Stochastic Drying and Creep Effects in Concrete Structures

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Abstract: The paper calculates how an aging concrete structure that has an uncertain constitutive law and random material properties responds to the random process of environmental humidity history. The time evolution of stochastic pore-humidity distributions is solved from a linear diffusion equation. The stresses produced by the shrinkage strains are calculated taking concrete creep into account. The environmental humidity process is described by three components: one Poisson square-wave random process and two random-phase processes, having periods of one year and one day, the latter of which is found to have a negligible effect. The response to the random-phase process is calculated by the spectral method combined with equal probability sampling. The Poisson-process component is important for very long times since it produces a response whose standard deviation grows as the square root of time; the nonstationary random-phase component stabilizes after about 10 years. The influence of the random-phase process reaches a depth of about 20 cm below the concrete surface; that of the Poisson process reaches only about 5 cm. The randomness of the material parameters and the uncertainty factor of the material model influence the entire structure. A numerical example of a cylindrical wall is given and simplified explicit formulas for estimating the response statistics are derived.

Introduction

The sources of the random scatter in creep and shrinkage of concrete are basically three: (1) Random variation of material parameters characterizing creep and shrinkage; (2) the uncertainties of the creep and shrinkage models; and (3) the influence of the random variation of environmental humidity and temperature, i.e., the influence of weather. The humidity influence in the third source has been investigated by means of the spectral method (Bažant and Wang 1984a, b, c; Tsubaki and Bažant 1982), based on the simplifying assumption that the random scatter due to sources 1 and 2 is negligible (i.e., the production of concrete is controlled perfectly and the constitutive law is known completely and the diffusion problem of drying is linear). Sources 1 and 2 were analyzed separately using the method of latin hypercube sampling, in which it was assumed that the environmental humidity was either constant (Bažant and Liu 1985) or represented a random parameter having a normal distribution independent of time (Bažant and Xi 1988). Actually the environmental humidity is a stochastic process in time.

The main purpose of this paper, whose essentials were reported at a conference (Bažant and Xi 1989), is to solve the random shrinkage stress problem with all three sources of randomness taken into account. Initial studies assumed the random variables in source 1 to be uncorrelated, but we consider them to be correlated, applying the method reported in Bažant and Xi (1988), and Xi and Bažant (1989). The random variables in source 2 are assumed to be independent random variables and those in source 3 stochastic processes. Similarly to Bažant and Wang (1984a, b, c), the stochastic process in time is analyzed by the spectral method, based on spectral decomposition of the humidity records. For the sake of simplicity, the problem is assumed to be linear, so that the principle of superposition is valid. Such a linear simplification will, of course, be sufficiently realistic only if the variable parameters covering nonlinearity are replaced by suitable effective (or average) values.

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Model of Structure

The model we seek consists of three components: (1) The diffusion problem governing the pore-humidity distribution throughout the structure; (2) the constitutive law relating the shrinkage strain to the pore-humidity changes; and (3) the set of equations for the stress problem.

Diffusion Equation

Although moisture transfer and thermal conduction in concrete are coupled (Bažant and Thonguthai 1978), for the sake of brevity we consider humidity fluctuation alone and assume the temperature to be a constant during the entire analysis. However, extension to variable temperature, with coupled heat and moisture transfer (Bažant and Thonguthai 1978; Bažant 1975a, b), would be possible since the diffusion problem remains of the same type. If we neglect the self-desiccation due to hydration, which causes only a mild drop in pore humidity (or the relative vapor pressure) h, the drying is governed by the diffusion equation

$$\frac{\partial h}{\partial t} = C(t)\nabla^2 h \qquad (1)$$

$C(t)$ = diffusivity of concrete, which strongly depends on the age, t, as well as on h (Bažant and Najjar 1972; Bažant 1982). However, to make an analytical solution feasible, we neglect the dependence on h, considering for C a certain effective (average) constant value. This simplification may still be expected to yield a realistic approximate model provided a suitable effective value of C is taken. (Note that this simplification is similar to that usually made in random dynamics, in which a constant mean stiffness depending on the expected mean amplitude is customarily considered for the entire time history as a replacement of the curved stress-strain or moment-curvature diagram.) The effective value of C of course depends on body size and geometry, as well as the domain and frequency. One can solve the problem for some first guess of the C-value; then, calculated based on the h-distribution, obtain a new average C-value; and iterate this procedure (similarly to Timoshenko's method of effective viscosity for vibrations of inelastic structures). Later, in a numerical example, we simplify the solution even further, taking C as constant.

Constitutive Law and Stress Analysis.

Pore-humidity changes cause local shrinkage strains $\varepsilon_{sh}(x, t)$. Although the dependence of ε_{sh} on h is nonlinear (Bažant 1975a), it again needs to be linearized to make superposition of the solutions feasible. In that case [in a spirit similar to Bažant and Chern (1985)]

$$\varepsilon_{sh} = \psi \varepsilon_s^0 f_s(h) g_s(t_e) \quad \text{(2)}$$

where ψ = a random variable reflecting the uncertainty of the shrinkage model; ε_s^0 = a constant; and $g_s(t_e) = E_c(t_0)/E_c(t_e)$, where $E_c(t_e)$ is taken as $E_c(t_0 + \tau_{sh})$ and approximately, $f_s(h) = h_0 - h(x, t)$, and according to the BP-KX model (Bažant et al. 1991, 1992), $\tau_{sh} = kD^2$ = shrinkage half-time, where D = effective thickness and k = constant [see Bažant et al. (1991)]. The foregoing linear approximation of $f_s(h)$ is acceptable within a relatively broad humidity range; such as from 0.5 to 1.0, as confirmed for example by the data in Neville (1981). Constant h_0 represents the initial pore humidity, usually $h_0 = 1$. The shrinkage law in incremental form is

$$d\varepsilon_{sh} = \psi \varepsilon_{s\infty} \frac{E_c(t_0)}{E_c(t_0 + \tau_{sh})} \frac{df_s(h)}{dh} \frac{dh}{dt} dt \quad \text{(3)}$$

The shrinkage stresses are strongly reduced by creep. As is well known from the linear theory of aging creep [e.g. Bažant (1982)], the creep effect may be analyzed by first obtaining the elastic solution and calculating the corresponding stresses from an incremental form of (3). Then, according to McHenry's analogy (Bažant 1975a), the stresses in the presence of creep are

$$\sigma(t) = \int_{t_0}^{t} \frac{R(t, t')}{E_c(t')} d\sigma^e(t', t_0) \quad \text{(4)}$$

where $R(t, t')$ = relaxation function of concrete, which can be easily obtained from the compliance function $J(t, t')$ (Bažant 1982); $\sigma^e(t', t_0)$ = elastic stress at age t' calculated (e.g. by finite element method) from shrinkage strains $\varepsilon_{sh}(t')$ for elastic modulus $E_c(t')$.

Random Material Parameters and Model Uncertainty

The random variability of material and model uncertainty have been introduced through parameters $\varepsilon_{s\infty}$ and ψ in (2) and (3). Only the uncertainty of the shrinkage model is considered; that of the relaxation function is neglected because of the lack of statistical data. To simplify the analysis, we assume the material parameters to be normally distributed, the admissibility of which was justified by previous studies [see Tsubaki (1988)]. Because the parameters are correlated, we consider a joint normal distribution based on the extensive study of Xi and Bažant (1989) (in which the correlations were verified by analyzing data according to hypothesis test). The material parameters under consideration are concrete cylinder strength f'_c, water-cement ratio w/c, aggregate-cement ratio g/c, and specific cement content c. Xi and Bažant (1989) give the 4 × 4 correlation matrix of the joint distribution of these parameters. The model uncertainty parameter ψ is assumed to be governed by an independent normal distribution. Note also that, for reasons of objectivity, the distribution of material parameters must of course be independent of (uncorrelated to) the random variation of environmental humidity.

