

GEOMETRIC DAMAGE TENSOR BASED ON MICROPLANE MODEL

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ABSTRACT: An appealing approach to formulate constitutive models for characterizing distributed damage due to microcracks and voids is continuum damage mechanics with the concepts of effective stress and strain equivalence. In that approach, in which damage is imagined to characterize the reduction of the net stress-transmitting cross-section area of the material, the constitutive model is separated into two independent parts, one for damage and the other for elastic and inelastic behavior (rheology) other than damage, which, if combined appropriately, give the overall constitutive behavior. However, the existing multidimensional formulations for damage are quite complex, and practical implementations capable of fitting experimental data are hard to obtain. The microplane models, by contrast, provide conceptual simplicity and close fits of multiaxial test data for concrete, soils, etc., although, as formulated in the past, various kinds of physical phenomena were mixed in the definition of the microplane stress-strain curves. In this work the microplane theory is reformulated in a manner that separates damage from rheology and makes the formulation fit the basic framework of continuum damage mechanics. Aside from a kinematic constraint between macrostrains and microstrains, the model satisfies a static constraint such that the effective microstresses are the resolved components of the effectiveness macrostresses.

INTRODUCTION

Inelastic phenomena are of two kinds: (1) Those that reduce material stiffness and cause strain-softening; and (2) those that do not. The former represent damage caused by nucleation, growth, and coalescence of microcracks and voids, and the latter consist of plastic slip, friction, creep, etc. Modeling each of these phenomena is difficult enough, but the complexity becomes formidable when both kinds of phenomena are combined, which occurs in many materials. In this paper we seek to separate these two phenomena to achieve simplification and clarity. We assume the microcracks and voids to remain homogeneously distributed so that the material can be microscopically treated as a continuum.

The real stress and strain fields at the microlevel are highly scattered and nonsmooth. Therefore, the continuum strains and stresses must be taken as the "average" values of these fields over a certain representative volume. However, applicability of damage in terms of such stresses and strains must be restricted to situations in which no localization phenomena take place, i.e., no macrocracking or shear-banding occurs. If the damage and strains

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can localize, unrealistic results can be obtained with this approach unless additional techniques such as crack band model, nonlocal models, or localization analysis are implemented in the context of the "smeared" approach to damage (Bažant and Cedolin 1979; Crisfield 1982; Pijaudier-Cabot and Bažant 1987; de Borst 1987; Ortiz 1987; Ortiz and Quigley 1989; Carol and Prat 1990). Alternatively, formulations can also be obtained by "discrete" crack models based on fracture mechanics with softening cohesive (crack-bridging) zone (Hillerborg et al. 1976; Hillerborg 1984).

The classical approach to distributed damage is the continuum damage mechanics. As originated by Kachanov (1958) for one-dimensional problems, the basic concept of continuum damage mechanics looks simple: a reduction of the net stress-carrying cross-section area fraction caused by the growth of microcracks and voids in the material microstructure. This is a purely geometric characteristic, reflecting the fact that the "effective" (or "true") stresses in the undamaged part of the cross section are higher than the stresses in the macroscopic continuum (the apparent stresses). A law can be postulated to predict the decrements of the current cross-sectional area in terms of stress, strain, and damage, and simple but useful results can be obtained for complex phenomena such as creep rupture of metals (Kachanov 1958; Rabotnov 1969a,b; Leckie 1978; Lemaitre 1984) and, more recently, the delayed failure of concrete under sustained uniaxial compression (Carol 1989).

The multiaxial generalization of continuum damage mechanics, however, poses a formidable challenge that, to a large extent remains unsolved. The net stress-carrying area fractions for the different directions in space cannot be easily integrated to give a vectorial or tensorial damage measure that would define the "effective" or "true" stress tensor from the macroscopic stress tensor and the apparent or reduced macroscopic stiffness tensor. A review of the existing multiaxial generalizations presented in the section headed "Existing Models for Distributed Damage" shows a number of possibilities, some of them rather complicated and not always very satisfactory in terms of data-fitting capabilities, especially when the classical macroscopic tensorial approach is pursued. This is the case when the constitutive model is formulated phenomenologically without recourse to any geometric concepts related to microstructure and the inelastic phenomena are formulated as functions of stress and strain invariants. Nevertheless, as will be shown in the section headed "Effective Stress and Damage Tensor Uncoupled from Rheology" some basic general structure for continuum damage models based on the concepts of true, or effective, stress and macro-micro strain compatibility can also be established in three dimensions. This basic structure provides a suitable framework for separating the constitutive model in two parts, one for damage and the other for rheology, each of which can be formulated independently and then combined. This provides a convenient flexibility for combining damage with various complex types of inelastic material behavior.

Compared to multiaxial continuum damage model theories, much closer fits of test data for concrete and considerable theoretical simplification (albeit at the expense of computer time requirements) is achieved by the microplane models. In general terms, already suggested for plasticity in a famous classical paper by G. I. Taylor (1938), this approach characterizes the material properties independently on planes of various orientation, which were initially called the slip planes (Batdorf and Budiansky 1949) but re-named microplanes when the concept was extended to materials incapable

of plastic slip (Bažant 1984). The basic material behavior is defined on the microplanes as a set of relations between the stresses and strains acting on that plane. These relations include all the rheologic phenomena involved in material behavior (elasticity, damage, plasticity) at the same time, in an inseparable manner. The microplane behavior is constrained to the macroscopic continuum behavior by a suitable macro-micro constraint having the form of a variational equation based for example on the principle of virtual work. In some studies (Bažant 1984; Bažant and Oh 1983, 1985; Bažant and Gambarova 1984; Bažant and Prat 1988; Carol et al. 1990) the microplane concept, originally developed for metal plasticity, was extended to materials exhibiting damage with strain softening. The resulting model, briefly described in the section headed "Kinematically Constrained Microplane Model," was shown to be capable of a very good description of the existing test data on tensile strain-softening of concrete due to microcracking as well as the nonlinear triaxial behavior of concrete, including strain-softening in compression and shear under different confining lateral stresses.

