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CRACKS INTERACTING WITH PARTICLES OR FIBERS IN COMPOSITE MATERIALS

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ABSTRACT: Micromechanics analysis of damage in heterogeneous media and composites cannot ignore the interactions among cracks as well as between cracks and inclusions or voids. Previous investigators can to this conclusion upon finding that states of distributed (diffuse) cracking (damage) cannot be mathematically represented merely as crack systems in a homogeneous medium, even though stable states with distributed damage have been experimentally observed in heterogeneous materials such as concrete. This paper presents a method for modeling interactions between a crack and many inclusions. Based on the Duhamel-Neuman analogy, the effect of the inclusions is equivalent to unbalanced forces acting on the contour of each inclusion in an infinite homogeneous solid. The problem is solved by superposition; it is decomposed into several standard problems of elasticity for which well-known solutions are available. The problem is finally reduced to a system of linear algebraic equations similar to those obtained by Kachanov for a system of interacting cracks without inclusions. The calculated estimates of the stress intensity factors differ from some known exact solutions by less than 10% provided the cracks or the inclusions are not very close to each other. Approximately, the problem can be treated as crack propagation in an equivalent homogeneous macroscopic continuum for which the apparent fracture toughness increases or decreases as a function of the crack length. Such variations are calculated for staggered inclusions. They are analogous to *R*-curves in nonlinear fracture mechanics. They depend on the volume fraction of the inclusions, their spatial distribution and the difference between the elastic properties of the inclusions and the matrix. Large variations (of the order of 100%) are found depending on the location of the crack and its propagation direction with respect to the inclusions.

INTRODUCTION

Most particulate or fiber-reinforced composites do not fail by propagation of a single microcrack. Typically, these materials are capable of sustaining significant loads while multiple microcracks propagate. In concrete loaded in uniaxial tension or compression, acoustic emission analyses (Legendre 1984; Maji et al. 1990) and X-ray microscopic observations (Darwin and Dewey 1989) show that distributed microcracks and damage localization exist in the material prior to failure. In these brittle heterogeneous composites, cracks are often initiated at the interface between the matrix and the aggregate pieces, and they propagate into the matrix eventually. Distributed cracking is also observed in fiber composites, the behavior of which in the planes normal to the fibers is similar to a two-dimensional particulate composite (Highsmith and Reifsnider 1982).

The key problem in developing a theory explaining such observations is how to take into account the effect of the heterogeneities. Pijaudier-Cabot

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and Dvorak (1990) recently proposed an approximation method for estimating the variation of the stress intensity factor and the inherent toughening effect at the tip of a crack that touches the interface between two elastic materials. In the case of concrete-like materials, which are the main motivation for this paper, most studies considered that the interactions among cracks or between aggregate pieces and cracks could reasonably be neglected, except in some special cases.

Zaitsev (1985) developed a rather comprehensive model in which the inclusion-crack interaction is neglected and each crack may interact only with its closest neighbor. However, the postpeak softening a response of concrete specimens could not be obtained with this method. More recently, Huang and Li (1989) and Hu et al. (1986) used similar ideas and proposed models in which the toughening (i.e., crack arrest) effect of the inclusions was incorporated. Although the mechanical interaction effects were still lacking, crack deflection mechanisms were represented statistically (Faber et al. 1983; Evans and Faber 1983). The effect of crack-inclusion interaction on dynamic crack propagation was studied by Sih and Chen (1980).

The effect of crack interaction has recently been considered in the studies of micromechanics of damage in concrete or ceramics (Horii et al. 1989; Ortiz 1988; Bažant et al. 1989; Kazemi and Pijaudier-Cabot 1989), and several approximation schemes for estimating crack-interaction effects have been proposed [see e.g., Kachanov (1987); Horii and Nemat-Nasser (1985)]. In particular, the importance of crack interaction at the onset of damage localization has been proven to be a fundamental aspect that justifies partial nonlocality of the constitutive relations at the macroscopic level, i.e., for the homogenized damaged medium (Pijaudier-Cabot and Berthaud 1990).

Some investigations have led to a striking conclusion: according to thermodynamics and stability analyses, most regular crack systems such as parallel equidistant cracks, periodic arrays of cracks and some colinear crack systems cannot be reached by a stable path under usual load or displacement control conditions (Bažant 1989; Bažant and Cedolin 1991; Bažant 1987b; Bažant and Tabbara 1989). Such models incorrectly predict that only a single crack ought to propagate. Thus, stable states of diffuse damage consisting of a system of tensile microcracks cannot exist according to these mathematical models in the first place, although they have been observed experimentally. Furthermore, the predicted shape of the softening postpeak load-displacement curve does not agree with experience and snap-back instability is predicted to occur earlier than seen in tests (Bažant 1987a). These discrepancies suggest that the mechanical effect of inhomogeneities cannot be ignored in modeling the evolution of damage and its progressive localization in concrete-like materials. This provided the motivation for the present study.

Solutions for some cases of the interaction between a crack and an inclusion in an elastic matrix exist [see e.g., Kunin and Gommerstadt (1985); Erodogan et al. (1974)]. They are based on a system of singular integral equations, which, however, appears to be intractable in the cases where several inclusions interact with the crack. Mura's equivalent inclusion method (Furuhashi et al. 1981) poses similar problems as it requires computation of integrals that may not converge absolutely when the inclusions are periodically distributed in an infinite medium.

In this paper [which is based on a conference paper by Pijaudier-Cabot et al. (1990)], we present an approximation scheme for solving the problem of

interaction between cracks and inclusions. The method can be viewed as an extension of Kachanov's superposition scheme (1987) for an interacting crack system without inclusions. Similar extensions could be made using the method of pseudotractions (Horii and Nemat-Nasser 1985).

The paper is organized as follows. First, the approximation method is developed, considering the simple case of one crack interacting with an inclusion, and verified by comparisons with solutions available in the literature. Second, an extension of this technique to the situation in which one crack interacts with several periodically distributed inclusions is carried out. Finally, the effect of the inclusions on crack propagation is interpreted in terms of an apparent fracture toughness of the homogenized composite. The ultimate objective is to develop a realistic model for the fracture process zone in composites.

The study is restricted to cases in which the bond between the matrix and the inclusion is perfect. Partial debonding and interfacial cracking will not be considered. This simplification is realistic especially for composites such as high-strength concrete or light-weight concrete.

Interaction between Crack and Inclusion

Consider an infinite two-dimensional solid subjected to remote uniform boundary tractions producing a uniform stress field $\boldsymbol{\sigma}_\infty$. The solid is made of a linear elastic material of stiffness matrix $\mathbf{D}_m$. It contains a crack of length $2c$ and an elastic circular inclusion (inhomogeneity) of radius R and stiffness matrix $\mathbf{D}_a$ [Fig. 1(a)]. The crack center is located at distance b from the center of the inclusion. The crack orientation is arbitrary. For such a solid, we seek an estimate of the stress intensity factors at the crack tips denoted as points A and B. For the sake of simplicity, we restrict attention to the case of circular inclusions, although the method we are going to develop is general and can, in principle, be extended to inclusions of arbitrary (smooth) shapes.