Input Humidity Model

Environmental humidity is a stochastic process (actually a superposition of several different stochastic processes). To establish a practicable but realistic humidity input model, we need to analyze climatic records. As an example, we take the climatic record of the Chicago area ("Observations"

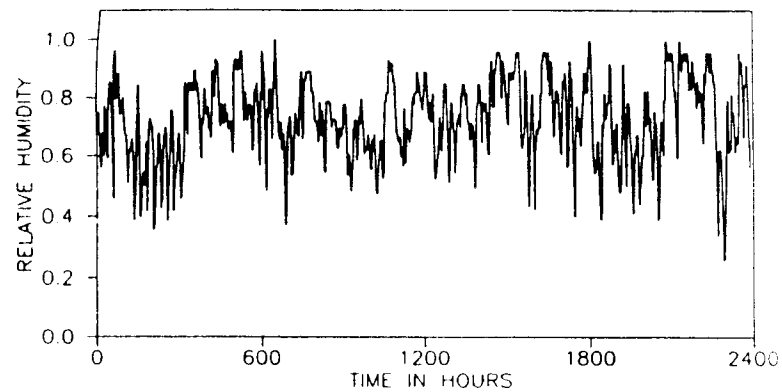

FIG. 1. Environmental Humidity Record, Midway Station, Chicago

1973–74), shown in Fig. 1. The unbiased estimate of sample covariance C_r and the estimate of spectral density Φ are obtained as

$$C_r = \frac{1}{N - r} \sum_{s=1}^{N-r} (H_s - \bar{H})(H_{s+r} - \bar{H}); \qquad \bar{H} = \frac{1}{N} \sum_{s=1}^{N} H_s$$

$$(r = 0, 1, 2, \ldots, N - 1) \quad \text{(5)}$$

$$\Phi = \frac{1}{2\pi} \int_{r=-k}^{k} \left(1 - \frac{r^2}{k^2}\right)\left(1 - \frac{r}{N}\right) C_r \cos(\omega r) \quad \text{(6)}$$

Subscripts r, s, and N refer to discrete times $t_0, t_1, \ldots t_N$; H_s = environment humidity record at time t_s; $\bar{H}$ = sample mean value. Eq. (6) is Parzen's modified formula for the spectral density estimate (Anderson 1971), where k can be chosen as one of the values of 20, 40, 60, 80 . . . The estimated results are shown in Figs. 2 and 3, from which we can detect that the environmental humidity consists of three different components: First, H_1 is the mean value $\bar{H}$, standing for the stable horizontal trend. Second, H_2 is a random phase process corresponding to the harmonic variation of humidity. Third, H_3 is a random-noise process, which may be simulated by the Poisson square-wave process (Madsen 1986) [in physics this is also called the simple thermal-electron current noise (Korn 1966)].

As is well known, the property of independence is most characteristic of the Poisson process. Its physical meaning is that the process remains unchanged after the occurrence of an event insofar as the future events are concerned. In this regard it is proper to point out that the assumption of H_3 being a process of Poisson type implies by no means that the current humidity has no effect on the humidity, say 1 hr, later; to the contrary, such an effect does exist and is taken into account by process H_2.

Fig. 3 shows the process H_3 generated by a Poisson square-wave process with intensity λ. The value of H_3 changes at each state change of the Poisson process, while between the state changes H_3 is constant. At each event τ_n, $H_2(\tau^+_n) = A_n$, where A_n should generally be independent identically distributed random variables with zero mean values. The process is stationary, ergodic, and can be expressed as follows:

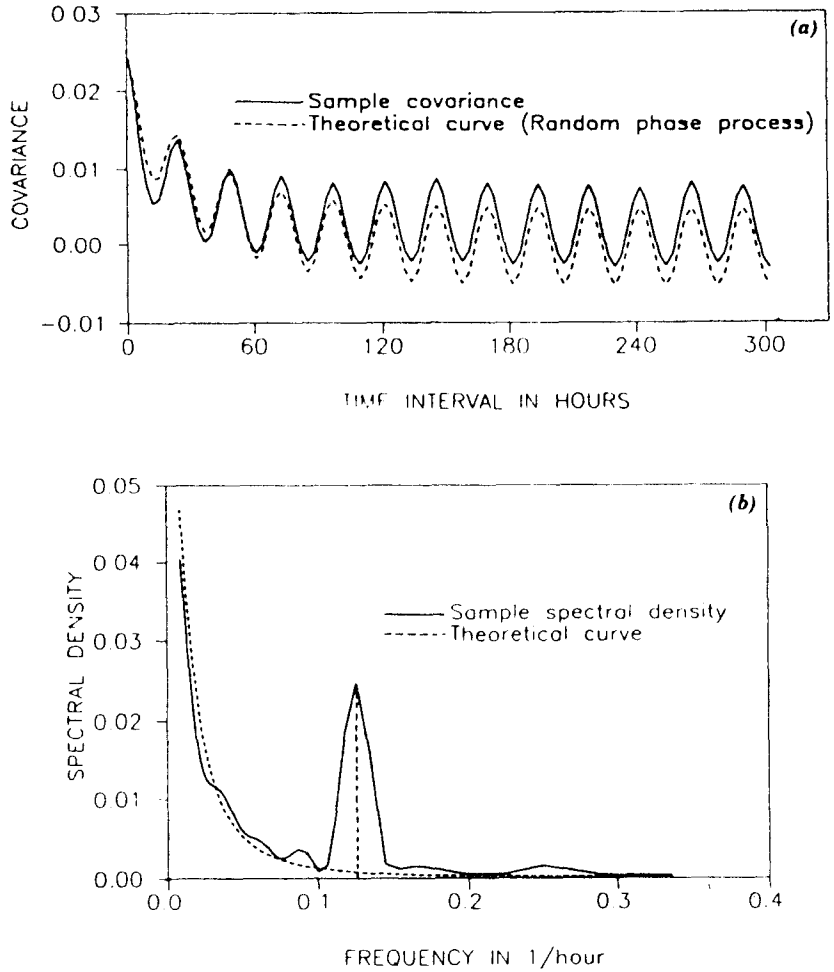

FIG. 2. Comparison of: (a) Sample Covariance with Theoretical Curve; (b) Sample Spectral Density with Theoretical Curve

$$H_3 = A_n w(t, \tau_n) \quad (7a)$$

$$w(t, \tau_n) = 1; \text{ for } \tau_{n+1} > t > \tau_n \quad (7b)$$

$$w(t, \tau_n) = 0; \text{ otherwise} \quad (7c)$$

Here $n = 1, 2, \ldots, N(t)$; $N(t)$ = a Poisson process with intensity λ; τ_n = arrival time of A_n (A_n are assumed to be uniformly distributed random variables). The interarrival times τ_1, $\tau_2 - \tau_1, \ldots$ are independent and identically distributed exponential random variables with the same parameter λ (Cinlar 1975). The expectation, covariance, autocorrelation, and power spectrum of H_3 are

$$E(H_3) = E(A_n) = 0 \quad (8a)$$

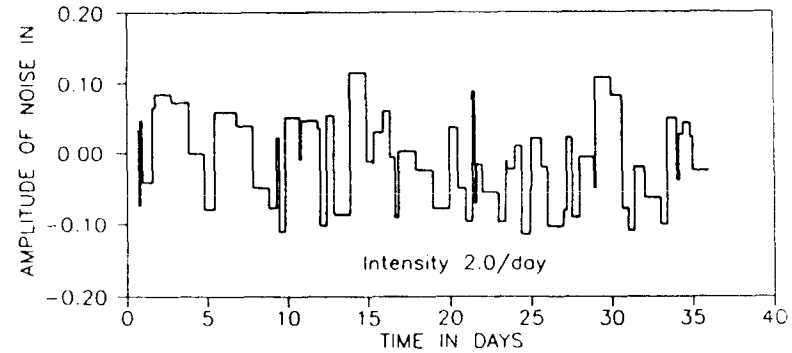

FIG. 3. Simulation of Poisson Square-Wave Process

$$\text{cov}[H_3(t)H_3(s)] = \text{var}(A_n)e^{-\lambda|t-s|} \quad (8b)$$

$$R_{H_3}(\tau) = \text{var}(A_n)e^{-\lambda|\tau|} \quad (9a)$$

$$\Phi_{H_3}(\omega) = \frac{2\lambda}{\omega^2 + \lambda^2}\text{var}(A_n) \quad (9b)$$