This paper [which was briefly summarized in a conference paper by Bažant and Carol (1990)] reconciles both types of constitutive models (continuum damage and microplane), and presents a new interpretation of the microplane formulation as a continuum damage model. This leads to the formulation of a fourth-order nonsymmetric geometric damage tensor. The term "geometric" reflects the fact that this tensor is independent of any particular rheologic model, and depends only on the reduction of the net stress-carrying area fraction in various directions in the material, which are geometric characteristics (the term "geometric," of course, does not imply that all the geometric characteristics, such as spatial crack and void configuration and statistical distributions, are taken into account). The proposed geometric damage tensor achieves a three-dimensional generalization of Kachanov's (1958) one-dimensional net stress-carrying area fraction. It is computed by integrating the geometric damage at each particular microplane over all the spatial directions. As a continuum damage model, it gives the relationship between the macroscopic and effective stress tensors and between the initial and current stiffness matrices, and can be combined with any rheologic model (elasticity, plasticity, viscoelasticity). As a microplane formulation, it also has the additional advantage of an a priori knowledge that very good data fitting can be obtained in practical applications. The derivation of the new damage tensor is presented in two different ways: In the section headed "Derivation of Geometric Damage Tensor from Concrete Model" it is derived from a version of the microplane model with particular microplane laws used previously for concrete (Carol et al. 1990), and in the section headed "Rheology-Independent Derivation of Geometric Damage Tensor" it is derived without recourse to any particular stress-strain laws.

EXISTING MODELS FOR DISTRIBUTED DAMAGE

Consider first the example of a simple one-dimensional continuum damage formulation. In this case damage such as microcracks or voids may be regarded as geometric, representing a reduction in some ratio α of the net stress-carrying area in the microstructure of the material [Fig. 1(a)]. For uniaxial stress (macrostress) σ , we may thus write

$$\sigma = \alpha \tau \dots \dots \dots (1)$$

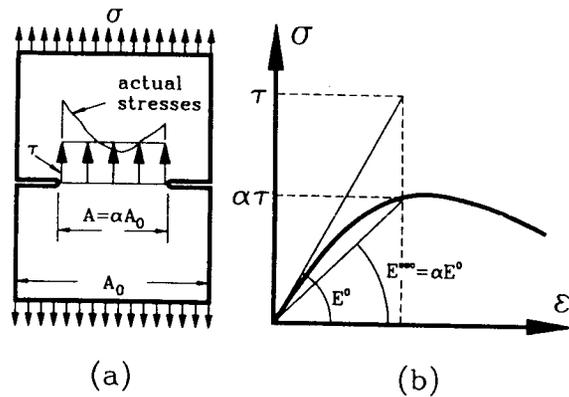


FIG. 1. Effective Stress and Damage in One Dimension: (a) Interpretation of τ and α (b) Stress-strain Diagram

in which τ is called the effective (or true) stress; and α varies from 1 to 0 when damage increases (alternatively, damage can also be represented by $\omega = 1 - \alpha$ varying from 0 to 1); τ represents the average stress in the stress-carrying portion of the cross-sectional area in the microstructure. Assuming a linear elastic relation, i.e., $\tau = E^0 \epsilon$, where E^0 is the elastic modulus of the undamaged material and ϵ the strain, one has

$$\sigma = \alpha E^0 \epsilon \quad (2)$$

This equation may also be written as

$$\sigma = E^{\text{sec}} \epsilon \quad (3a)$$

$$E^{\text{sec}} = \alpha E^0 \quad (3b)$$

where E^{sec} = secant modulus [Fig. 1(b)]. Now, a damage evolution law can be assumed for α . If it is defined in terms of ϵ and introduced in (2) or (3), an explicit one-dimensional stress-strain relation for damage ensues. As seen in this simple example, the formulation of a continuum damage model includes three steps.

1. Characterization of damage by means of a variable (α or ω) or set of variables.
2. Dependence of the effective stress or secant modulus on the damage variable ($\tau = \sigma/\alpha$ or $E^{\text{sec}} = \alpha E^0$)
3. Evolution laws for the damage variable; e.g. $\dot{\alpha} = F(\sigma, \dot{\epsilon}, \alpha \dots)$ (F = function or functional).

In triaxial generalization, some early models included damage-related concepts (e.g. stiffness degradation) that did not explicitly introduce a damage variable as such, but instead derived the evolution laws from a "fracturing surface" with attributes similar to the classical yield surface in plasticity, defining this surface in terms of stress and strain invariants with hardening-softening rules, internal variables, etc. (Dougill 1976; Bažant and Kim 1979). That was a phenomenologic approach to the problem whose

most sophisticated forms became impractical because of the high complexity and the presence of too many parameters and empirical functions, giving little insight into the physical processes actually occurring at the microstructural level.

Most of the existing theories dealing with three-dimensional stiffness degradation incorporate some kind of damage variable in their definition and can be satisfactorily described in terms of the foregoing three-step scheme. The simplest possible characterization of three-dimensional damage is a scalar variable, which can be thought of as the damage caused by a field of spherical microvoids with the same stress-carrying area reduction in all spatial directions (Leckie 1978; Lemaitre and Chaboche 1978; Mazars 1985; Resende 1987; Frantzikonis and Desai 1987). To capture the anisotropic nature of damage caused by microcracks with predominant orientation, damage tensors of various orders have been proposed: vectors (Davidson and Stevens 1973; Krajcinovic and Fonseka 1981; Suaris and Shah 1984; Costin 1985; Costin and Stone 1987); second-order tensors (Vakulenko and Kachanov 1971; Dragon and Mróz 1979; Kachanov 1980; Murakami and Ohno 1980; Betten 1983; Oda 1983; Murakami 1987; Suaris 1987); fourth-order tensors (Chaboche 1979, 1981; Ortiz 1985; Simó et al. 1987; Yazdani and Schreyer 1988), or even eighth-order tensors (Chaboche 1978).

The influence of the damage variable on the secant stiffness or effective stresses, along with the definition of the laws for damage evolution (steps 2 and 3), depend much on what kind of damage variable is adopted. Using a scalar damage variable, these assumptions can be made analogous to the one-dimensional case, but then no induced anisotropy can be accounted for. Some authors use the secant stiffness itself (or a related quantity such as the increment of compliance over the initial or elastic one) as a fourth-order damage tensor (Ortiz 1985; Simó et al. 1987; Yazdani and Schreyer 1988). With this approach, the dependence in step 2 is defined implicitly, but the stiffness tensor does not have a direct physical meaning as a damage variable, and the corresponding evolution laws (step 3) have to be established again, mainly on a phenomenologic basis.

