t$ is nearly singular (a state near the peak of the stress-strain diagram), then a finite $\delta\epsilon$ can occur even for a vanishingly small $\delta\epsilon^0$. Therefore, if $\mathbf{D}_t$ is positive definite, so must be $\mathbf{Q}_t^{-1}$, and if $\mathbf{D}_t$ is singular, so must be $\mathbf{Q}_t^{-1}$. Hence, in view of equation (10), singularity of matrix $\mathbf{Z}_t$ implies the tangential moduli matrix $\mathbf{D}_t$ to be singular, i.e.,

$$\det \mathbf{D}_t = 0. \tag{12}$$

This is equivalent to the well-known bifurcation condition for an infinite localization band in an infinite solid (Rudnicki and Rice, 1975). Equation (11) is satisfied at the peak point of the stress-strain diagram and represents the condition which separates the strain-hardening regime of material ($\mathbf{D}_t$ positive definite) from the strain-softening regime ($\mathbf{D}_t$ indefinite).

So we must conclude that, during a loading process in which the displacements are continuously increased at infinity, localization of homogeneous strain into an ellipsoidal region must begin as soon as $\mathbf{D}_t$ loses positive definiteness, i.e., as soon as strain softening begins, same as localization into an infinite band.

Let us now discuss the type of bifurcation. In view of equation (10), equation (11) for eigenvector $\delta\epsilon^*$ of matrix $\mathbf{Z}$ may be written as

$$\mathbf{D}_t\mathbf{x}^* = \mathbf{0}, \tag{13}$$

in which $\mathbf{x}^* = \mathbf{Q}_t^{-1}\,\delta\epsilon^*$ or

$$\delta\epsilon^* = \mathbf{Q}_t\mathbf{x}^*, \tag{14}$$

and $\mathbf{x}^*$ is the eigenvector of matrix $\mathbf{D}_t$ corresponding to zero eigenvalue.

Two cases may now be distinguished: (1) Either the eigenvector $\delta\epsilon^*$ lies in the sector of loading, or (2) it does not.

If it does, then the bifurcation state would be a state of neutral equilibrium, which represents the limit of stability. In previous work (Bažant 1988d, Bažant and Lin 1989), however, it was found that, except for the limit case of an infinite layer (which is equivalent to an ellipsoid for which the two ratios of its axes tend to infinity), the limit of stability does not occur when det $\mathbf{D}_t = 0$ but only later, when matrix $\mathbf{D}_t$ becomes indefinite (i.e., at a certain finite distance after the peak of the stress-strain diagram).

It follows that the eigenvector $\delta\epsilon^*$ must lie outside the loading sector. So the actual increment $\delta\epsilon^{ep}$ along the equilibrium path cannot coincide with $\delta\epsilon^*$ because it would imply unloading, which we have ruled out. Therefore, the actual $\delta\epsilon^{ep}$ musts lie at the boundary of the loading sector and must differ from the eigenvector $\delta\epsilon^*$. Hence, $\mathbf{Z}_t\, \delta\epsilon^p \neq 0$ and, furthermore, $\mathbf{D}_t$ $\mathbf{x}^{ep} \neq \mathbf{0}$ where $\mathbf{x}^{ep} = \mathbf{Q}_t^{-1}\, \delta\epsilon^p$. This means that the increment $\delta\epsilon^{ep}$ along the equilibrium path must be happening at increasing boundary displacements (or increasing strains) at infinity, i.e., neutral equilibrium does not exist. This is obviously the Shanley type of bifurcation, in which the bifurcation state and all the immediate postbifurcation states are stable (Bažant 1988a, b).

For an infinite localization band in an infinite solid, by contrast, the first bifurcation is not of Shanley type. It occurs at neutral equilibrium and represents the onset of instability.

Comments

As shown by Rudnicki (1977) for Drucker-Prager plasticity and by Bažant (1988d) and Bažant and Lin (1989) for general material properties, the loss of stable equilibrium with localization into an ellipsoidal region can occur only when matrix $\mathbf{D}_t$ becomes indefinite, i.e., when the material enters a strain-softening state. This occurs at a finite distance after the peak point of the stress-strain diagram, at which the tangential modulus is nonzero and negative. We now find, however, that along the equilibrium path the localization into an ellipsoidal region occurs already when matrix $\mathbf{D}_t$ becomes semi-definite, i.e., at the peak-stress state, which always precedes the state of stability loss.

The classical bifurcation condition of Hill (1962), which serves as the basis of the method of linear comparison solid, also indicates that singularity of matrix $\mathbf{D}_t$ is the condition of first bifurcation; see also Rudnicki and Rice (1975), Rice (1976), Leroy and Ortiz (1988), and de Borst (1988). Hill's bifurcation condition, however, has been proven only for localization into an infinite planar band in an infinite space. The boundary conditions of such a localization mode cannot be accommodated for finite bodies. The present analysis proves that Hill's bifurcation condition (i.e., det $\mathbf{D}_t = 0$) is also correct for localization into ellipsoidal regions, although the mode of localization is different.

Note that Hill's case of localization into an infinite layer is obtained as the limiting case of ellipsoidal localization when two axes of the ellipsoid tend to infinity. The fact that this limiting case agrees with Hill's case was numerically verified in Bažant (1988d) and Bažant and Lin (1989).

The foregoing analysis, which led to the bifurcation condition det $\mathbf{D}_t = 0$, shows only when localization can occur. To show that it must occur, Bažant's (1988a,b) path-stability criterion requires it to prove that, for the conditions of prescribed displacements at infinity, the value of $\delta^2 W$ is smaller for the localizing path than for the nonlocalized path (in which the strain field remains uniform). The calculation of $\delta^2 W$ along the equilibrium path can be done numerically and will not be illustrated in the present paper.

The preceding analysis shows that the case of stability loss with localization into an ellipsoidal domain can never occur in a continuous loading process in which the displacement at infinity are controlled. Localization along the equilibrium path, without any loss of stability, always precedes such instability. So, can the solution of stability loss have, in this case, any practical application? It can—either if the tangential moduli matrix $\mathbf{D}_t$ changes suddenly during the loading process (which happens, e.g., for bilinear stress-strain diagrams), or if the uniform strain state in the strain-softening range at incipient instability is reached by some other type of process. Examples are processes in which some displacements inside the body are controlled, or processes in which a finite sudden change of $\mathbf{D}_t$ is caused by a change of temperature, change in moisture content or pore pressure, crystallographic conversion, chemical conversion, irradiation, etc., or processes in which $\mathbf{D}_t$ is changed due to hysteretic cycles.

Conclusion

A bifurcation of equilibrium path characterized by localization of initially homogeneous strain into an ellipsoidal region takes place when the matrix of incremental moduli becomes singular. This coincides with Hill's classical bifurcation condition for localization of strain into an infinite layer, which represents a condition of neutral equilibrium and stability limit. By contrast, the ellipsoidal bifurcation is of Shanley type, occurring in a stable manner while the controlled displacements at infinity increase. If the incremental material moduli vary continuously during the loading process, this bifurcation always precedes the loss of stability of equilibrium in the ellipsoidal localization mode. Such a loss of stability of an initially uniform strain field can nevertheless occur simultaneously with the first bifurcation if the matrix of incremental moduli changes suddenly.

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