with $\lambda = t - s$. Finally, by combining the statistical functions of H_2 and H_3, the total input humidity model H, its autocorrelation function R_H, and the spectral density Φ_H can be written as

$$H = \bar{H} + H_1 \cos(\omega_1 t + \phi_1) + A_n w(t, \tau_n) \quad (10)$$

$$R_H(\tau) = 0.5H_1^2 \cos(\omega_1\tau) + \text{var}(A)e^{-\lambda|\tau|} \quad (11a)$$

$$\Phi_H(\omega) = 0.5H_1^2\pi[\delta(\omega - \omega_1) + \delta(\omega - \omega_1)] + \frac{2\lambda}{\lambda^2 + \omega^2}\text{var}(A_n) \quad (11b)$$

The coefficients of the foregoing equations can be determined from the estimate of the sample in the following way: When t is large enough, $R_H(\tau) = 0.5\, H_1^2 \cos(\omega_1\tau)$, where amplitude H_1 can be determined; when $\tau = 0$, $R_H(0) = 0.5\, H_1^2 + \text{var}(A_n)$, where $\text{var}(A_n)$ can be obtained; then, from the peak point of the spectrum diagram in Fig. 2(*b*), frequency ω_1 is obtained; and when $\omega = 0$, $\Phi_H(0) = 2\,\text{var}(A)/\lambda$, from which λ can be evaluated.

For the coefficient values identified in this manner, Fig. 2 shows the theoretical curves compared with the estimated curves of the climatic sample. As we see, the theoretical curves approximate the sample curves quite well. This confirms that the humidity process includes not only the periodic term assumed in previous studies but also a noise term. This term greatly influences the standard deviation of structural response, as is seen later.

One limitation of our example should be noted. The period identified from our humidity sample is one day, and no period that would be longer than that is found. This is of course because only two years of weather record were included in our statistical example, which certainly would not be long enough to detect long periods, such as one year, especially if the amplitude variation over such a long period is small. In real situations, though, an environmental humidity period of one year must be expected to be present, as justified and demonstrated by observations in Diamantidis

et al. (1983). Therefore, based on these observations, we add to the input humidity model an additional term with a period of one year.

Method of Solution

We have two kinds of random input variables: random parameters and random processes in time. The random parameters are represented by the material parameters and the uncertainty factors in the material model. The random material parameters are independent of time, and the random realizations of their values occur when the material is created. The random-process input is represented by the environmental humidity (and environmental temperature). This process determines the time-dependent boundary condition in space.

The method of latin hypercube sampling is especially suited for time-independent random variables such as random material parameters and model uncertainty, provided that sample units of equal probability content can be defined. A successful sampling procedure has been developed to ensure that both uncorrelated and correlated normal variables have the same probability content (Bažant and Xi 1988). Even if the random phase angle ϕ and random noise H_3 (a time-dependent random variable representing the input humidity process characterized by A_n) are included in the sample, it is still possible to devise a way to use the efficient latin hypercube procedure. Random phase angle ϕ is a uniformly distributed random variable, that is, the probability $P(\phi)$ of ϕ taking a certain value is a constant. Variables A_n in process H_3 are also uniformly distributed random variables; however, the probabilities $P(A_n)$ are rather different from $P(\phi)$. Variable ϕ receives its random-value input, and then keeps that value. On the other hand, variables A_n receive new random values randomly at every state change of Poisson process H_3, and the times at which these state changes happen (called the arrival times) are random, too.

Let us now formulate the equal-probability sampling method, which can be applied to the present time-dependent random variables. We consider an infinitesimal time interval $(t, t + \Delta t)$ that represents the time point at which a humidity value is input during the calculation. The probability of a state change in this interval is (for a Poisson process) negligible. The probability of no state change in interval Δt is $P(0) = \exp(-\lambda\Delta t) =$ constant. At the same time, the probability of H_3 keeping an unchanged value in $H_3 = A_n = A^*$ is $P(A^*) = \Delta A_n/(B - A) =$ constant, where $A^* \varepsilon \Delta A_n$ and $1/(B - A) =$ probability density function of the uniform distribution. Therefore, the probability of H_3 keeping an unchanged value in Δt is $P(A_n) = P(A^*)P(0) =$ constant because the interarrival times and A_n are different and mutually independent random variables. So the probability of one sampling unit is of the form

$$P_u = P\left(f'_c, \frac{w}{c}, \frac{g}{c}, c\right)P(\psi)P(\phi)P(A_n) = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

where $P(f'_c, w/c, g/c, c)$ = joint probability of material parameters, $P(\psi)$ = probability of model uncertainty; $P(\phi)$ = probability of the random phase H_2; and $P(A_n)$ = probability of the noise process H_3. Eq. (12) means that the probability content of every sampling unit and even of every time step of the same sampling unit (i.e., the same computer run) is the same. This satisfies the basic requirement of latin hypercube sampling, which has been shown to give optimum sampling efficiency.

Although the sampling method outlined seems to suffice for solving the entire problem, it is nevertheless computationally more efficient to also apply at the same time the spectral method, provided the problem is linearized. The spectral method is widely used in random vibration for nonaging systems with an ergodic input process, and it has recently been generalized for aging systems with an ergodic input, such as concrete structures (Bažant 1986). The assumption of ergodicity of input and the question of applicability to aging structures were discussed at length in Bažant (1986), and so the discussion need not be repeated here.

In the spectral method, in contrast to the direct sampling method, frequency-domain calculations replace time-domain calculations. Since the power spectral density of response $Y(t)$ is a frequency-domain representation of the second moments of $Y(t)$, the analysis for ergodic $Y(t)$ can be done either in the time domain or in the frequency domain. Preference for the choice of either method depends on the forms of information given as the input and the forms needed for the output.

A great simplification brought about by the spectral method is that, for an ergodic input process, it is not necessary to sample the random variables characterizing the input process, because one can obtain all of the statistical characteristics of response from only one output sample curve. For a nonaging system, the variance of response is given by

$$\text{var}(Y) = E(Y^2) - \mu_y^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

in which

$$E(Y^2) = R_y(0) \quad \ldots\ldots\ldots\ldots\ldots\ldots (14a)$$

$$R_y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_y(\omega)e^{i\omega\tau}\, d\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots (14b)$$

Here $R_y(\tau)$ = autocorrelation function of system response; $\Phi_y(\omega)$ = response spectral density; μ_y = mean value of response, which is zero for a fluctuating input of zero mean. Either the autocorrelation function or the spectral density of the response can be found from the output data. The confidence band for the response can be obtained from the variance. For the case of an aging system, it has been shown (Bažant 1986) that for the case of a periodic ergodic input $X(t) = s_0 \exp(i\omega t)$

$$\text{var}[Y(t)] = s_0^2|H(x, t, t_0, \omega)|^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

where $H(x, t, t_0, \omega)$ = frequency-response function of the structure, which depends on t and t_0 because of aging of concrete.