In a certain sense the intermediate option of a second-order tensor as a damage variable seems appealing in the face of other possibilities. Consider for instance the "fabric tensor" proposed by Oda (1983) for the analysis of cracked rock masses, which was an improvement over other previous similar proposals. This tensor is defined as an integral over all the possible crack orientations and sizes of an expression including the crack density function. It is a symmetric dimensionless tensor, and some of its invariants can be interpreted in terms of the overall geometric effect caused by the field of microcracks. The concept of a second-order damage tensor also seems to be supported by theoretical considerations relative to frame invariance (Leckie and Onat 1980), as well as by experimental microstructural measurements of the corresponding "damage ellipsoid" in quite different materials (e.g. measurements of cancellous bone structure and granular soil fabric; Cowin 1985). Unfortunately, the attractive concept of a second-order fabric tensor contrasts with the absence of equally clear and direct relations for the degradation of the macroscopic secant stiffness tensor and for the effective stresses in the cracked medium. Cowin (1985) deduces the most general dependence of the secant stiffness on the second-order fabric tensor permitted by the condition of frame invariance, which turns out to be a complex polynomial expression, with the somewhat disappointing peculiarity that at least a quadratic (and likely even higher) dependence on the first invariant

of the fabric tensor must be involved in the relation if orthotropic symmetry is to be preserved in the cracked material. Some other authors, on the other hand, handle the problem of reduction of the net stress-carrying area due to microcracks through simulation of a finite deformation field with the equivalent effects of area reduction in various directions in space, or through certain geometric and tensorial considerations. However, either they do not use a second-order tensor as a damage variable [Krajcinovic and Fonseka (1981), who used a vector], or they get an awkward nonsymmetric effective stress tensor, which must further be symmetrized in some arbitrary way in order to obtain a practical formulation (Murakami and Ohno 1980; Betten 1983).

A different approach is suggested by studies of a homogeneous medium containing periodic arrays of inclusions or penny-shaped cracks. In such studies, the overall elastic modulus is obtained by calculating the approximate stress fields in the solid, by the so-called self-consistent method or other related techniques (Budiansky and O'Connell 1976; Nemat-Nasser et al. 1982; Horii and Nemat-Nasser 1983; Attigboe and Darwin 1986; Kachanov 1987; Aboudi 1987). This approach can be useful when the overall modulus of a continuum (e.g. rock, a composite material) containing an a priori known field of cracks or inclusions needs to be computed. However, in most of those theories the field of microcracks and the configuration of inclusions must be known in some detail; in constitutive modeling a damage variable characterizes these effects in a smeared way. Also, the relationships obtained for step 2 are usually very complicated (and sometimes they even do not have an explicit form but are a set of nonlinear equations to be solved numerically for each practical case); they should be explicitly tractable expressions to be introduced as one component of the constitutive model. As a result [although some authors did use this approach to derive or support the choice of relations included in step 2; (Chaboche 1979; Oda 1983)], this approach in general seems to be only of limited usefulness for the development of complete macroscopic constitutive relationships.

EFFECTIVE STRESS AND DAMAGE TENSOR UNCOUPLED FROM RHEOLOGY

To define the effective stress in three dimensions the following tensorial expression similar to (1) for one dimension has been introduced (Rabotnov 1969a,b; Chaboche 1979; Lemaitre and Chaboche 1985; Simó et al. 1987):

$$\sigma_{ij} = \alpha_{ijkm} \tau_{km} \dots \dots \dots (4)$$

where the repetition of lower-case indices implies summation over 1, 2, 3; σ_{ij} and τ_{km} = Cartesian components of the macroscopic and effective (or true) stress tensors σ and τ ; and α_{ijkm} = components of a fourth-order dimensionless tensor α characterizing damage (which can also be expressed as $\alpha = I - \omega$, where I = fourth-order identity tensor; and ω = alternative damage tensor).

Symmetry is assumed for the macroscopic stresses σ as well as for the effective stress tensor τ . Thus, the components α_{ijkm} must preserve the interchangeability of index i with j and k and m , and can be grouped into a 6×6 matrix in the context of the six-component vectorial representation of symmetric stress tensors. However, no interchangeability of indices i, j with k, m is assumed, which means that no symmetry is required for the

6×6 matrix. This nonsymmetry is needed so as not to exclude the possibility of representing phenomena such as internal friction.

To establish more precisely the meaning and nature of the damage tensor α and effective stresses τ , the hypothesis of strain equivalence is also assumed (Chaboche 1979; Lemaitre and Chaboche 1985; Simó et al. 1987). According to this hypothesis, the effective stresses τ are defined as the stresses that would exist in the undamaged material subjected to the same strains ϵ as those that produce the macroscopic stress σ in the damaged material. With this definition the components of the 6×6 damage tensor α can be geometrically interpreted as a net (effective) stress-carrying area fraction in every coordinate direction, with each stress component considered to act independently on a plane of that direction.

In the relationship between τ and ϵ , any rheologic model can be used; e.g., using linear elasticity ($\tau_{km} = D_{kmpq}^{el} \epsilon_{pq}$) as the rheologic model, one immediately obtains the expression for the secant or reduced elastic stiffness

$$\sigma_{ij} = D_{ipq}^{sec} \epsilon_{pq} \dots \dots \dots (5a)$$

$$D_{ijkm}^{sec} = \alpha_{ijrs} D_{rspq}^{el} \dots \dots \dots (5b)$$

These equations are the three-dimensional counterpart of (3).

In general, a nonlinear rheologic model may be used instead of linear elasticity to compute the effective stresses τ from prescribed strain ϵ (which can in general be done according to a step-by-step algorithm). Because the value of α can be computed independently according to the damage laws, the macroscopic stress σ can then be obtained using (4). Thus, the overall constitutive model can be decomposed in two independent parts, one for damage and another for rheology. Different models can be used for each part and then combined. Establishing the laws for the evolution of α , one can get an entire new class of models from the combination of the α evolution law with any rheologic model for τ . This opens new interesting possibilities and presents the challenge of developing a satisfactory law for the evolution of the damage tensor α , probably the weakest point at present and the main goal of this paper.

It should be pointed out that the idea of combining damage and rheology is not new, of course. For example Leckie (1978) and Lemaitre and Chaboche (1978) combined viscoplasticity and damage, and Bažant and Chern (1985) combined concrete creep with damage due to smeared cracking. The new idea is to combine damage and rheology by means of the microplane model, which has the benefit of yielding the form of the damage tensor.