The stress and displacement fields for this problem can be solved by superposing the solutions of two simpler problems [Fig. 1(a)]:

- Subproblem I: The solution for the infinite solid without any crack containing the given inclusion and loaded by the remote tractions corresponding to $\boldsymbol{\sigma}_\infty$.
- Subproblem II: The solution for the infinite cracked solid loaded by distributed normal and tangential forces $\mathbf{p}(\mathbf{x})$ on the crack faces Γ_c, that cancel the stresses on the crack line obtained in I.

By superposition, the equilibrium condition for the crack surface may be written as

$$\boldsymbol{\sigma}\cdot\mathbf{n}(\mathbf{x}) + \mathbf{p}(\mathbf{x}) = 0 \quad \text{on} \quad \Gamma_c \qquad (1)$$

in which $\boldsymbol{\sigma}$ denotes the stress field solution of subproblem I calculated at the imaginary crack surface Γ_c and $\mathbf{n}(\mathbf{x})$ is the outward normal to Γ_c at a point with cartesian coordinates $\mathbf{x}$. Ideally, (1) should be satisfied exactly at every point of Γ_c and superposition would then yield an exact result. For the sake of simplicity, we assume that (1) is satisfied only approximately, in the average sense, that is

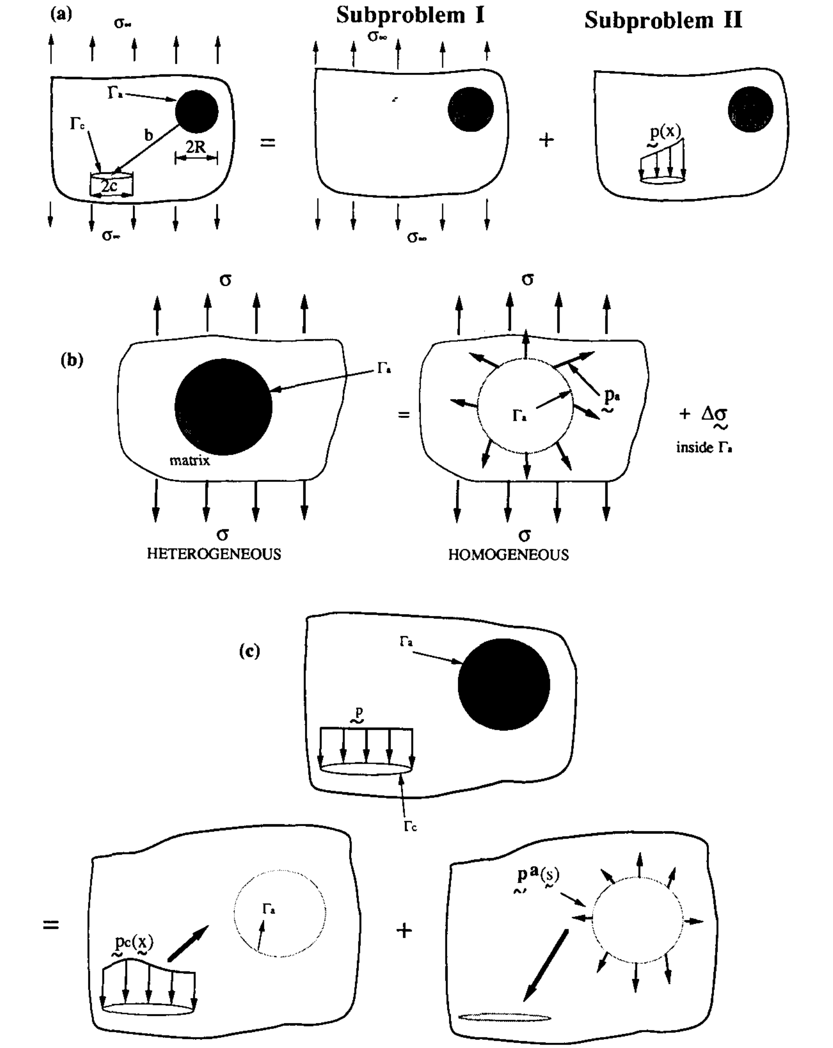

FIG. 1. Crack Interacting with Inclusion: (*a*) Superposition Scheme; (*b*) Duhamel-Neuman Analogy; (*c*) Superposition in Subproblem II

$$\langle \boldsymbol{\sigma} \cdot \mathbf{n}(\mathbf{x}) + \mathbf{p}(\mathbf{x}) \rangle = 0 \qquad (2)$$

in which the brackets $\langle\ \rangle$ denote the averaging over Γ_c. This simplification is inspired by Kachanov's (1987) approximation scheme for interacting crack systems in homogeneous solids without inclusions, which has been showed to be satisfactory in most situations. In Kachanov's scheme as well as here, the averaging is justified by the St. Venant principle: the errors represent a self-equilibrated stress field that must be decaying very rapidly with the distance from the crack and is, therefore, negligible for a sufficient separation of the crack and inclusion. Moreover, even if the crack tip is close, its K_I

value depends on the energy release rate from the entire structure rather than just the stresses in the vicinity.

Subproblem I

For the sake of simplicity, attention is restricted to plane elasticity. The perturbation stress due to the presence of one inclusion is given by the well-known Eshelby's solution [see e.g., Mura (1987)]. Since we intend to deal with many inclusions as well as interacting cracks, it appears preferable to devise a simpler, iterative, solution. From the stress field $\boldsymbol{\sigma}$, which is a solution of subproblem I, we can calculate the unbalanced stress field $\Delta\boldsymbol{\sigma}$ inside the inclusion of contour Γ_a:

$$\Delta\boldsymbol{\sigma} = (\mathbf{D}_a - \mathbf{D}_m):\boldsymbol{\epsilon} \qquad (3a)$$

with

$$\boldsymbol{\epsilon} = \mathbf{D}_m^{-1}:\boldsymbol{\sigma} \qquad (3b)$$

while in the matrix outside Γ_a, the stresses $\Delta\boldsymbol{\sigma}$ vanish. The unbalanced stresses $\Delta\boldsymbol{\sigma}$ can be equilibrated by applying tractions $\Delta\boldsymbol{\sigma}.\ \mathbf{n}_a$ on interface Γ_a. Since these tractions do not exist in reality, the opposite unbalanced interface tractions must act on the interface Γ_a in the composite

$$\mathbf{p}_a = -\Delta\boldsymbol{\sigma} \cdot \mathbf{n}_a \quad \text{on } \Gamma_a \qquad (4)$$

in which $\mathbf{n}_a$ is the unit outward normal of the boundary curve Γ_a of the inclusion, and $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are the strain and stress tensor inside the inclusion. The stress field in subproblem I may be written as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^* \quad \text{outside } \Gamma_a \qquad (5a)$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \Gamma\boldsymbol{\sigma} \quad \text{inside } \Gamma_a \qquad (5b)$$

in which $\boldsymbol{\sigma}^*$ = an equilibrium stress field when stiffness $\mathbf{D}_a$ of the inclusion is changed to $\mathbf{D}_m$, i.e., when the properties of the infinite solid are uniform. Eqs. (3)–(5) can also be obtained from the Duhamel-Neuman analogy [see e.g., Lin (1968); Muhkelishvili (1953)], which is widely used in thermoviscoelasticity and creep and is illustrated in Fig. 1(*b*). This analogy transforms a problem of elasticity of a heterogeneous solid into an equivalent problem of a homogeneous solid that can be decomposed into a superposition of standard problems for which analytical solutions (e.g., complex potentials) exist. Obviously, the unbalanced stress field $\Delta\boldsymbol{\sigma}$ is the unknown in the equivalent problem. Its determination calls for an iterative procedure.