The approach to deal with the aging of concrete is as follows. The first step is to consider a nonaging system. We solve (1) by taking diffusivity as a constant, which may be easily done numerically in time steps. Then we obtain from (3) the shrinkage strain increments $\Delta\varepsilon_{sh}$ at each time step and various points of the structure. Based on $\Delta\varepsilon_{sh}$ we can then determine the elastic stress increments $\Delta\sigma^e$ by finite element analysis. The second step is to consider the aging effect. To obtain the statistical characteristics of the response we need to average over the times at which the structure was built, called the birth times. The variation of birth time may be imagined as a shifting of the input history against the instant when the structure was built (Bažant 1986). If the input is a periodic process with period T, then the response is also periodic and the mean response is still ergodic, although the autocorrelation and variance of the response are not. However, due to

periodicity, the birth-time averaging over a long time range is equivalent to the averaging over period T. As a special case, for input of the random phase process the shifting of the input can be regarded as giving the random phase different initial values, which means the birth-time averaging over a long time range is equivalent to ensemble averaging of random phase. This approach is demonstrated in detail in our example.

For the noise-input process, one generally may first fix t and t_0 to obtain the integral in (4), and then carry out the birth-time averaging and repeat it for each time step.

Response Analysis

Consider the example of an infinitely long cylindrical wall, similar to that in Bažant and Wang (1984a, b). The external and internal radii of the cylinder are $b = 21$ m and $a = 20$ m; that is, the thickness of the wall is 1 m, which is typical for nuclear-reactor containment shells. The initial pore humidity inside concrete is 1.0, and the mean value of environmental humidity is 0.7. The random-phase process of a one-day period is $A_1 \cos[2\pi(t - t_0) + \phi_1]$, with $A_1 = 0.1$, and the random-phase process of a one-year period we assume as $A_{365} \cos[(2\pi/365)(t - t_0) + \phi_{365}]$, with $A_{365} = 0.08$ (Diamantidis et al. 1983), in which ϕ_1 and ϕ_{365} have uniform distributions with constant density $1/2\pi$; $t_0 = 28$ days = time at which exposure to the environment begins. The Poisson square-wave process is defined by (7); A_n has a uniform distribution, with density 1/0.24. The intensity of the Poisson process is 2.0/day, and so the interarrival time has an exponential distribution with parameter $\lambda = 2.0$. The distribution parameters for model uncertainty are taken from Xi and Bažant (1989): ψ_1 has as its mean 0.944 and as its variance 0.0046. The mean values of concrete properties are $f'_c = 45.2$ MPa; $w/c = 0.46$; $g/c = 2.07$; and $c = 450$ kg/m^3, and the associated coefficients of variation are assumed as 0.1 (Madsen and Bažant 1982).

The responses to the input humidity components have been analyzed one by one and then be summed. A computer program has been written to generate all of the random values for the independent and correlated random variables according to the latin hypercube sampling criterion. For these generated values, the response histories and spatial distributions of humidity and stresses across the cylinder wall have been solved using the finite difference method for humidity and the finite element method for stresses. Only the circumferential stress response σ is presented in this study, since it is larger than the radial and longitudinal stress responses.

Figs. 4 and 5 show the sampling results of pore humidity and the stress responses of the structure to every input random humidity process. It can be seen that the response to the one-year period dominates.

Response to Input of One-Year Period

Although the sampling method can give the mean and standard deviation of response, the computer time is very long. The methods of spatial analysis shown in Bažant and Wang (1984a, b) and Tsubaki and Bažant (1982), and involving numerical integrals or Bessel functions, are also complex. To simplify the calculations, consider a very thick wall, which can be analyzed as a half-space. The given surface humidity is $A_{365} \cos(\omega_{365} t - \phi)$, and the response is

$$H(x, t) = H_0(x, t) + H_1(x, t) \quad (16a)$$

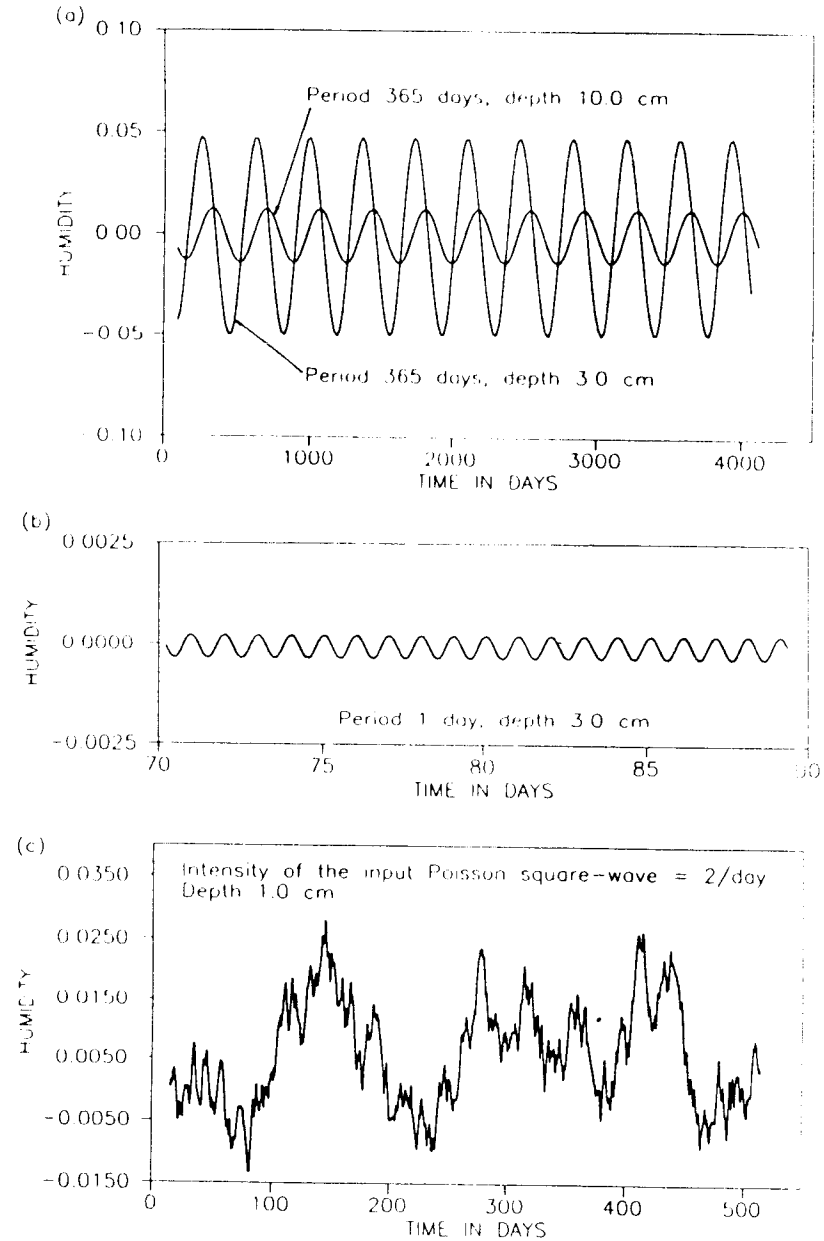

FIG. 4. Humidity Response to: (a) Random-Phase Process of One-Year Period; (b) Random-Phase Process of One-Day Period; (c) Poisson Square-Wave Process