KINEMATICALLY CONSTRAINED MICROPLANE MODEL

As already mentioned, the new damage tensor proposed here was derived by reformulating the latest version of the microplane model for concrete in an explicit form (Carol et al. 1990). Therefore, only the basic hypotheses and main features need to be given here.

The model is based on the concept of a microplane. A microplane is an arbitrary plane on which the constitutive properties are defined. Instead of defining them by means of a relation between tensors σ_{ij} and ϵ_{ij} , these properties are defined by means of a relation between the stress and strain components on a microplane, which is conceptually much simpler because there are fewer stress and strain components and the problems of tensorial invariance do not arise on the microplane level. The macroscopic stress-

strain relationship is then obtained by integrating under some suitable micro-macro constraint over the microplanes of all spatial orientations n . This integration guarantees the condition of tensorial invariance to be satisfied automatically. There are two simple alternatives for the macro-micro constraint relating the behavior of all the microplanes to the macroscopic behavior: the static and kinematic constraints, both suggested by Taylor (1938). In the static constraint, the microplane stresses represent the projections of the macroscopic stress tensor on the plane considered; the kinematic constraint does the same for strains. Note that both types of constraints cannot apply at the same time if general stress-strain laws are considered at the microplane level.

Although the static constraint was used exclusively in the works dealing with metal plasticity (Batdorf and Budiansky 1949; Lin 1968), the kinematic constraint was recognized to be necessary to ensure that the microplane system be stable in strain-softening behavior (Bažant 1984; Bažant and Oh 1983, 1985). If the kinematic constraint is assumed, the normal and shear strains ϵ_N and ϵ_T , on the microplane with unit normal n_i [Figs. 2(a and b)] can be obtained as projections of the strain tensor ϵ_{ij} . Previous studies (Bažant and Prat 1988) revealed that in order to obtain any desired Poisson ratio ν ($-1 \leq \nu \leq 0.5$) as well as capture the effect of the hydrostatic pressure on the incremental stiffness, the normal microplane strain ϵ_N needs to be further split into its volumetric and deviatoric components ϵ_V and ϵ_D . This split, however, has no direct physical meaning; rather, the physical meaning is that the microplane stress depends not only on ϵ_N but also on the lateral normal stress ϵ_L along the directions lying within the microplane (the mean value of ϵ_L can be easily expressed in terms of ϵ_V and ϵ_D). The resulting expressions for the microplane strains are

$$\epsilon_V = \frac{\epsilon_{kk}}{3} = \frac{\delta_{km}}{3} \epsilon_{km} \dots \dots \dots (6a)$$

$$\epsilon_D = \left(n_k n_m - \frac{\delta_{km}}{3} \right) \epsilon_{km} \dots \dots \dots (6b)$$

$$\epsilon_{Ti} = \frac{1}{2} (\delta_{ik} n_m + \delta_{im} n_k - 2n_i n_k n_m) \epsilon_{km} \dots \dots \dots (6c)$$

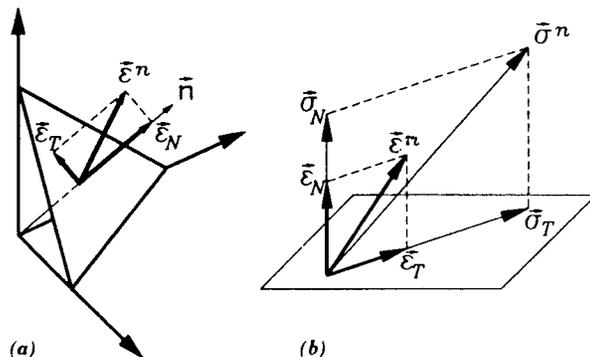


FIG. 2. Stress and Strain Components on Microplane

where δ_{ij} = Kronecker delta; and n_i = direction cosines of the unit vector n normal to the microplane. Note that ϵ_D and ϵ_T can also be obtained as the normal and shear components of the projection of the deviatoric strain tensor $e_{ij} = \epsilon_{ij} - \epsilon_V \delta_{ij}$ on the microplane, but that the deviatoric part of the strain projection, which is $\epsilon_{Ti} = \epsilon_{ij} n_j - \epsilon_N n_i$, differs from (6). Eq. (6) for ϵ_{Ti} is obtained as the symmetric part of this tensor (the asymmetric part has no effect on stresses).

In view of the kinematic constraint imposed, the microplane stresses in general do not and cannot represent the resolved components of the (macroscopic) stress tensor. Equilibrium must nevertheless be imposed in an approximate overall sense. For this purpose, application of the principle of virtual work was proposed by Bažant (1984). This principle yields the following condition of static equivalence of the stress (macrostress) tensor σ_{ij} with the normal ($\sigma_N = \sigma_V + \sigma_D$) and shear stresses (σ_T), on the microplanes:

$$\frac{4\pi}{3} \sigma_{ij} \delta \epsilon_{ij} = 2 \int_{\Omega} (\sigma_N \delta \epsilon_N + \sigma_T \delta \epsilon_T) f(n) d\Omega \dots \dots \dots (7)$$

The left-hand side represents the virtual work of the macrostresses in an elementary unit sphere; and the right-hand side represents the virtual work of the microplane stresses on the microplane strains on all the elementary facets on this sphere, representing the microplanes. The integration needs to be carried out only over the area of a unit hemisphere, Ω , because the stresses and strains at its diametrically opposite points are equal. Function $f(n)$ may be used to introduce initial material anisotropy. For isotropic materials, $f(n) \equiv 1$.

Substituting microplane strain variations according to (1) and (2), and setting $\sigma_N = \sigma_V + \sigma_D$, where σ_N , σ_V , and σ_D are the normal, volumetric, and deviatoric stress components on the microplanes, we obtain

$$\left\{ \int_{\Omega} \left[(\sigma_V + \sigma_D) n_i n_j + \frac{\sigma_T}{2} (n_i \delta_{ij} + n_j \delta_{ji} - 2n_i n_j n_i) \right] f(n) d\Omega - \frac{2\pi}{3} \sigma_{ij} \right\} \delta \epsilon_{ij} = 0 \dots \dots \dots (8)$$

This is a variational equation that must be true for any variation $\delta \epsilon_{ij}$. This occurs if and only if the expression within braces, $\{ \}$, vanishes. Noting also that $\int_{\Omega} n_i n_j d\Omega = 2\pi \delta_{ij}$, one obtains the expression

$$\sigma_{ij} = \sigma_V \delta_{ij} + \int_{\Omega} \sigma_D n_i n_j d\Omega + \int_{\Omega} \frac{\sigma_T}{2} (n_i \delta_{ij} + n_j \delta_{ji} - 2n_i n_j n_i) d\Omega \dots \dots (9)$$

Notice that, in fact, not all the components of $\delta \epsilon_{ij}$ in (8) can have independent variations, since ϵ is a symmetric tensor. In principle, this would complicate somewhat the elimination of this term from both sides of the equation (Carol et al. 1990). However, thanks to the symmetry (between i and j) in the expressions used [(6)], the resulting expression between brackets, $\{ \}$, in (8) is also symmetric with respect to the interchange of indices i and j , which makes this simplification possible (the same remark also applies to other similar simplifications of tensorial variables from other expressions with similar structure later in the paper).