1. The starting solution is $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}_\infty$ everywhere. It gives the first estimate of $\mathbf{p}_a$ according to (4). The curve Γ_a is subdivided into segments of length ds and the tractions $\mathbf{p}_a$ are replaced by concentrated forces $\mathbf{p}_a(\mathbf{s})\ ds$ acting at the center points of coordinate $\mathbf{s}$ of the segments. Then one may use the well-known two-dimensional solution for a concentrated force $\mathbf{p}$ applied at point $\mathbf{s}$ of an infinite homogeneous elastic space denoted as $\mathbf{f}[\mathbf{p}(\mathbf{s})]$; see e.g., Timoshenko and Goodier (1970) or Mukhelishvili (1953). The normal and shear components of the stress tensor $\mathbf{f}$ with respect to the rotated cartesian axes (x', y') at a point of cartesian coordinate (x, y) are

$$f'_x = p[1 - \nu - 2(1 + \nu)\sin^2(\theta)]\cos(\theta)(4\pi r)^{-1} \qquad \text{[Continued]}$$

$$f'_y = p[-3 - \nu - 2(1+\nu)\sin^2(\theta)]\cos(\theta)(4\pi r)^{-1}$$

$$f'_{xy} = -p[1 - \nu + 2(1+\nu)\cos^2(\theta)]\sin(\theta)(4\pi r)^{-1} \quad (6)$$

in which r = distance between points (x,y) and $\mathbf{s}$; axis x' coincides with the direction of p; and θ = the angular deviation of the line connecting points (x,y) and $\mathbf{s}$ from the direction of p. Superposition of these solutions yields the stress $\boldsymbol{\sigma}_a$ caused by tractions $\mathbf{p}_a(\mathbf{s})$ in an infinite homogeneous elastic space:

$$\boldsymbol{\sigma}_a = \int_{\Gamma_a}^{\mathbf{f}} [\mathbf{p}_a(\mathbf{s})]ds \quad (7)$$

A new stress field $\boldsymbol{\sigma}^*$ inside Γ_a is obtained as:

$$\boldsymbol{\sigma}^* = \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_\infty \quad (8)$$

2. The new unbalanced pressures $\mathbf{p}_a$ are then recalculated from (3)–(4). Eq. (7) yields the new field $\boldsymbol{\sigma}_a$.

3. Step 2 is iterated until the change $\mathbf{p}_a^{l+1} - \mathbf{p}_a^l$ of the unbalanced interface tractions from iteration becomes small enough. This is determined on the basis of the norm $\|\mathbf{p}_a(\mathbf{s})\| = \int_{\Gamma_a} |\mathbf{p}_a(\mathbf{s})|ds$ where $|\mathbf{p}_a(\mathbf{s})|$ is the length of vector $\mathbf{p}_a(\mathbf{s})$. The convergence criterion is that

$$\frac{\|\mathbf{p}_a(\mathbf{s})\|^{l+1}}{\|\mathbf{p}_a(\mathbf{s})\|^{l}} - 1 \leq e \quad (9)$$

in which e = a given small tolerance; $e = 0.01$ was used in computations and usually less than five iterations were needed. The convergence is very fast, and for small enough e this iterative procedure can approximate the exact solution (for uniform p) as closely as desired. It can be shown that the iterates of P_a form a geometric progression.

Subproblem II

Consider now that there is a crack in the matrix near the inhomogeneity and that the crack faces Γ_c are loaded by a uniform pressure distribution $\langle \mathbf{p}(\mathbf{x})\rangle$. The boundary at infinity is stress free. Again, we can apply the Duhamel-Neuman analogy in order to compute the interaction stress field due to the presence of the inclusion, and subsequently the distribution of internal pressure on the crack faces. For this, we use the superposition scheme depicted in Fig. 1(c).

First, the body without the inclusion is loaded by an unknown average pressure $\langle \mathbf{p}_c(\mathbf{x})\rangle$. This causes interface tractions $-\Delta\boldsymbol{\sigma}_c \cdot \mathbf{n}_a$ on the imagined contour of the inclusion as given by (4).

Next, we consider the uncracked heterogeneous body loaded by these unbalanced pressures on Γ_a. From subproblem I we can get the solution stress field and the pressure distribution on the imagined contour of the crack $\mathbf{p}_c^a(\mathbf{x})$:

$$\mathbf{p}_c^a(\mathbf{x}) = \left\{ \int_{\Gamma_a} \mathbf{f}[-\Delta\boldsymbol{\sigma}_c \cdot n_a(\mathbf{s})]ds \right\} \cdot \mathbf{n} \quad (10)$$

Superposition yields

$$\langle \mathbf{p}\rangle \approx \mathbf{p}_c(\mathbf{x}) + p_c^a(\mathbf{x}) \quad \text{on } \Gamma_c \quad (11)$$

Note that this superposition method, with the average pressure approximation on the crack surface, is similar to Kachanov's (1987) approximate solution for interacting cracks except that instead of two cracks we deal with one crack and one inclusion. In (11), the right-hand side terms are not constant. If we restrict the present analysis to configurations in which the interactions are small, the superposition equation may be approximated by:

$$\langle \mathbf{p}\rangle = \langle \mathbf{p}_c(\mathbf{x})\rangle + \langle p_c^a(\mathbf{x})\rangle \quad \text{on } \Gamma_c \quad (12)$$

Under these two assumptions, the superposition equation [(11)] has a single vector unknown $\langle \mathbf{p}_c(\mathbf{x})\rangle$:

$$\langle \mathbf{p}\rangle = (\mathbf{1} + \boldsymbol{\Lambda}_a)\cdot\langle \mathbf{p}_c(\mathbf{x})\rangle \quad (13)$$

with

$$\boldsymbol{\Lambda}_a \cdot \langle \mathbf{p}_c(\mathbf{x})\rangle = \frac{1}{2c}\int_{\Gamma_c} \left\{ \int_{\Gamma_a} \mathbf{f}[-\Delta\boldsymbol{\sigma}_c \cdot \mathbf{n}_a(\mathbf{s})]ds \right\} \cdot \mathbf{n}dx \quad (14)$$

in which $\mathbf{1}$ is the 2×2 identity matrix, and $\boldsymbol{\Lambda}_a$ is a full 2×2 matrix which couples the mode I crack opening and the mode II crack opening. It can be regarded as a transmission factor that represents the average influence of the inclusion on the crack. Note at this point that if $\boldsymbol{\sigma}_c$ is not computed from the constant pressure distribution $\langle \mathbf{p}_c(\mathbf{x})\rangle$, the unknown in the problem would need to be solved iteratively (as in subproblem I) as $\boldsymbol{\Lambda}_a$ depends on $\mathbf{p}_c(\mathbf{x})$.