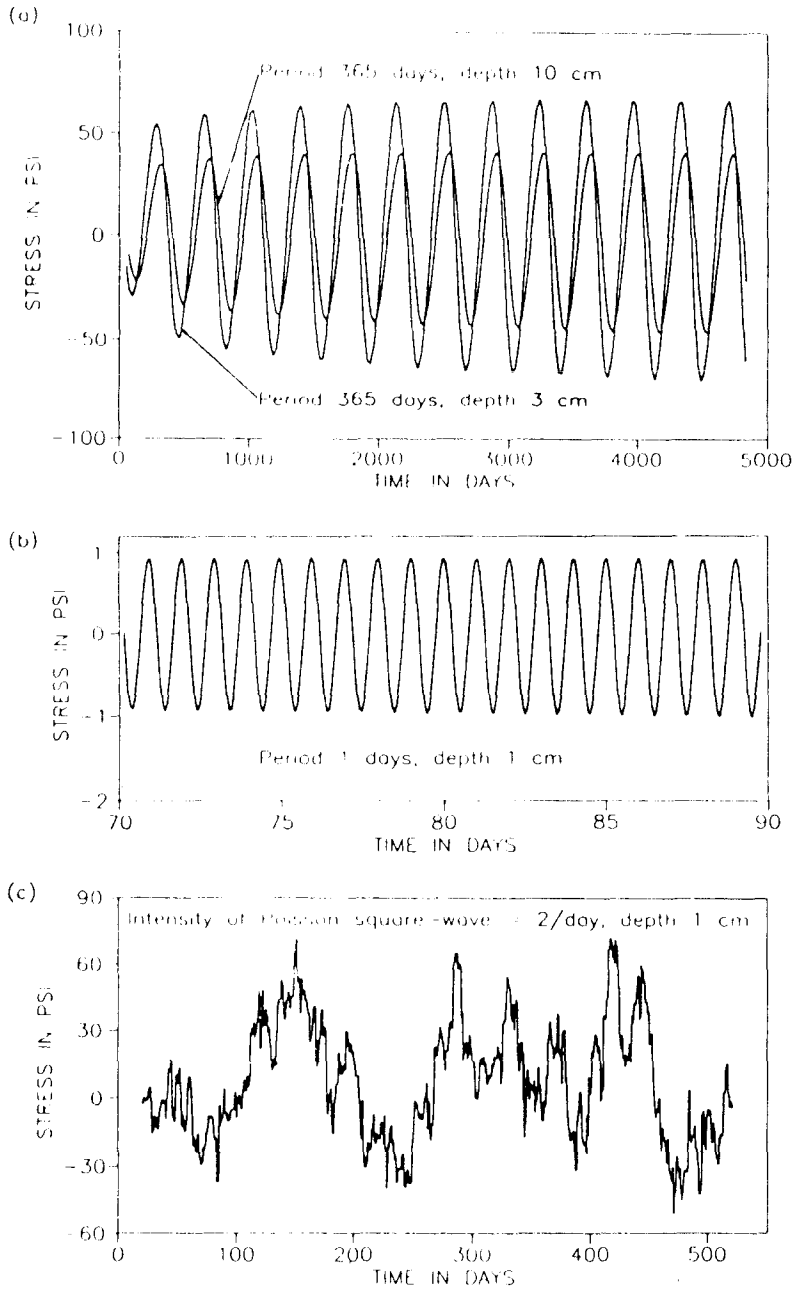

FIG. 5. Stress Response to: (*a*) Random-Phase Process of One-Year Period; (*b*) Random Phase Process of One-Day Period; (c) Poisson Square-Wave Process

$$H_0(x, t) = A_{365}e^{-kx}\cos(\omega_{365}t - kx - \phi) \quad \text{(16)}$$

$$H_1(x, t) = -\frac{2A_{365}}{\sqrt{\pi}}\int_0^{\zeta}\cos\left[\omega_{365}\left(t - \frac{x^2}{4k\mu^2}\right) - \phi\right]e^{-\mu^2}\,d\mu;\ \zeta = \frac{x}{2\sqrt{kt}} \quad \text{(17)}$$

where k = wave number; and $k = \sqrt{\omega_{365}/2C}$. Term $H_1(x, t)$ is a transient disturbance that dies away as t increases, leaving only the term $H_0(x, t)$, which represents a steady-state oscillation of period $2\pi/\omega_{365}$. The wavelength is $L = 2\pi/k = \sqrt{4\pi CT}$, where T = period. In our case T = 365 days; C = 0.3 cm²/day; and so L = 37.09 cm. From (16), the amplitude of the humidity oscillation diminishes with x as e^{-kx}. At $x = L$, the amplitude is reduced by the factor $e^{-2\pi} = 0.0019$. This means that the half-space is a good approximation for walls thicker than about one wavelength. Our wall is 1 m thick, which is about $2L$. Therefore, we may use $H(x, t) = H_0(x, t)$.

The lag in the phase of the humidity wave, $kx = x\sqrt{\omega_{365}/2C}$, increases with depth x. Fig. 6(*a* and *b*) shows comparisons of the cylinder solution by finite difference method and the simplified half-space solution. We can see that these two solutions agree perfectly, including the periodic characteristic, the attenuated amplitude, and the progressive lag with increasing depth. So (16) is valid also for a cylinder. For the stress distribution in the cylinder we may assume a similar form as for humidity:

$$\sigma = A_\sigma\cos[\omega_{365}(t - t_0) - kx - \phi] \quad \text{(18)}$$

Indeed, the sampling results in Fig. 5(*a*) show that the stress history has the same harmonic characteristic and the same progressive lag as the humidity. The only difference is the amplitude, A_σ, which is attenuated with depth at a different rate. More importantly, A_σ is not deterministic but depends on the random variables of the initial state, and A_σ is also a function of $t - t_0$ and t_0 because of the aging aspect of (4).

To obtain the first and second moment for σ, we first analyze A_σ and ϕ in (18) separately. Since the birth-time averaging is equivalent to ensemble averaging over ϕ, we can use the spectral method to determine the effect of random phase ϕ on the basis of (18), in which x is a chosen constant, t_0 is a given constant, and A_σ is one of the realizations of the random amplitude. Eq. (18) becomes a random-phase process whose spectral density and autocorrelation function are

$$\Phi_\sigma(\omega) = 0.5\pi A_\sigma^2[\delta(\omega - \omega_{365}) + \delta(\omega + \omega_{365})] \quad \text{(19)}$$

$$R_0(\tau) = 0.5A_\sigma^2\cos(\omega_{365}\tau); \qquad \tau = t_2 - t_1 \quad \text{(20)}$$

The mean value of σ is zero, and from (13) or (14) the variance is $A_\sigma^2/2$; the standard deviation is $S_\sigma = A_\sigma/\sqrt{2}$. Note that S_σ, like amplitude A_σ, is not a deterministic value but a random variable; its mean value and standard deviation are