The formulation needs to be supplemented by particular microplane stress-strain laws for σ_V , σ_D , and σ_T as functions of ϵ_V , ϵ_D , and ϵ_T . As shown by

Bažant and Prat (1988), these can be assumed to be algebraic explicit relations (rather than differential equations) separate for each component, i.e. (for loading)

$$\sigma_V = F_V(\epsilon_V) \dots \dots \dots (10a)$$

$$\sigma_D = F_D(\epsilon_D) \dots \dots \dots (10b)$$

$$\sigma_{T_i} = \frac{F_T(\gamma)}{\gamma} \epsilon_{T_i} \dots \dots \dots (10c)$$

where $\gamma = \sqrt{\epsilon_{T_i} \epsilon_{T_j}}$ (the response, however, is in general path-dependent because various combinations of loading and unloading on various microplanes arise, even for macroscopically monotonic loading). A further simplifying hypothesis, that the shear stress σ_{T_i} is parallel to the shear strain ϵ_{T_i} , is also made. In that case it suffices to specify the constitutive relation for the magnitude $\sigma_T = \sqrt{\sigma_{T_i} \sigma_{T_j}}$, rather than all the components σ_{T_i} . In previous works (Bažant and Prat 1988; Carol et al. 1990) the simplified relations in (10) did not prevent achieving good fits of a wide range of test data for concrete.

The computation of a prescribed-strain macroscopic load step can be performed explicitly: the microplane strain increments are computed from the macroscopic strain increment using (6). Then the new microplane stresses are obtained using the microplane laws, (10) (for each microplane one must know whether unloading started and, if so, which unloading path it is following). And, finally, the microplane stresses are integrated according to (9) to obtain the new macroscopic stresses.

DERIVATION OF GEOMETRIC DAMAGE TENSOR FROM CONCRETE MODEL

The microplane constitutive laws from (10) can be rewritten in the form of elastic-damage laws with secant moduli E_V^{sec} , E_D^{sec} , and E_T^{sec}

$$\sigma_V = E_V^{sec} \epsilon_V \dots \dots \dots (11a)$$

$$\sigma_D = E_D^{sec} \epsilon_D \dots \dots \dots (11b)$$

$$\sigma_{T_i} = E_T^{sec} \epsilon_{T_i} \dots \dots \dots (11c)$$

where $E_V^{sec} = f_V(\epsilon_V)$, etc. ($f_V =$ function). Introducing (11) and the expressions of microplane strains from (6) in the right-hand side of (9), replacing the macroscopic stress in the left term by its elastic-damage expression with secant stiffness tensor from (5) for σ_{ij} , and eliminating ϵ_{ij} from both sides of the resulting equation one obtains the macroscopic secant stiffness tensor expressed as an integral of the secant stiffness at microplane level over the unit hemisphere

$$D_{kmpq}^{sec} = \frac{E_V^{sec}}{3} \delta_{km} \delta_{pq} + \frac{3}{2\pi} \int_{\Omega} E_D^{sec} n_k n_m \left(n_p n_q - \frac{\delta_{pq}}{3} \right) d\Omega$$

$$+ \frac{3}{2\pi} \int_{\Omega} \frac{E_T^{sec}}{4} (n_k n_p \delta_{mq} + n_k n_q \delta_{mp} + n_m n_p \delta_{kq}$$

$$+ n_m n_q \delta_{kp} - 4n_k n_m n_p n_q) d\Omega \dots \dots \dots (12)$$

Now the following substitutions may be introduced: (1) The secant stiffness

at macroscopic level is replaced by the product of the initial elastic stiffness and the damage tensor [(5) for D_{ijpq}^{sec}]; (2) similar replacements are made for the microplane secant moduli:

$$E_V^{sec} = \alpha_V E_V^0 \dots \dots \dots (13a)$$

$$E_D^{sec} = \alpha_D E_D^0 \dots \dots \dots (13b)$$

$$E_T^{sec} = \alpha_T E_T^0 \dots \dots \dots (13c)$$

where α_V , α_D , and $\alpha_T =$ microplane damage variables reflecting the net stress-carrying area fraction at microplane level; and E_V^0 , E_D^0 , and $E_T^0 =$ initial elastic moduli; (3) the initial moduli E_V^0 , E_D^0 , and E_T^0 are replaced by their expressions in terms of the macroscopic elastic parameters E and ν and the additional parameter η_0 (Bažant and Prat 1988)

$$E_V^0 = \frac{E}{1 - 2\nu} \dots \dots \dots (14a)$$

$$E_D^0 = \eta_0 E_V^0 \dots \dots \dots (14b)$$

$$E_T^0 = \frac{1}{3} \left[\frac{5(1 - 2\nu)}{1 + \nu} - 2\eta_0 \right] E_V^0 \dots \dots \dots (14c)$$

and (4) the elastic stiffness matrix is replaced by its expression:

$$D_{kmpq}^e = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \delta_{km} \delta_{pq} + \frac{E}{(1 + \nu)} \delta_{kp} \delta_{mq} \dots \dots \dots (15)$$

