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TECHNICAL NOTE

JUSTIFICATION AND IMPROVEMENT OF KIENZLER AND HERRMANN'S ESTIMATE OF STRESS INTENSITY FACTORS OF CRACKED BEAM

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Abstract—Kienzler and Herrman recently showed that good and remarkably simple approximations of the stress intensity factors of cracks in beams can be obtained by bending theory estimations of the strain energy release as the crack is widened into a band of finite width. Their method has been based on equating this energy release to the energy release rate due to crack length extension, which has been postulated as a hypothesis. This note presents a justification of this hypothesis and further shows a possibility of improvement by introduction of an additional factor. The improvement is demonstrated by numerical comparison with the exact solution of a cracked beam. The aforementioned factor, however, can be determined only by optimum fitting of the exact solution.

INTRODUCTION

A REMARKABLY simple method for close approximation of stress intensity factor K_I in cracked or notched beams was recently discovered by Kienzler and Herrmann[1] (see also Herrmann and Sosa[2]). The method was derived from a certain postulated hypothesis regarding the energy release as the crack is widened into a fracture band. The purpose of this note is to present a justification of this hypothesis and also show a different derivation of this new method. This derivation is simpler and at the same time indicates that the hypothesis used by Kienzler and Herrmann is not exact but merely a good approximation. The present method avoids sophisticated elegant concepts, such as material forces, which were introduced by Kienzler and Herrmann, but are not used in the present derivation. They do not seem to be necessary for obtaining the result.

KIENZLER AND HERRMANN'S METHOD

We may illustrate this new method by considering a cracked beam that is subjected to bending moment M. The beam has bending stiffness EI_1, and the notched cross-section has bending stiffness EI_2, where $I_1 = bH^3/12$ and $I_2 = b(H-a)^3/12$; I_1, I_2 = moments of inertia, H = beam depth, a = notch depth, b = beam thickness. Kienzler and Herrmann[1] consider the energy release ΔU of the beam as the notch thickness b is widened from zero to Δh. From the theory of bending one has $\Delta U = M^2(1/EI_1 - 1/EI_2)\,\Delta hb/2$, and so

$$\frac{\partial U}{\partial h} = M^2\frac{b}{2}\left(\frac{1}{EI_1} - \frac{1}{EI_2}\right) \tag{1}$$

where U = strain energy of the beam. To calculate the energy releases rate, and from that the stresses intensity factor, Herrmann and Kienzler write (for $b = 1$):

$$\frac{\partial U}{\partial a} = 2\frac{\partial U}{\partial h} \tag{2}$$

where $\partial U/\partial a = -bG$, and G is the energy release rate of the beam per unit advance of the crack (and per unit length of the crack front edge). The stress intensity factor is $K_I = (EG)^{1/2}$. According to Kienzler and Herrmann's[1] Figs 2–4, the values of K_I calculated in this manner compare quite well with accurate solutions from handbooks.

Equation (2) represents a crucial step. This step, however, has not been justified theoretically. It has been postulated as a hypothesis.

JUSTIFICATION AND IMPROVEMENT

Formation of a crack in an uncracked body may be imagined to completely relieve the strain energy from the triangular areas 021 and 023 which are limited by the so-called "stress diffusion lines" (see e.g. Knott[3]), as shown in Fig. 1(a), (b). When the crack length a increases by Δa, the additional energy release, therefore, comes from the strips 2683 and 2641 of width a (Fig. 1a) from which $\partial U/\partial a$, G and K_I can be calculated. For bodies with an initially homogeneous stress field, this method gives correct formulas for K_I, except that the value of the proper slope k can be determined only by comparison

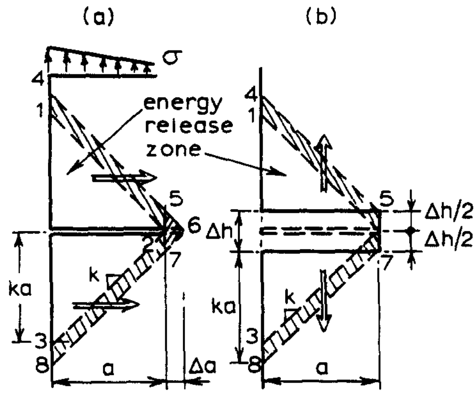

Fig. 1

with the exact solution of elasticity and is found to depend on the geometry of the body. Generally, the order of magnitude of k is 1.