$$\mu_{S_\sigma} = \frac{1}{\sqrt{2}}E(A_\sigma) \quad \text{(21a)}$$

$$S_{S_\sigma}^2 = \frac{1}{2}\operatorname{var}(A_\sigma) \quad \text{(21b)}$$

Consider now the effect of amplitude A_σ by the sampling method. Keeping

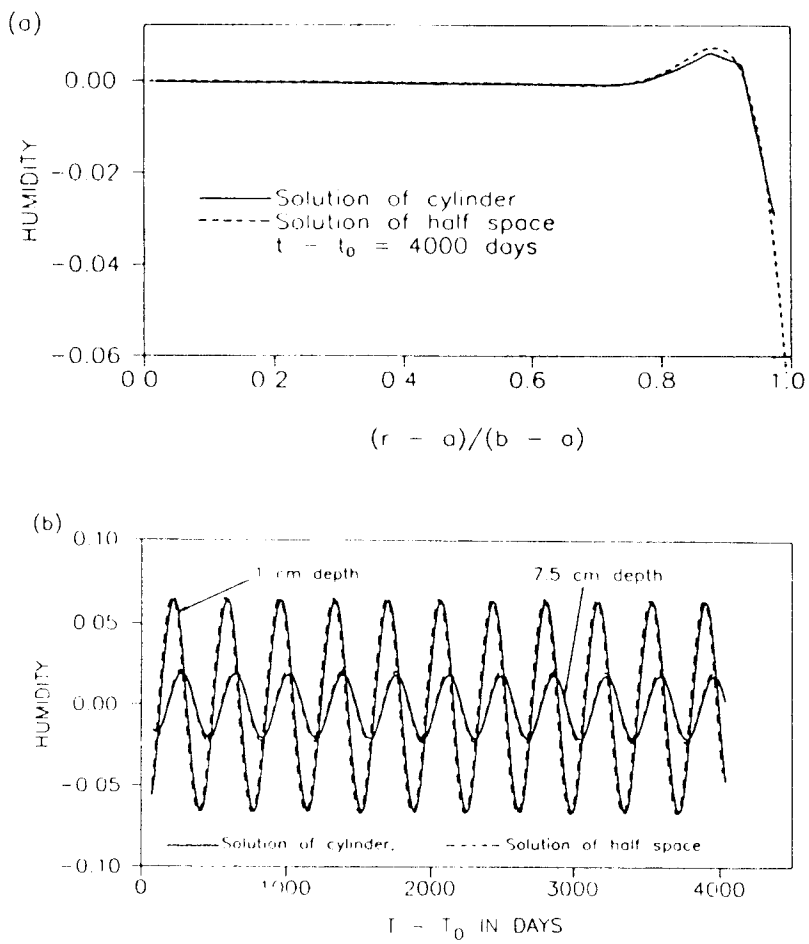

FIG. 6. Comparison of: (*a*) Humidity Distribution in Cross Section; (*b*) Humidity History at Different Depths

ϕ constant, samplings are made only over the material parameters and model uncertainty. Fig. 7(*a*) shows the mean stresses, which are seen to decrease as the depth increases. The influence the random process ends at the depth of about 20 cm. Fig. 7(*b*) shows the envelope of the stress amplitude. It can be noticed that variable A_σ is time-dependent. At the beginning of response, the stresses need a long time to change from zero to their maximum value, which occurs at about 10 years. After that, the stress response gets reduced slowly with time as a result of aging. The mean and standard deviation of A_σ have time-dependent behaviors.

Based on (18), we can get a simplified expression for the confidence band. Since the mean of (18) is zero, the confidence band of stress, $\mu_\sigma \pm S_\sigma$, becomes $\pm S_\sigma$. Since S_σ is a random variable, one can also obtain its confidence band, $\mu_{s\sigma} \pm S_{s\sigma}$, and combine it with (21). Because only the

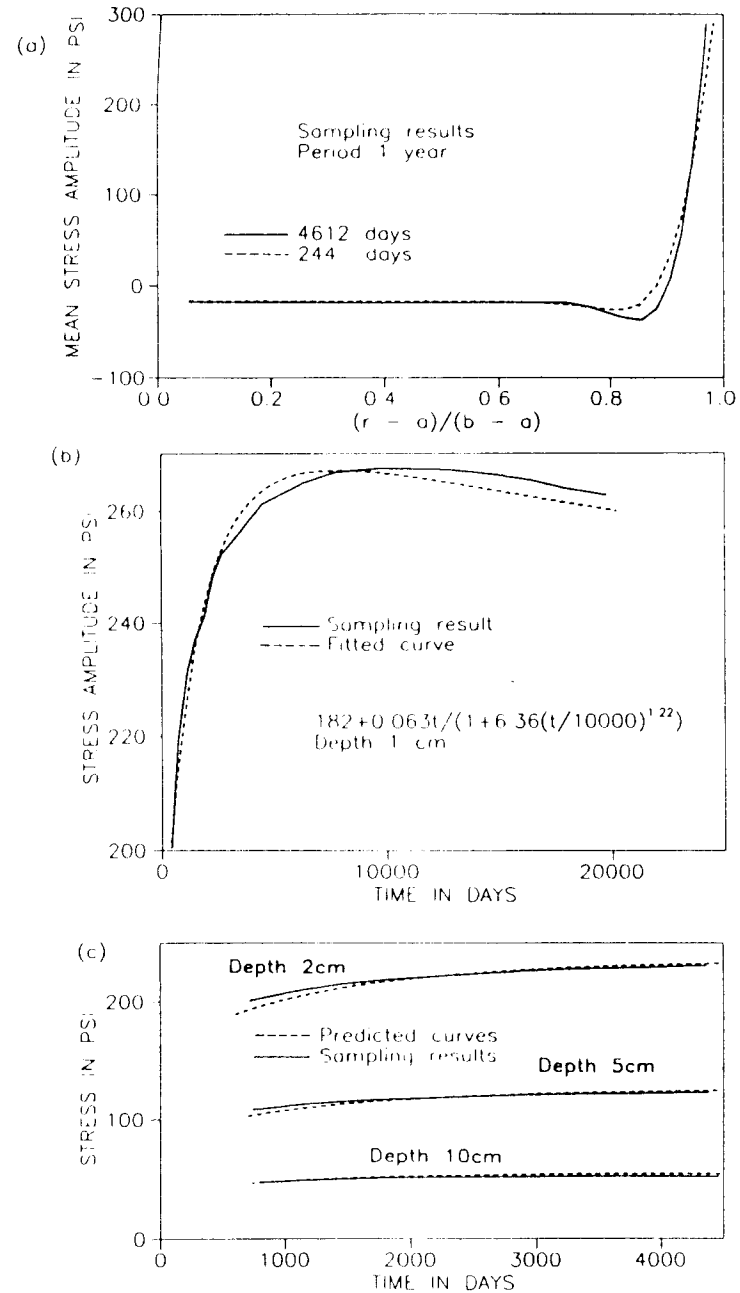

FIG. 7. (*a*) Mean Stress Amplitude in Cross Section; (*b*) Relation of Stress Amplitude and Time; (*c*) Comparison of Predicted Confidence Bands with Sampling Results

envelope of stress deviation is of interest, and so the positive sign is taken, i.e., $\mu_{s\sigma} + S_{s\sigma}$; this yields

$$\mu_\sigma \pm S_\sigma = \pm S_\sigma = \pm(\mu_{s_\sigma} + S_{s_\sigma}) = \pm\frac{1}{\sqrt{2}}[E(A_\sigma) + \mathrm{var}(A_\sigma)] \quad \ldots\ldots (22)$$

The factor $1/\sqrt{2}$ in (22) reflects the influence of the random phase. The problem of stress deviation is thus transformed into a problem of amplitude deviation. By curve-fitting of sampling results in Figs. 7(*a*) and 7(*b*), one can obtain the expressions for the dependence of the mean and standard deviation of A_σ on x and t. From this, one finally gets the following simplified equation for the stress confidence band corresponding to the random phase process:

$$\mu_\sigma \pm S_\sigma = \pm\frac{1.15}{\sqrt{2}}\left(-75.02 + \frac{604}{1 + 0.315x}\right)\left(0.685 + \frac{\xi}{4216 + 0.354\xi^{1.22}}\right) \quad \ldots\ldots (23)$$

with $\xi = (t - t_0) - 9.84(x - 1)$. Fig. 7(*c*) shows that (23) gives satisfactory predictions compared to the results obtained purely by sampling.

Response to Input of One-Day Period

Figs. 4(*b*) and 5(*b*) show that the magnitude of response to the input of the one-day period is very small (10^{-4} for pore relative humidity, 10^{-1} psi for stress) even at the depth of only 3 cm. The reason is the very low value of diffusivity of moisture in concrete. Consequently the one-day period can be neglected for most purposes.

Response to Poisson Process Input

When the sampling method is used to obtain the mean and variance of noise response, a long running time on the computer is required. The Poisson square-wave process, which can be regarded as a sort of shot noise (Nigam 1983), consists of an influence function in the form of Dirac delta function. For computation, it is necessary to approximate the influence function as a sequence of narrow-band square-wave functions, which arrive one by one. The response of the structure at time t is the sum of the responses to every individual square wave.