Once all these expressions are introduced into (12) and simplified, an expression for α_{ijkl} is obtained in terms of α_V , α_D , and α_T . Now an interesting idea comes to mind: parameters E , ν , and η_0 can vanish from the expression for α_{ijkl} when the following assumption is made:

$$\eta_0 = \frac{1 - 2\nu}{1 + \nu} \dots \dots \dots (16)$$

As we will see later, this hypothesis appears to be acceptable for practical purposes, and one gets

$$\alpha_{ijpq} = \frac{\alpha_V}{3} \delta_{ij} \delta_{pq} + \frac{3}{2\pi} \int_{\Omega} \alpha_D n_i n_j \left(n_p n_q - \frac{\delta_{pq}}{3} \right) d\Omega$$

$$+ \frac{3}{2\pi} \int_{\Omega} \frac{\alpha_T}{4} (n_i n_p \delta_{jq} + n_i n_q \delta_{jp} + n_j n_p \delta_{iq}$$

$$+ n_j n_q \delta_{ip} - 4n_i n_j n_p n_q) d\Omega \dots \dots \dots (17)$$

The fact that this tensor is purely geometric, i.e., does not involve E and ν , is the main contribution of this paper. We thus gain a new form of the microplane model in which the basic material behavior is not defined as a set of laws for microplane stresses [(10)] but as a set of laws for the evolution of the microplane damage variables α_V , α_D , and α_T . Then, the macroscopic damage tensor α_{ijkl} is obtained by integration of the microplane damage variables over the hemisphere using (17).

RHEOLOGY-INDEPENDENT DERIVATION OF GEOMETRIC DAMAGE TENSOR

The foregoing derivation of the damage tensor started from a particular version of the microplane model. At least, the following assumptions were implicitly present in the derivation: (1) The microplane laws are of the form $\sigma = \alpha E^0 \epsilon$ [(11) and (13)]; (2) the macroscopic stress-strain relationship is of the same type [(5)], involving linear elasticity in its definition [(15)]; (3) the initial moduli at microplane level are related to the elastic parameters E, ν , [by (14) and (16)]; and (4) the shear stresses and strains on a microplane are parallel [(10)]. No references to these hypotheses remain in the final expression in (17), which depends on geometric dimensionless quantities only. Thus, from a practical point of view it is obvious that (17) can be used as a damage model in combination with any rheologic model in the general context described in the section headed "Effective Stress and Damage Tensor Uncoupled from Rheology." However, from a theoretical viewpoint the question remains whether the damage tensor derived under these restrictive assumptions could also have been derived using any alternative companion rheologic model or the derivation is exclusive for this case.

Let us start a rheology-free derivation of α_{ijkm} from the assumed basic micro-macro kinematic and equilibrium relations in (6) and (9), which are both rheology independent. On the microplane level we may define the effective (true) stresses, τ_V, τ_D , and τ_T , such that

$$\sigma_V = \alpha_V \tau_V \dots \dots \dots (18a)$$

$$\sigma_D = \alpha_D \tau_D \dots \dots \dots (18b)$$

$$\sigma_T = \alpha_T \tau_T \dots \dots \dots (18c)$$

This may be also regarded as a rheology-free definition of damage at microplane level, alternative to (13).

Having assumed the kinematic micro-macro constraint and introduced general microplane stress-strain laws, the microplane stresses cannot (except by chance) represent the resolved components of any stress (macrostress) tensor. The microplane effective stresses in general need not represent the resolved components of any effective stress tensor, i.e., no such tensor needs to exist. However, for a certain special form of the damage ratio tensor, this could occur. Such a property would bring about considerable simplification. To explore it let us introduce the hypothesis that the microplane effective stress components are in fact such resolved components of an effective stress tensor at the macroscopic level, τ_{ij} . This means that, in analogy to (6) for strains, we can write

$$\tau_V = \frac{\tau_{kk}}{3} = \frac{\delta_{km}}{3} \tau_{km} \dots \dots \dots (19a)$$

$$\tau_D = \left(n_k n_m - \frac{\delta_{km}}{3} \right) \tau_{km} \dots \dots \dots (19b)$$

$$\tau_T = \frac{1}{2} (\delta_{ik} n_m + \delta_{im} n_k - 2n_i n_k n_m) \tau_{km} \dots \dots \dots (19c)$$

Let us now substitute (18) and (19) into the right-hand side of (9), and (4) into the left-hand side. After some rearrangements we can factor out τ_{pq}

$$\left\{ \frac{\alpha_V}{3} \delta_{ij} \delta_{pq} + \frac{3}{2\pi} \int_{\Omega} \alpha_D n_i n_j \left(n_p n_q - \frac{\delta_{pq}}{3} \right) d\Omega + \frac{3}{2\pi} \int_{\Omega} \frac{\alpha_T}{2} (n_i n_p \delta_{iq} + n_i n_q \delta_{ip} + n_j n_p \delta_{iq} + n_j n_q \delta_{jp} - 4n_i n_j n_p n_q) d\Omega - \alpha_{ipq} \right\} \tau_{pq} = 0 \dots \dots (20)$$

This equation should hold for any tensor τ_{pq} . This is possible if and only if the expression within braces, $\{ \}$, vanishes. From this condition we obtain the expression for α_{ipq} already given in (17). Furthermore, if (19) were violated, it would not be possible to factor out τ_{pq} in (20), and (17) would not result. Therefore, the static constraint in (19) is not only sufficient but also necessary to achieve uncoupling of damage from rheology, the latter being defined by the dependence of τ_V, τ_D , and τ_T on the microplane strains. It must be emphasized that this second derivation was not made with recourse to any particular rheologic model, but only on the basis of geometric, kinematic, and equilibrium considerations. This demonstrates from a theoretical viewpoint the general validity of the proposed geometric damage tensor.

Note also that the model represented by the new damage tensor [(17)] coupled with a particular rheologic model is equivalent to a microplane formulation in which there is not only a kinematic constraint for strains but also a static constraint for a certain kind of stresses, namely, the effective stresses. This is a more appealing formulation than that in which there is only a kinematic constraint. The kinematic constraint is reminiscent of parallel coupling of elements in a rheologic model; the static constraint is reminiscent of series coupling. From the modeling of composite materials it is known that the parallel coupling models give the stiffest possible response; whereas the series coupling models give the softest possible response (an example for elastic composites are the Voigt and Reuss bounds). The real behavior can usually be best described by some combination of both types of coupling. It thus appears satisfactory that the foregoing formulation combines both kinematic and static constraints.