Substitution of (13) into (2) yields:

$$\langle \mathbf{p}_c(\mathbf{x})\rangle = -(\mathbf{1} + \boldsymbol{\Lambda}_a)^{-1}\cdot\langle \boldsymbol{\sigma}\cdot\mathbf{n}\rangle \quad (15)$$

The stress distribution on Γ_c is also computed using the right-hand side of (11) and, for example, the stress intensity factors for mode I crack opening are:

$$K_I(\pm c) = \frac{1}{\sqrt{\pi c}}\int_{-c}^{+c} \sqrt{\frac{c \pm x}{c \mp x}}\, \mathbf{p}_c(\mathbf{x})\mathbf{n}dx \quad (16)$$

As an example, Fig. 2 shows the results for the mode I stress intensity factors for a crack in an epoxy matrix located near a metallic inclusion. The remote loading is uniaxial tension parallel to the crack faces and plane strain is assumed. For simplicity, we analyze cases where (1) The crack is radial to the inclusion [Fig. 2(a)]; and (2) the crack is tangential to the inclusion [Fig. 2(b)]. In both situations the average tangential pressure distribution is zero and (15) has a scalar unknown. The radius of the inclusion is such that $R/c = 2$ and the material properties are $E_a/E_m = 23$, $\nu_a = 0.3$, $\nu_m = 0.35$ where E_a, E_m and ν_a, ν_m are the Young's moduli and Poisson's ratios of the inclusion and matrix, respectively. In the figures, K_I is normalized with respect to the stress intensity factor K_{Io} for a crack in an infinite homogeneous solid, which is $K_{Io} = \sigma_\infty\sqrt{\pi c}$. The approximation is compared to the analytical solution of Erdogan et al. (1974). For a radial crack [Fig. 2(a)], the approximation turns out to be very accurate. The error is only a few percent except if the crack and the inclusion are very close. When the crack is tangential to an inclusion [Fig. 2(b)] the present averaged superposition equations become rather inaccurate if the crack is close to the inclusion ($a/c <$ 4). The reason is that the stress fields in subproblems I and II have a large

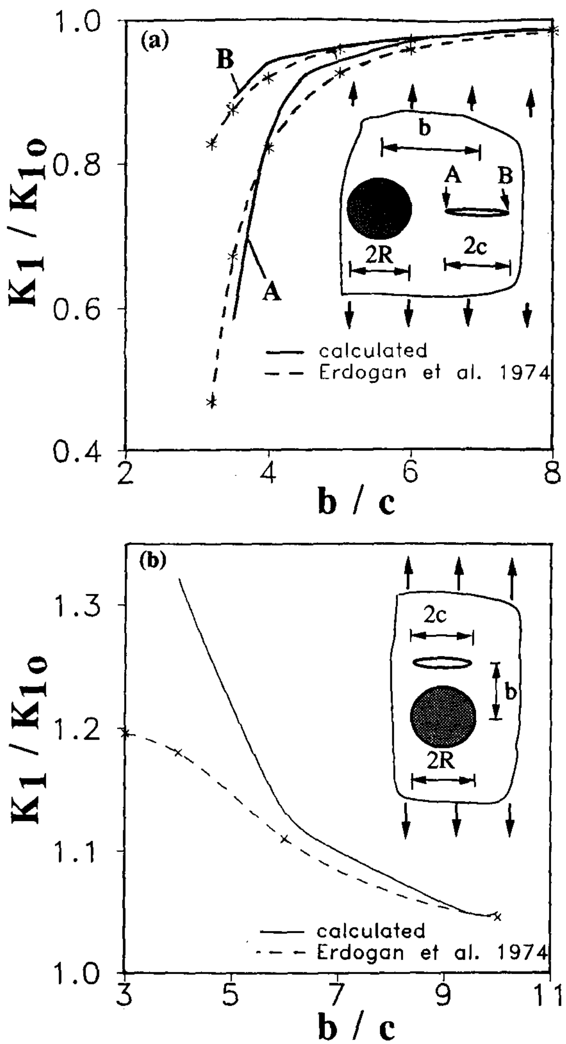

FIG. 2. Stress Intensity Factor for Crack in Epoxy near Inclusion: (*a*) Radial Crack; (*b*) Tangential Crack

variation over the imagined crack length.

Fig. 3 shows the results for a crack in an epoxy matrix located near a void. The same two configurations as in Fig. 2 are considered and the material stiffness of epoxy is equal to that in Fig. 2. Again, the quality of the approximation is quite acceptable unless crack and void become very close. Compared to the results in Fig. 2, the variation of stress intensity factors is the opposite. When the crack tip A approaches the void [Fig. 3(*a*)], the stress intensity factor K_I increases and tends to infinity, but when the tip approaches a stiffer inclusion, K_I decreases. The same remark holds when the

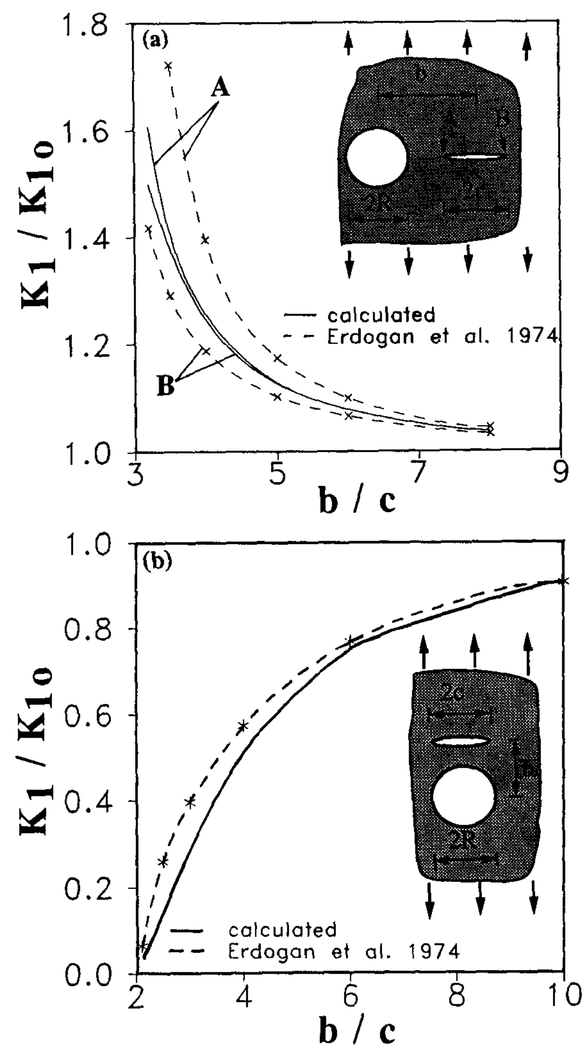

FIG. 3. Stress Intensity Factor for Crack in Epoxy near Void: (*a*) Radial Crack; (*b*) Tangential Crack

crack is tangential to the void or inclusion, although the stress intensity factors remain finite.