Let $t_0, t_0 + \tau_1, t_0 + \tau_2, \ldots$ be the times at which the system receives the square waves, such that $t_0 < t_0 + \tau_1 < \ldots$. As the result of the square-wave pulse at time $t + \tau_n$, the stress at a given depth x changes its value by a random amount Y_n. Variable Y_n represents the nonaging response to the humidity pulse; Y_n needs to be modified by an aging relaxation function, as indicated in (4). With the notation $\tau = t_0 + \tau_n$, the aging response is $Y_n^* = Y_n R(t, \tau)/E_c(\tau)$. The actual stress $\sigma(t, x)$ at time t and at depth x is the sum of all Y_n^* that occurred from t_0 to t. Assuming the stress to be 0 at time t_0, we thus have

$$\sigma(t, x) = \sum_{n=1}^{N(t)} Y_n^* w(t - \tau); \qquad \tau = t_0 + \tau_n \quad \ldots\ldots (24)$$

$w(t - \tau)$ = a step function called the shape function; $w(t - \tau) = 1$ if $t - \tau > 0$; otherwise, $w(t - \tau) = 0$; $N(t)$ and τ_n have the same meaning as in (7). This type of model is called the filtered Poisson process (Parzen 1962).

The probability structure of the filtered Poisson process can be obtained from the mth cumulant function

$$k_m[\sigma(t_1, x), \sigma(t_2, x), \ldots, \sigma(t_m, x)]$$
$$= \int_0^{t_1} E[Y^* w(t_1, \tau), \ldots, Y^* w(t_m, \tau)]\lambda\, d\tau \quad \ldots\ldots (25)$$

with $t_1 < t_2 < \ldots < t_m$. In particular, for $m = 1$ and $m = 2$, the mean μ_σ, variance $\mathrm{var}(\sigma)$, and covariance K_σ are

$$\mu_\sigma(t) = \int_0^t E[Y^* w(t, \tau)]\lambda\, d\tau \quad \ldots\ldots (26a)$$

$$\mathrm{var}[\sigma(t)] = \int_0^t E[Y^{*2} w^2(t, \tau)]\lambda\, d\tau \quad \ldots\ldots (26b)$$

$$K_\sigma(t_1, t_2) = \int_0^{t_1} E[Y^* w(t_1, \tau) Y^* w(t_2, \tau)]\lambda\, d\tau; \qquad t_1 < t_2 \quad \ldots\ldots (27)$$

For the mean value, by substituting $w(t, \tau) = w(t - \tau)$ and $Y^* = YR(t, \tau)/E_c(\tau)$ into (26), we obtain

$$\mu_\sigma(t) = \lambda \int_0^t E\left[Y\frac{R(t, \tau)}{E_c(\tau)} w(t - \tau)\right] d\tau = \lambda \int_0^t E(Y)\frac{R(t, \tau)}{E_c(\tau)} w(t - \tau)\, d\tau \quad \ldots\ldots (28)$$

in which E denotes the expectation (mean) operator; and E_c = elastic modulus at age τ. The aging effect is manifested by the dependence of $E[Y(\tau)]$ on $E_c(\tau)$ and the dependence of $R(t, \tau)$ on both t and τ separately [rather than only on the time lag $(t - \tau)$]; $\tau = t_0 + \tau_n$ = time at which the random square wave arrives; $E[Y(\tau)]/E_c(\tau)$ represents the mean stress increment for a unit value of elastic modulus; it is a nonaging value, independent of time. If we calculate Y based on $E_c(28)$, and then divide by $E_c(28)$, we get the same result:

$$\mu_\sigma(t) = \lambda \int_0^t \frac{E\{Y[E_c(\tau)]\}}{E_c(\tau)} R(t, \tau) w(t - \tau)\, d\tau \quad \ldots\ldots (29a)$$

$$\mu_\sigma(t) = \lambda \int_0^t \frac{E\{Y[E_c(28)]\}}{E_c(28)} R(t, \tau) w(t - \tau)\, d\tau \quad \ldots\ldots (29b)$$

$$\mu_\sigma(t) = \lambda \frac{E\{Y[E_c(28)]\}}{E_c(28)} \int_0^t R(t, \tau)\, d\tau \quad \ldots\ldots (30)$$

in which we dropped the step function $w(t - \tau)$ because $R(t, \tau) = 0$ for $t < \tau$.

In the same manner, we obtain covariance and variance

$$K_\sigma(t_1, t_2) = \frac{\lambda}{E^2(28)} E\{Y^2[E_c(28)]\} \int_0^t R(t_1, \tau) R(t_2, \tau)\, d\tau \quad \ldots\ldots (31)$$

$$\mathrm{var}\{\sigma(t)\} = \frac{\lambda}{E_c^2(28)} E\{Y^2[E_c(28)]\} \int_0^t R^2(t, \tau)\, d\tau \quad \ldots\ldots (32)$$

From (30)–(32) we can see the influence of aging on the statistics of

response. We consider the upper bound of (30)–(32). Always $R(t, \tau) < E_c(28)$, and regardless of the τ-value, one has for most of the time range $R(t, \tau) < E_c(28)$; so $\int_0^t R(t, \tau)\, d\tau < \int_0^t E_c(28)\, d\tau = E(28)t$; and $\int_0^t R^2(t, \tau)\, d\tau < \int_0^t E_c^2(28)\, d\tau = E_c^2(28)t$. Hence the upper bounds for the statistics are

$$\mu_\sigma(t) = \lambda E(Y)t; \qquad \mathrm{var}[\sigma(t)] = \lambda E(Y^2)t \quad \ldots\ldots (33)$$

$$K_\sigma(t_1, t_2) = \lambda E(Y^2)t_1; \qquad t_1 < t_2 \quad \ldots\ldots (34)$$

From (30)–(32) or (33)–(34), one can see that the problem of stress deviation is transformed into a problem of stress increment deviation, which can be determined from only one response of the structure. The random stress increment Y_n depends in a complicated way on the random material properties, model uncertainty, depth x, age t_0 at the start of exposure, and, particularly, the amplitude and length of each input square wave. The largest contributions come from the depth x, and the amplitude, as well as the lengths of the input square waves. Even if the amplitude of one input square wave is very small, the increment of the stress response can still be large if the square wave is very long.

A convenient way to obtain the mean and mean square of Y_n in (26) is to employ the spectral method in the time domain, keeping all the other variables constant while one response curve, caused only by input H_3 at certain depth x, is calculated. Fig. 8 shows the stress increment at every state change of H_3 at $x = 1$ cm. The mean values are always zero, and so the variance is equal to the mean square, which is 43.23 at $x = 1$ cm and 1.16 at $x = 5$ cm. The standard deviation S_y drops markedly with increasing depth x; this means the noise component of humidity affects only a shallow surface layer of the structure, up to about 5 cm depth. By fitting these data and combining them with (26), one gets for the stress confidence band corresponding to the Poisson square-wave input model the expression $[-7.58 + 16.82/(1.0 + 0.189x)]\sqrt{2(t - t_0)}$. The deviation of the material parameters and model uncertainty can be taken into account easily by a multiplicative factor $(1 + \rho)$; and the final expression is

$$\mu_\sigma \pm S_\sigma = \pm(1 + \rho)\left(-7.58 + \frac{16.82}{1 + 0.189x}\right)\sqrt{2(t - t_0)} \quad \ldots\ldots (35)$$

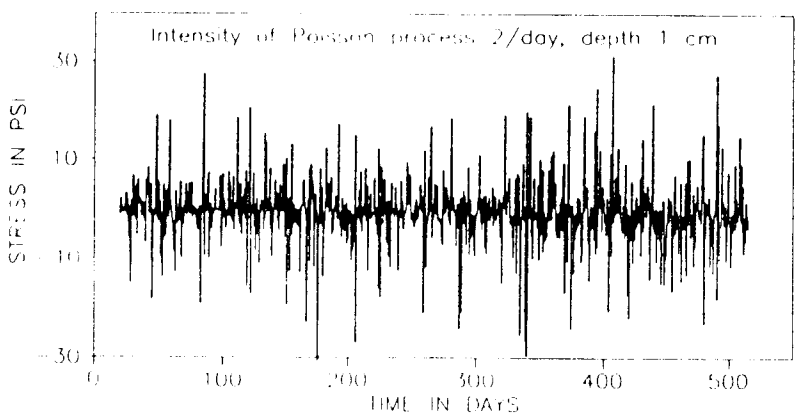

FIG. 8. Increments of Stress Response to Poisson Square-Wave Process

where ρ = a coefficient of variation; in our case $\rho = 0.15$, which is explained later.