NUMERICAL IMPLEMENTATION AND DEMONSTRATION FOR CONCRETE

Previous studies (Bažant and Prat 1988; Carol et al. 1990) showed that a good and remarkably broad description of the existing test data on the constitutive properties of concrete, encompassing multidirectional tensile strain-softening due to cracking as well as nonlinear triaxial response in the hardening and strain-softening ranges, could be achieved by the microplane model. In those studies all the micromechanics phenomena involved in the microplane laws for σ_V, σ_D , and σ_T were mixed. By contrast, in the new model a clear separation has been established between damage effects, represented through the new damage tensor, and rheology, represented by a companion model that can be simple since it does not represent damage (or strain-softening). In this new scheme a separation of tasks can be established between both parts of the model, each characterizing different aspects of the material behavior observed in tests.

In the first stage of development presented here, attention is focused on the new microplane damage tensor, and the companion rheologic model is chosen as the simplest possible, i.e. linear elasticity. Choosing a more sophisticated rheologic model, e.g., elastoplastic, would no doubt broaden the possibilities of representing complex material behavior.

The evolution laws for microplane damage variables were established on the basis of the laws used for σ_V , σ_D , and σ_T by Carol et al. (1990); they were of the form $\sigma(\epsilon) = \alpha(\epsilon)E^0\epsilon$. Eliminating from these expressions the factor $E^0\epsilon$, we obtain the equation $\alpha = \exp[-(\epsilon/a)^p]$, which gives for α a smooth variation from 1 to 0 when the strain increases monotonically; see Fig. 3 for the effect of different values of p .

Fig. 4 depicts the general laws for α_V , α_D , and α_T . In the case of α_V , as seen in this figure, the basic behavior just described is used only for tension (positive ϵ_V), with parameters a_1 and p_1 and state parameter ϵ_V^{min} , which depends on the history. No damage is assumed to occur in hydrostatic compression, where we always have $\alpha_V = 1$. This is because the volumetric behavior observed in compression tests shows no stress peak and exhibits unloading-reloading branches with almost the initial slope [see Bažant and Prat (1988)], which seems to be modeled by the companion rheologic model of plasticity type better than by the damage tensor itself. If unloading-reloading occurs in tension, the curve given by (2) acts as an envelope, and the maximum strain reached so far is the state parameter depending on history. A jump is assumed for α_V if ϵ_V becomes positive after some damage occurred in tension or vice versa. This kind of jump (and the same kind of jump occurring for α_D) can represent in some way the effect of a strain-controlled microcrack opening and closing at microplane level.

In the case of α_D , two independent sets of material parameters and state parameters depending on history are used for tension and compression: a_1 , p_1 (both the same as used for α_V), ϵ_D^{min} for tension; and a_2 , p_2 , ϵ_D^{max} for compression. A jump can also occur for α_D when crossing from tensile to compressive parts of the diagram. Finally, a_3 , p_3 , and γ^{max} are used for the tangential (shear) law, which only shows the positive side since γ is the magnitude of the tangential (shear) strain vector. In contrast to Bažant and Prat (1988) and Carol et al. (1990), no dependence of the tangential behavior on any volumetric variable is assumed. This is an aspect of behavior directly caused by internal friction rather than by damage itself, and therefore might be better handled by the companion rheologic model.

The model was implemented in a computer code in the manner described by Carol et al. (1990), with 28 spatial orientations of microplanes (Bažant and Oh 1985) for which history is recorded and damage evaluated. The

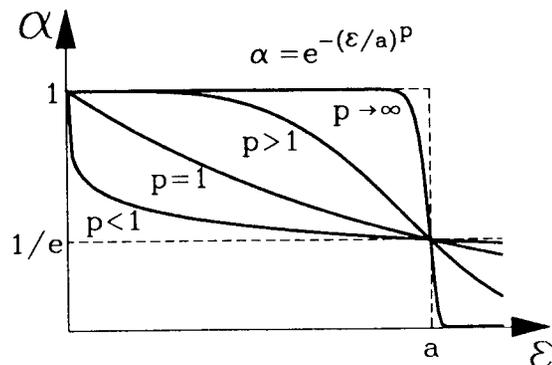


FIG. 3. Basic Exponential Curve for Damage Evolution

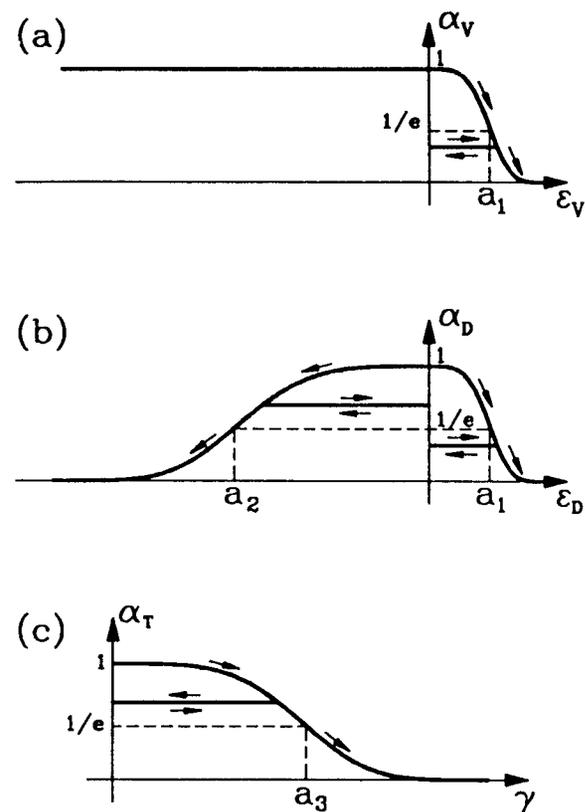


FIG. 4. Microplane Damage Laws: (a) Volumetric Damage α_V ; (b) Normal Deviatoric Damage α_D ; (c) Tangential (shear) Damage α_T

stresses are computed explicitly from specified strain increments of finite size, and subroutines co-opted from a "single-point" or "full-finite element (FE)" computer program are also used. In fact, no major differences with regard to the previous version of microplane model exist from this viewpoint except that the laws are established in terms of microplane damage variables instead of microplane stresses and that, after the damage tensor has been obtained, additional steps are needed to obtain the new values of τ_{ij} and to evaluate the corresponding tensorial product in (4) that gives the new σ_{ij} .

There are six material parameters of the model for the microplane damage tensor: a_1 , p_1 , a_2 , p_2 , a_3 , and p_3 . However, the values $p_1 = 0.5$; $p_2 = 1.5$; and $p_3 = 1.5$ can be assigned for most of practical cases, thus reducing the number of material parameters to be determined from test results to only three. The number of history-dependent state variables for 28 microplane directions is 85 (one for the volumetric curve, same for all microplanes; two for the deviatoric curve; and one for the shear curve on each microplane). Furthermore, the material parameters of the rheologic model (E and ν in this case) and possibly its internal variables (none in this case) may also be counted.