Interaction between Crack and Several Inclusions

We look now at an elastic solid that contains N elastic inclusions and one crack. The inclusions are arbitrarily distributed in the matrix. The inclusion contours are denoted as Γ_i ($i = 1, \ldots, N$) and for the sake of simplicity all the inclusions are assumed to be made of the same material of stiffness $\mathbf{D}_a$.

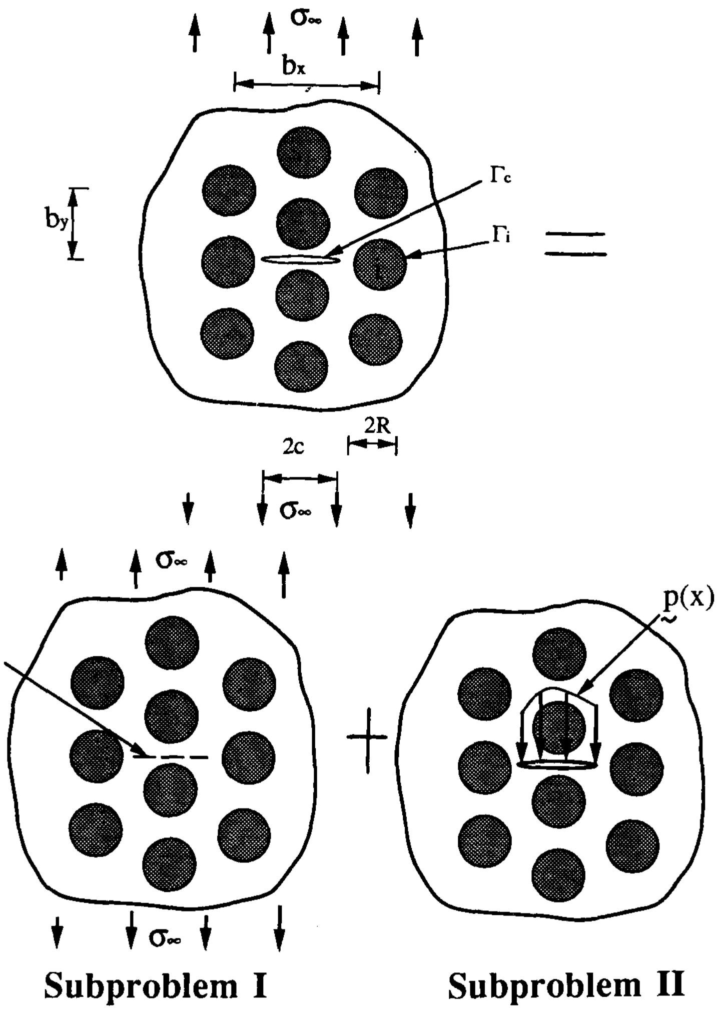

FIG. 4. Crack Interacting with Periodic Array of Inclusions: Superposition Scheme

The superposition method is now applied as follows (see Fig. 4).

First in subproblem I, we solve again for the stress field in the composite without the crack loaded by tractions corresponding to $\boldsymbol{\sigma}_\infty$. Then, in subproblem II, the composite is free from the remote boundary tractions and it is loaded by an unknown internal pressure $\mathbf{p}(\mathbf{x})$ on the crack contour Γ_c. The superposition equation [(2)] is again applied in the average sense.

Subproblem I

When the uncracked composite contains several inclusions, the interactions are an important factor in the evaluation of the local stress and strain fields. As we will see, the Duhamel-Neuman analogy is also easy to implement.

Since the problem remains elastic, the effect of each inclusion can be superposed as a first approximation neglecting the interactions. The following iterative procedure, similar to that described before, yields the effect of the interactions on the local stress field in the matrix.

1. The initial stress field is

$$\boldsymbol{\sigma}^* = \boldsymbol{\sigma}_\infty + \sum_{i=1}^{N} \boldsymbol{\sigma}_i \quad (17)$$

in which the $\boldsymbol{\sigma}_i$ = the stress due to the presence of inclusion i alone in the matrix (Eshelby's solution). The unbalanced pressures $\mathbf{p}_i$ on the contour Γ_i of each inclusion i are calculated from $\boldsymbol{\sigma}^*$ according to (4)–(5). The stress $\boldsymbol{\sigma}_i$ due to $\mathbf{p}_i$ is then calculated as if each inclusion i were alone in the infinite solid, i.e.

$$\boldsymbol{\sigma}_i = \int_{\Gamma_i} \mathbf{f}[\mathbf{p}_i(\mathbf{s})]ds \quad (18)$$

A new total stress field is computed from (17) using superposition.

2. From $\boldsymbol{\sigma}$ (5), the unbalanced pressures $\mathbf{p}_i$ on each contour Γ_i are recalculated using (4). Then again the stress $\boldsymbol{\sigma}_i$ due to $\mathbf{p}_i$ is calculated from (18) as if the inclusions were alone, and by superposing $\boldsymbol{\sigma}_i$, the new total stress field obtained from (17).

3. Step 2 is iterated until the unbalanced tractions $\mathbf{p}_i^l$ ($i = 1, \ldots N$) resulting from $\boldsymbol{\sigma}$ in iteration number l differ negligibly from those at iteration number $l - 1$. This is determined according to the convergence criterion in (9).

The foregoing algorithm converges quite rapidly. Normally, convergence is reached in less than five iterations provided the inclusions are not too stiff compared to the matrix ($E_a/E_m < 7$) (but for perfectly rigid inclusions the present iterative method does not work). When the inclusions are periodically distributed, the unbalanced pressures $\mathbf{p}_i$ should be identical on each contour Γ_i ($i = 1, \ldots, N$), and in that case the convergence criterion does not need to be applied for each inclusion.

Fig. 5 gives an example of the calculated stress distribution of stress in a two-dimensional composite with periodically spaced circular inclusions of radius R. The remote loading is a unit uniaxial tension in the y-direction. The inclusion centers are located on a square grid of spacing $b_x = b_y = 3R$. The material properties are $E_a/E_m < 3$ and $\nu_a = \nu_m = 0.2$. Plane stress is assumed and the central inclusion is assumed to interact only with its 48 closest neighbors. The stresses σ_{xx} and σ_{yy} are computed along the axis of symmetry of two adjacent inclusions, and obviously $\sigma_{xy} = 0$. Convergence was achieved in 3 iterations, with tolerance $e = 0.01$. The results are graphically undistinguishable from those obtained by the equivalent inclusion method (Furuhashi et al. 1981).