Application of the foregoing method to beams would be more problematic since the stress field before formation of a crack is nonuniform and the transverse crack advances in the direction of nonuniformity of the stress field. This means that the average initial strain energy density in the aforementioned triangular zones changes, and so does the effective slope k. This problem, however, is circumvented by Kienzler and Herrmann's idea; instead of crack extension (Fig. 1a), they consider crack widening (Fig. 1b). The widening occurs in the direction of constant stress, and so it is not accompanied by a change of the size of the triangular zones. Therefore, the average strain energy density in the triangular zones remains approximately constant, and so does the effective slope k.

The widening of the crack into a crack band of width Δh (Fig. 1b) causes the stress relief zone to change from area 1231 to area 45784 as shown by arrows in Fig. 1(b). Since the triangular area 56725 in Fig. 1(b) is second-order small if Δa is small, the increments of the stress relief zones 123876541 and 1237541 in Fig. 1(b), (c), (d) are identical provided that $\Delta h/2 = k\Delta a$. Therefore

$$-bG = \frac{\partial U}{\partial a} = 2k\frac{\partial U}{\partial h}. \tag{3}$$

The case $k = 1$ (eq. 2) is identical with the hypothesis (postulate) of Kienzler and Herrmann[1, p. 41]. However, as the foregoing argument shows, there seems no reason to assume $k = 1$. Indeed, numerical results show that more accurate results can be obtained if the empirical constant k is allowed to differ from 1.

NUMERICAL COMPARISONS

Figure 2 shows the plot of the nondimensionalized stress intensity factor vs the relative crack length a/h for a single-edge-notched beam of depth h under uniform bending moment M. The exact solution of elasticity[4] is shown as the data points. The solid curve shows the approximate solution from Kienzler and Herrmann's Fig. 4[1], based on eqs (1) and (2). The dashed curve shows the present adjustment, achieved by taking $b = 1.32$ as the optimum, instead of $k = 1$. The dashed curve is graphically obtained from the solid curve by vertically scaling the ordinates with the factor $\sqrt{k} = \sqrt{1.32}$.

Although Kienzler and Herrmann's approximate solution in Fig. 2 may be quite adequate for most engineering purposes, the present adjustment represents an improvement with no loss in simplicity. Kienzler and Herrmann's Fig. 2 plots their formula for a center-cracked beam, and their Fig. 3 for a double-edge-cracked beam, both under bending moment. For the former case, optimum improvement is achieved with $k \simeq 1.17$, and for the latter case with $k \simeq 0.91$. Diverse

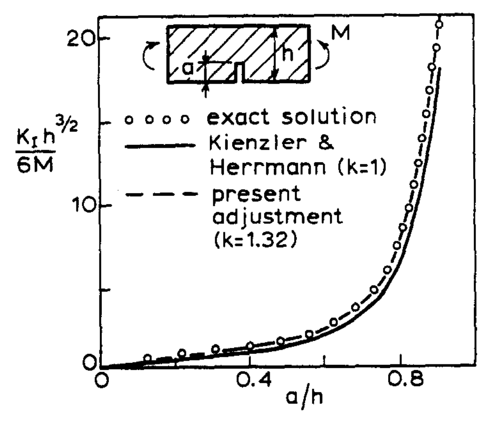

Fig. 2

values of k achieve improvements of the results calculated by Herrmann and Sosa[2] for beams under axial force instead of bending moment.

Overall, we see that the optimum values of k vary and cannot be obtained by elementary reasoning.

CONCLUSIONS

(1) Kienzler and Herrmann's approximate method can be justified by adapting the method of "stress relief zone" to the widening of a crack into a band.

(2) This justification indicates that an improvement can be achieved by introducing an additional factor k, and numerical results confirm that this indeed is the case. Factor k, however, can be determined only by optimum fitting of the exact solution of elasticity.

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