Although the standard deviation of the stress increment is only slight, one can see from (33) that the standard deviation of stress increases steadily with time t. At a depth of 5 cm, it reaches 182 psi when $t = 40$ years. Thus the standard deviation of the response to random noise will grow greatly with time and will become dominant in the shallow surface layer of concrete. This is due to the cumulative effect of the random stress increment deviation.

Old concrete structures have experienced a large variety of environmental histories. Their random consequences cannot be described reasonably without a random-noise component, because the deviations would approach a steady value after about 10 years. But as we know, the older the structure, the larger is the disparity between various possible responses. So the Poisson process input is an inevitable part of the stochastic model.

Response to Initial Pore Humidity and Mean Environmental Humidity

If the randomness of material parameters and model uncertainty are not taken into account, the structural response to the initial humidity in concrete and the mean value of environmental humidity would be deterministic. Fig. 9 shows the sampling results of the stress response history and the stress distribution in the cross section up to 4,800 days. The numerical results indicate that although the mean value and standard deviation are functions of x and y, their coefficient of variation is almost independent of x and t. In our case, $\rho = S_\sigma/\mu_\sigma = 0.15$ at any time and any depth. This is true not only for the response to the mean environmental humidity and the initial pore humidity, but also for any of the responses discussed before. This property has in fact been used in fitting of the curves of prediction equations (23) and (35). However, ρ is not a constant; it depends on the statistical properties of the initial random variables (material parameters and model uncertainty). The sampling method could be used to determine the statistical distribution of the stress response, provided that the sample size is large enough. Sensitivity analysis, such as the partial correlation coefficient method (McKay 1979; Mises 1964), could be used to obtain the deviations of ρ with respect to the deviations of every single initial random variable and of their combinations.

As an approximate measure, one can first input the mean values of every initial random variable to obtain an estimate of the mean of the stress response (one computer run); then

$$\mu_\sigma \pm S_\sigma = \mu_\sigma(1 + \rho) \quad \ldots\ldots (36)$$

where $\rho = 0.15$; and μ_σ = mean of stress response.

According to the principle of superposition, the entire response of the linear system is the sum of the responses: (1) To the initial value of pore humidity; (2) to the mean value of the environmental humidity; (3) to the harmonic humidity variations of a one-year period; and (4) to the random noise-like Poisson square-wave process. Thus the entire stress-response confidence band is the overall mean plus or minus the total standard deviation from all the uncertainty sources. These simplified formulas, based on simulating the stress response as a filtered Poisson process, can provide the confidence band at any time and any depth. The calculaton is so simple that it can be done using a hand calculator.

Note that the present method can also be applied to the thermal stress

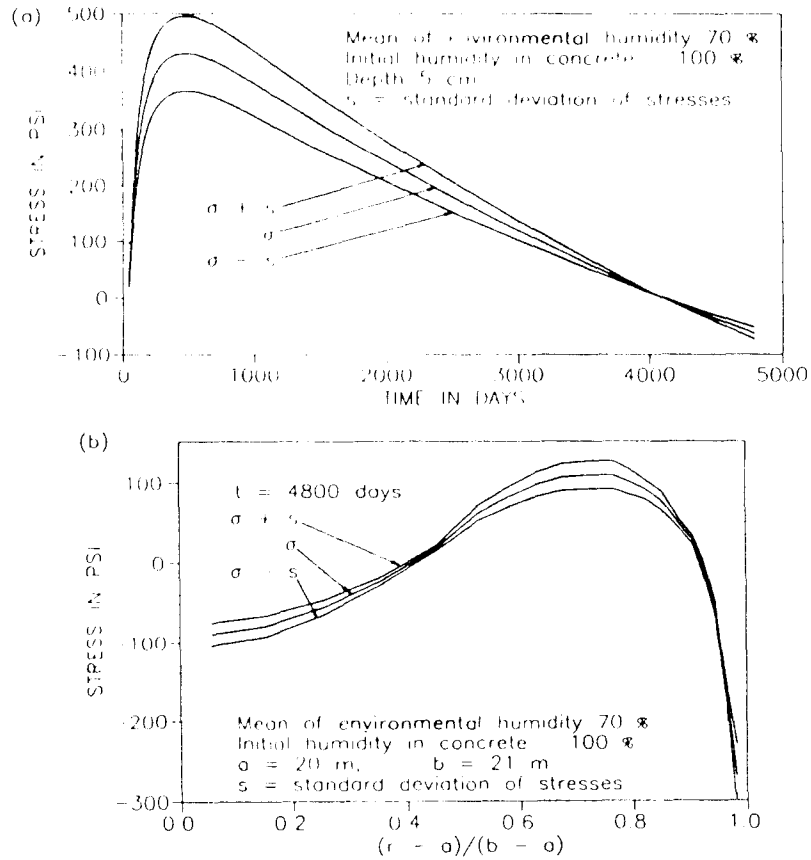

FIG. 9. (*a*) History of Stress Response to Mean of Environmental Humidity and Initial Humidity in Concrete; (*b*) Stress-Response Distribution in Cross Section

problem if one keeps the pore humidity constant and considers the environmental temperature as a stochastic process. However, since the thermal conductivity of concrete is several thousand times larger than moisture diffusivity, the one-day period (temperature difference between day and night) can no longer be neglected. Thus, the long-period solution cannot be simplified as a half-space solution, and the noise response will play a more important role in the concrete wall.

Conclusions

The random process of environmental humidity can be decomposed into three processes: two random-phase periodic processes, with periods of one day and one year, and one Poisson square-wave process.

Due to the extremely low value of moisture diffusivity of concrete, the response to the environmental humidity component of a one-day period is so feeble that it can be neglected without losing much in accuracy.

The stress response to the environmental humidity of one-year period is a nonstationary random-phase process. Its influence region is a layer about 20 cm deep. Its standard deviation reaches its maximum value at about 20 years and then decreases slowly as a result of aging.

Due to the extremely low value of moisture diffusivity of concrete, the stress response to the noise component of environmental humidity, which represents a filtered Poisson process, influences only a shallow layer of concrete about 5 cm deep. The standard deviation of stress accumulates and its upper-bound response is proportional to $\sqrt{t}$; so in this layer the stress response develops high random scatter at long times.

The randomness of material parameters and the model uncertainty influence the whole structure. At depths over 20 cm below surface, the coefficient of variation of the stress response is independent of the depth and time, and depends only on the values of material parameters and model uncertainty.

For the present example of an infinitely long cylinder with a thick wall, the response to a random-phase environmental humidity process of a one-year period need not be solved from the diffusion equation for a cylinder but can be replaced by a much simpler solution of a half-space. Also, the stress response to Poisson square-wave process of environmental humidity can be modeled as a filtered Poisson process. This leads to simplified but satisfactory explicit formulas for predicting the confidence band of the stress response.

The present combination of sampling method with the spectral method can be applied to creep and shrinkage of any concrete structure with random material parameters and model uncertainty, subjected to a random environmental humidity process.

Acknowledgment

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