Subproblem II

The crack is loaded by a uniform internal pressure $\mathbf{p}(\mathbf{x})$ on its contour Γ_c. From superposition,

$$\mathbf{p}(\mathbf{x}) = \mathbf{p}_c(\mathbf{x}) + \sum_{k=1}^{N} \mathbf{p}_{\mathrm{int}}^k(\mathbf{x}) \quad \text{on } \Gamma_c \quad (19)$$

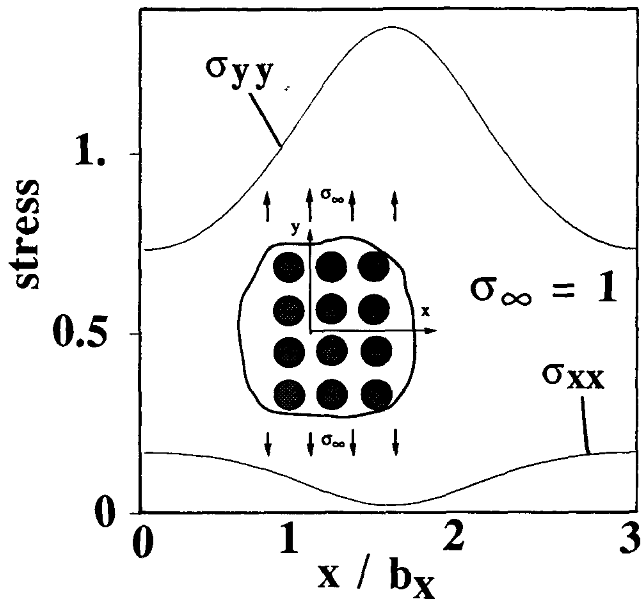

FIG. 5. Local Stress Field in Composite with Periodically Distributed Inclusions

In this equation, which is similar to (11), $\mathbf{p}_c$ = the distribution of the internal pressure applied on Γ_c, and $\mathbf{p}^k_{int}$ ($k = 1, \ldots, N$) = the interaction terms due to the presence of the inclusions. $\mathbf{p}^k_{int}$ is computed at the imagined location of the crack as if the composite were uncracked. Again, there are two types of contributing terms in $\mathbf{p}^k_{int}$.

The first type of contribution arises from the effect of the loading $\mathbf{p}_c$ on the inclusion k which is assumed to be alone in the matrix with the crack (same as in the previous section of the paper). This term denoted as $\mathbf{p}^k_k$ is computed according to (10):

$$\mathbf{p}^k_k = \left\{ \int_{\Gamma_k} \mathbf{f}[-\Delta\boldsymbol{\sigma}_c \cdot \mathbf{n}_k(\mathbf{s})]ds \right\} \cdot \mathbf{n} \quad \text{(20)}$$

in which $\boldsymbol{\sigma}_c$ = the stress field due to the crack loaded by $\mathbf{p}_c(\mathbf{x})$, calculated for the infinite solid without inclusions; $\Delta\boldsymbol{\sigma}_c$ = the unbalanced stresses computed on the imagined contour Γ_k of the inclusion k; and $\mathbf{n}_k$ = the outward unit normal vector of Γ_k.

The second type of contribution is the interaction between the inclusion $j(j \neq k)$ and inclusion k, and its influence on the crack faces. Each inclusion in the composite is subjected to the stress $\boldsymbol{\sigma}_c$. The value of $\mathbf{p}^k_j$ can be computed in the same manner as in subproblem I but the stress fields $\boldsymbol{\sigma}_c$ is substituted to the remote field $\boldsymbol{\sigma}_\infty$. From (17) and (18) we obtain:

$$\mathbf{p}^k_j(\mathbf{x}) = \left\{ \int_{\Gamma_k} \mathbf{f}[-\Delta\boldsymbol{\sigma}_j \cdot \mathbf{n}_k(\mathbf{s})]ds \right\} \cdot \mathbf{n} \quad \text{(21)}$$

in which $\Delta\boldsymbol{\sigma}_j$ = the stress field due to the unbalanced pressure $\mathbf{p}_j$ acting on contour Γ_j of normal vector $\mathbf{n}_j$:

$$\boldsymbol{\sigma}_j = \int_{\Gamma_j} \mathbf{f}[-\Delta\boldsymbol{\sigma}_c \cdot \mathbf{n}_j(\mathbf{s})]ds \quad \text{(22)}$$

Superposition yields:

$$\mathbf{p}^k_{int}(\mathbf{x}) = \sum_{j=1}^{N} \mathbf{p}^k_j(\mathbf{x}) \quad \text{(23)}$$

and after substitution into (19),

$$\mathbf{p}(\mathbf{x}) = \mathbf{p}_c(\mathbf{x}) + \sum_{k=1}^{N} \sum_{j=1}^{N} \mathbf{p}^k_j(\mathbf{x}) \quad \text{(24)}$$

We assume again that (24) needs to be satisfied only in the average sense:

$$\langle \mathbf{p}(\mathbf{x}) \rangle = \left[\mathbf{1} + \sum_{k=1}^{N} \sum_{j=1}^{N} \boldsymbol{\Lambda}^k_j \right] \cdot \langle \mathbf{p}_c(\mathbf{x}) \rangle \quad \text{(25)}$$

in which $\boldsymbol{\Lambda}^k_k$ = the transmission factor due to inclusion k considered to be alone with the crack; and $\boldsymbol{\Lambda}^k_j$ = the transmission factor due to interaction between inclusion k and inclusion j.

If $\boldsymbol{\sigma}_c$ is the stress field due to the crack alone subjected to the uniform internal pressure $\langle \mathbf{p}_c(\mathbf{x}) \rangle$, (25) is linear in $\langle \mathbf{p}_c(\mathbf{x}) \rangle$ and has a single vector unknown. According to this assumption, the transmission factors do not depend on the shape of the distribution of $\mathbf{p}_c(\mathbf{x})$. This simplifying assumption is acceptable if the distances between any two inclusions are not too small, as we will see next in comparisons with the results from the literature.

Fig. 6 shows an example of the calculated variation of the mode I stress intensity factor K_I at the tip of a crack located between two circular voids as a function of the crack length and of the spacing between the voids. The remote loading is uniaxial tension perpendicular to the crack. The center of the crack is equidistant from the centers of the adjacent voids. The results are compared with the known analytical solution given in Tada et al. (1985).

If the distance between the voids is large compared to their radius, the approximation is seen to be adequate (error less than 10%). However, when the crack length increases, the effect of the voids becomes localized in a small segment of the crack surface Γ_c and the agreement with the analytical solution is less than satisfactory. This discrepancy is mainly due to the two successive averagings of the distributions of internal pressures on the crack faces [averaging of $\mathbf{p}(\mathbf{x})$ first and of $\mathbf{p}_c(\mathbf{x})$ second]. Another limitation is that the approximation loses its accuracy when the voids get too close to each other.

The example in Fig. 7 shows the variation of K_I for a crack propagating in a composite containing a square array of identical circular inclusions ($b_x = b_y$). The center of the crack is at equal distances from two neighbor inclusions along the y-axis and the crack propagates in the x-direction due to tensile stress σ_{yy} (see Fig. 5). Plane stress is considered, with $\nu_a = \nu_m = 0.2$ and $E_a/E_m = 3$. The spacings between the inclusions are equal, $b_x = b_y$.