As an example, a uniaxial test reported by van Mier in 1984, in which both longitudinal stress and transverse strains were measured, is reproduced with the model. The parameters used to fit these data are: $E = 2,406$ MPa; $\nu = 0.18$; $a_1 = 0.0004$; $a_2 = 0.006$; and $a_3 = 0.0018$. The remaining parameters have their general values already given. The results are represented in Fig. 5 by solid lines. The dots are the experimental data and the dashed lines are the results obtained with the previous version of microplane model for the same example. It can be observed that both curves are very similar in each diagram, which should not be surprising since both models are almost equivalent in this example, in which no volumetric dependence

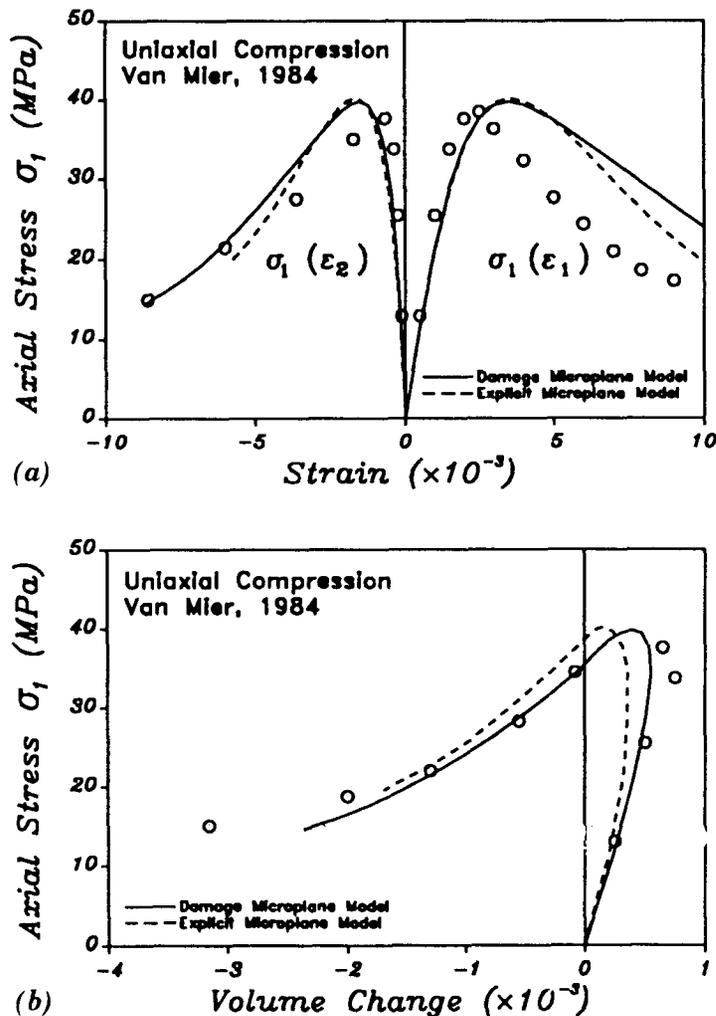


FIG. 5. Comparison with Uniaxial Test by van Mier (1984)

was assumed for σ_T in the computations with the previous microplane model (Carol et al. 1990). The only difference arises from the parameter η_0 , which existed in the old model but now has disappeared from the formulation. In the section headed "Derivation of Geometric Damage Tensor from Concrete Model" it was shown that the full equivalence between both models occurs for $\eta_0 = (1 - 2\nu)/(1 + \nu)$ in the old model. For $\nu = 0.18$ this means about $\eta_0 = 0.54$; the value taken in the previous computations for this example was 0.85. Due to that difference, the models are not completely equivalent, and the parameters a_1 , a_2 , and a_3 cannot have the same values. It is clearly shown in the example, however, that the restriction over the value of η_0 does not prevent the model from yielding a very good fit of experimental data. This is not surprising, since Bažant and Prat (1988) found the effect of η_0 in the closeness of fit to be minor and that the optimized η_0 -values were mostly quite close to those from (16). Finally we must emphasize that Fig. 5 does not represent an experimental verification of the present model. Rather, this verification rests on the fact that: (1) The Predictions of the present and previous models are quite close; while (2) the experimental verification of the previous model was extensive.

CONCLUSIONS AND BROADER IMPLICATIONS

For the general microplane model characterized by a kinematic micro-macro constraint it is possible to formulate a fourth-order damage tensor in a form that is uncoupled from the rheologic constitutive properties of the material. In this formulation the damage is solely of geometric origin, being due to the reductions of the net stress-carrying cross-sectional area fractions for various orientations in the microstructure. The damage tensor characterizes the values of these area fractions for all spatial directions, and permits them to have any values. The damage tensor formulated in this manner fits well the framework of continuum damage mechanics.

Despite the assumed kinematic constraint between macrostrains and microstrains, it is possible to satisfy at the same time and additional static constraint such that the effective microstresses on the microplanes are the resolved components of the effective macrostresses.

The fact that the constitutive model is composed of two independent parts (one for damage, another for rheology) allows better insight, facilitates the fitting of test data, and increases versatility.

A recent microplane model for tensile and compressive strain softening and general nonlinear triaxial behavior of concrete, which has been amply verified by test data, can be easily adapted to the geometric damage form without any appreciable detriment to the closeness of test data fits.

The possibility of uncoupling geometric damage from constitutive (rheologic) properties is attractive for development of more general constitutive models combining damage with creep and rate effects, with complex unloading and reloading paths, and with cyclic loading. It might be possible to model such behavior adequately using some existing plastic, viscoelastic, viscoplastic, or nonlinear cyclic material model without damage, and then incorporating into it the geometric damage tensor deduced here (17)]. By separating damage, the constitutive (rheologic) model, which relates here the effective (microscopic, true) stress tensor (instead of the macroscopic stress tensor) to the strain tensor, can probably be much simpler than it would have to be if the damage were mixed with nondamage constitutive properties. By virtue of the geometric damage concept constitutive modeling

can be restricted to the material between the microcracks and voids in the microstructure, leaving the overall effect of the growth of the microcracks and voids as the only behavior to be described by the geometric damage tensor.

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