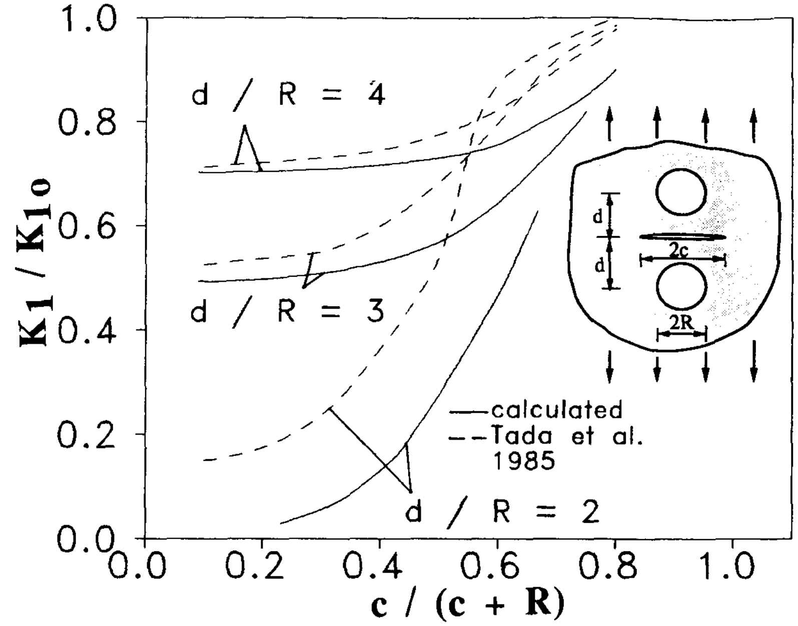

FIG. 6. Stress Intensity Factor for Crack Interacting with Two Circular Voids

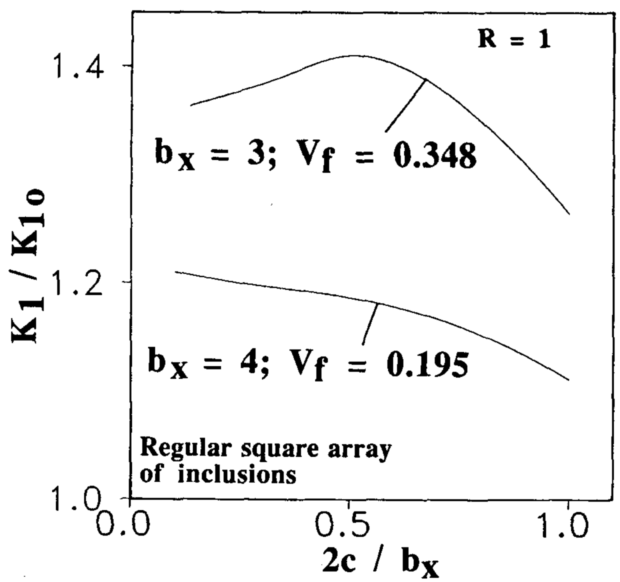

FIG. 7. Stress Intensity Factor at Tip of Crack Propagating in Composite with Square Array of Inclusions for Two Different Volume Fractions of Inclusions

Two volume fractions V_f of inclusions are chosen: $V_f = 0.195$: $b_x/R = 4$, and $V_f = 0.35$: $b_x/R = 3$. Denoting K_{Io} = stress intensity factor if there were no inclusions, we see that the effect of the inclusions is to cause the ratio K_I/K_{Io} to decrease with the crack length c, except when the spacing b_x is too small. This means that the apparent stress intensity factor increases during crack propagation. So the composite behaves as if the crack followed an R-curve. Furthermore, the stress intensity factor increases with the volume fraction of inclusions. According to this result, cracks in a densely packed composite must occur earlier than in a loosely packed composite. Finally, we can see that even for a low-volume fraction of inclusions, the amplification of the stress intensity factor compared to K_{Io} is quite important.

The present approximate method could no doubt be combined with Kachanov's method (1987) and thus be generalized for a system of cracks in a composite. However, programming the computation of the various transmission coefficients seems to be too tedious.

Apparent Fracture Toughness of Composite

As we have observed from Fig. 7, inclusions may cause the composite to behave as a homogeneous solid with a rising R-curve. The knowledge of such an apparent R-curve would permit a much simpler calculation of fracture in composites. In such an approach, the interaction between cracks is uncoupled from the interaction between the cracks and the inclusions. Similar assumptions have been made by Mori et al. (1988) and Gao and Rice (1988), who used a perturbation method to analyze fiber-reinforced composites in which the values of the elastic moduli of the matrix and the inclusions are sufficiently close. More precisely let K_{co} be the fracture toughness of the matrix. According to Griffith's criterion, crack propagation occurs when $K_I = K_{co}$. For a crack length c in a macrohomogeneous composite loaded with tensile stress σ_∞, we may write $K_I = K_{Io} F(c)$ where $K_{Io} = \sigma_\infty \sqrt{\pi c}$, and where $F(c)$ is a certain amplification function that is computed from the crack-inclusions interaction. The estimation of K_I yields the apparent fracture toughness K_c of the composite

$$K_c = \frac{K_{co}}{F(c)} \quad (26)$$

In most studies [see e.g., Zaitsev (1985) and Zaitsev et al. (1986)]. $F(c)$ was assumed to remain constant or to change only when the crack touches an inclusion (Huang and Li 1989). Fig. 8 presents the approximate variation of fracture toughness for a crack propagating symmetrically in a composite made of regular staggered circular inclusions embedded in an elastic matrix. The radii of the inclusions are equal and denoted as R ($R = 1$). The volume fraction of inhomogeneities is $V_f = 0.7$. Plane stress is assumed with $E_a/E_m = 3$ and $\nu_a = \nu_m = 0.2$. The remote boundary traction is uniaxial tension perpendicular to the crack faces (mode I crack opening).

Three configurations have been analyzed [Fig. 8(a)]. In configuration 1, the crack propagates toward the centers of two inclusions. In configuration 3, the center of the crack is at equal distances from two rows of inclusions. Configuration 2 is intermediate between configurations 1 and 3.

Fig. 8(b) shows the variation of the apparent fracture toughness with the

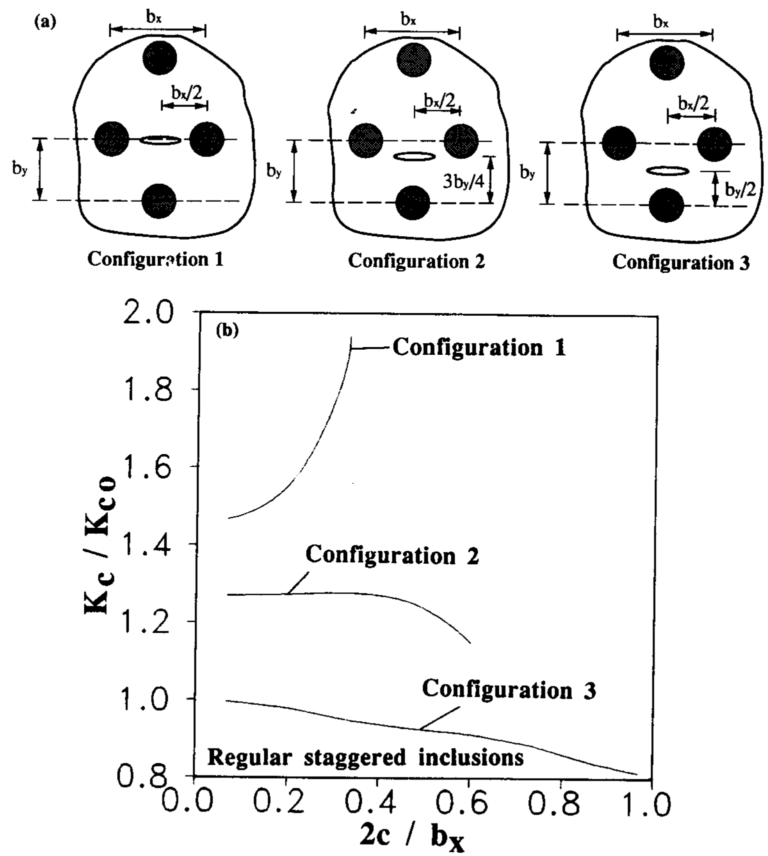

FIG. 8. Apparent Fracture Toughness of Composite with Staggered Inclusions: (*a*) Configurations Analyzed; (*b*) Fracture Toughness versus Crack Length

crack length according to (26). We see that these variations may be radically different depending on the configurations analyzed. Configurations 1 and 3 give the highest and lowest values of the apparent mode I fracture toughness, respectively. The more drastic variation is obtained when the crack propagates toward an inclusion; this corresponds to the maximum possible toughening.

These variations of apparent fracture toughness have a great influence on stability of interactive crack systems. As we see, the mechanical effect of the inclusions cannot be neglected in crack propagation studies as the fracture toughness of the equivalent medium may vary by as much as 100%. It should be stressed that these curves are valid only if the crack does not touch the inclusions. Otherwise, the singular stress field at the tips of the crack would need to be modified.

To exemplify the influence of the spatial distribution of the inclusions at a constant volume fraction, Fig. 9 shows the variation of apparent toughness for a regular ($b_x = b_y = 3R$) staggered distribution of inclusions and a non-

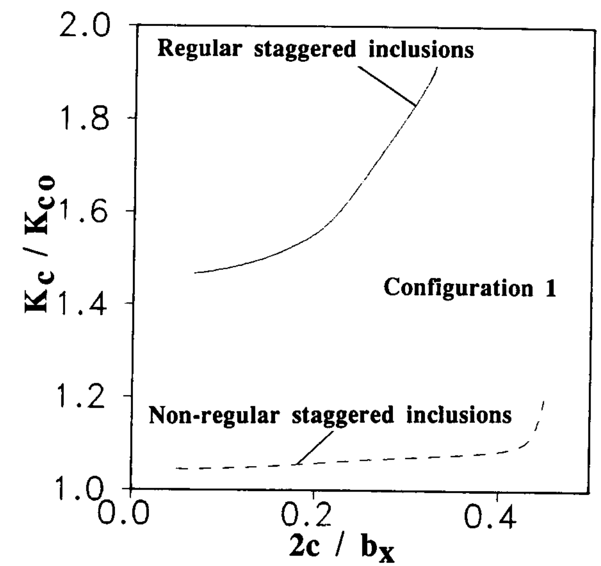

FIG. 9. Influence of Spatial Distribution of Inclusions on Apparent Toughness of Composite

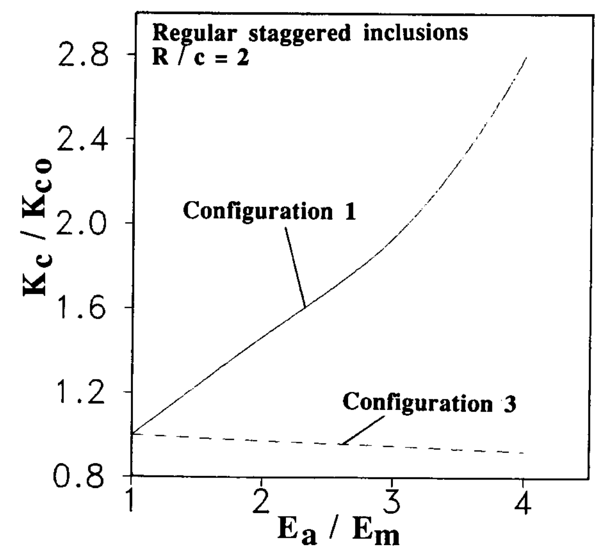

FIG. 10. Influence of Relative Elastic Young's Modulus of Inclusions on Apparent Toughness of Composite

regular staggered distribution of inclusions ($b_x = 4R$, $b_y = 2.25R$). The inclusion spacings are such that the volume fraction is the same, $V_f = 0.7$. Configuration 1 is chosen with the same material properties as in Fig. 8. Again, there is a large difference between the two cases. The nonregular staggered distribution (dashed curve) provides the lowest apparent fracture toughness. This suggests that inclusions that are radial to the crack have the largest influence since b_x has been increased. The toughening effect, which is important for the regular distribution, is delayed as the tips of the crack are more distant from the inclusions.

The effect of the variation of the ratio E_a/E_m of the elastic moduli of the inclusion and the matrix is shown in Fig. 10. The apparent fracture toughness of the composite has been computed for the crack length $2c = R$, with $\nu_a = \nu_m = 0.2$. The composite contains a regular staggered distribution of inclusions with $V_f = 0.7$. The solid line corresponds to configuration I and the dashed line corresponds to configuration 3 [see Fig. 8(a)]. We obtain the upper and lower bounds of variation of toughness for a crack opened under mode I as a function of the ratio E_a/E_m. For configuration 1 this curve is certainly not linear. It should be pointed out that for large values of E_a/E_m, convergence could not be reached in subproblem I ($E_a/E_m > 7$). The range of variation of E_a/E_m showed in Fig. 10 corresponds to the usual values for concrete.

From the present analysis one might get the impression that the length of crack extension needed to reach the asymptotic value of an R-curve is about as long as the inclusion spacing. No doubt this can be true only for periodic inclusion arrays. For random arrays, this length could be much longer.

Conclusions

1. The interaction between a crack and several inclusions can be analyzed by superposing known solutions of standard problems of elasticity. The method uses first Duhamel-Neuman analogy in order to transform the problem into a problem of elasticity of a homogeneous body in which the inclusions are replaced by the matrix and the boundary conditions are modified. A superposition scheme is proposed, similar to Kachanov's method for interacting cracks. The solution of the problem of interaction of one crack with many inclusions is reduced to the solution of a linear algebraic equation with transmission factors characterizing the interactions of the crack with each inclusion and of any two inclusions. Comparisons with exact results from the literature show that in most cases the method is sufficiently accurate for practical purposes (with an error better than 10%) when the inclusions and the crack are not too close to each other.

2. The variation of the apparent fracture toughness of the equivalent homogeneous medium (representing the inverse of the calculated variation of the mode I stress intensity factor at the tip of a crack propagating in the composite) is analogous to the R-curve in nonlinear fracture mechanics. Calculations show that the apparent fracture toughness depends on the volume fraction of the inclusions, on their spatial distribution, and finally on the elastic properties of the constituents of the composite. The largest (mode I) toughness is obtained when the crack propagates toward an inclusion and the lowest toughness corresponds to a crack propagating between two inclusions. The difference between these two cases can be of the order of 100%.

3. Finally, the results show that, for a given composite and for a fixed crack configuration, the mechanical effect of the interaction between the crack and the inclusions is not negligible. This effect is important for explaining stability of simultaneous propagation of many interacting cracks in a heterogeneous medium, as well as for determining the conditions under which stable states of diffuse damage can exist.

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