CONTINUOUS RETARDATION SPECTRUM FOR SOLIDIFICATION THEORY OF CONCRETE CREEP

By Zdeněk P. Bažant,[1] Fellow, ASCE, and Yunping Xi[2]

ABSTRACT: The basic creep of concrete is the time-dependent strain caused by a sustained stress in absence of moisture movements. It is the strain observed on sealed specimens. Similar to other properties of concrete, it is dependent on the age of concrete, as a consequence of long-time chemical reactions associated with the hydration of cement. This paper formulates the solidification theory with a continuous retardation spectrum, and shows how this spectrum can be readily and unambiguously identified from arbitrary measured creep curves and how it then can be easily converted to a discrete spectrum for numerical purposes. The identification of the continuous spectrum is based on Tschoegl's work on viscoelasticity of polymers. Attention is limited to basic creep.

INTRODUCTION

The basic creep of concrete is the time-dependent strain caused by a sustained stress in absence of moisture movements. It is the strain observed on sealed specimens. Similar to other properties of concrete, it is dependent on the age of concrete, as a consequence of long-time chemical reactions associated with the hydration of cement.

The aging aspect of basic creep of concrete can be mathematically handled by two different approaches (*Mathematical* 1988): (1) The classical, direct approach which treats the material parameters involved in the creep model as empirical functions of age (Bažant and Wu 1974a,b); (2) the recently proposed approach called the solidification theory (Bažant and Prasannan 1989a,b), in which the material parameters for creep are considered to be age-independent but the volume fraction of the age-independent material increases with age. Only the latter approach has a solid foundation from the viewpoint of chemical thermodynamics. It also has an important practical advantage, namely, the characterization of creep by a nonaging model, which is much simpler. The aging in the solidification theory is introduced separately by means of a variation of the volume fraction of the solidifying viscoelastic material constituent.

In the formulation of solidification theory, there are two separate problems. The first is how to describe the variation of the volume fraction of the solidified nonaging material constituent. The second is how to characterize nonaging creep for the purposes of large-scale numerical analysis and correlate this characterization to the physics of the problem. Such a correlation is possible only if the creep of the nonaging constituent is described in a rate-type form consisting of first-order differential equations.

Such a rate-type form can be based on the Kelvin chain or the Maxwell chain. The Kelvin chain is a rheologic model composed of a series coupling of many Kelvin units, each of which consists of a parallel coupling of a spring and a dashpot. The Maxwell chain is a rheologic model composed of a parallel coupling of many Maxwell units, each of which consists of a series coupling of a spring and a dashpot. Roscoe (1950) proved that the Kelvin and Maxwell chains can each describe any given linear viscoelastic behavior with any desired accuracy. This means that there is no need for considering other more complicated rheologic models, and provides a justification for assuming one of the two basic rheologic models. Kelvin chain is more convenient because its parameters can be more easily identified from creep tests. The Maxwell chain parameters can be more easily identified from relaxation tests, but these are harder to carry out.

In the original formulation of the solidification theory (Bažant and Prasannan 1989a,b), the creep of the nonaging constituent is described by a Kelvin chain with a finite number N of Kelvin units. Each Kelvin unit number μ is characterized by its spring modulus E_μ and retardation time $\tau_\mu = \eta_\mu/E_\mu$, where η_μ = dashpot viscosity. The plot of $1/E_\mu$ versus τ_μ ($\mu = 1, \ldots, N$), which is in viscoelasticity called the retardation spectrum, fully characterizes the material creep properties. For a finite number N of Kelvin units, as used in all the previous studies of concrete creep, the spectrum is discrete (because the τ_μ-values are distributed along the time axis discretely). However, as is well known from classical (nonaging) viscoelasticity, identification of

[1]Walter P. Murphy, Prof. Civ. Engrg., Tech. Inst., Northwestern Univ., 2145 Sheridan Rd., Evanston, IL 60208-3109.

[2]Asst. Prof., Drexel Univ., Philadelphia, PA 19104; formerly, Postdoctoral Res. Assoc., Northwestern Univ., 2145 Sheridan Rd., Evanston, IL.

Note. Discussion open until July 1, 1995. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on January 27, 1993. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 121, No. 2, February, 1995. ©ASCE, ISSN 0733-9399/95/0002-0281–0288/$2.00 + $.25 per page. Paper No. 5521.

a broad discrete spectrum from test data is an ill-posed problem because different retardation times can give almost equally good fits of the measured creep curves. Thus, the discrete retardation times must be chosen, although with some restrictions (*Mathematical* 1988). The arbitrariness of the choice of retardation times is disturbing. Moreover, a simple method to determine the retardation spectrum has been available only for creep curves in the form of the power law, log-law, or log-power law (Bažant and Prasannan 1989b).

The purpose of the present paper, whose basic results were summarized at a recent conference (Bažant and Xi 1993), is to formulate the solidification theory with a continuous retardation spectrum, and show how this spectrum can be readily and unambiguously identified from arbitrary measured creep curves and how it then can be easily converted to a discrete spectrum for numerical purposes. The identification of the continuous spectrum will be based on Tschoegl's (1971, 1989) work on viscoelasticity of polymers. Attention will be limited to basic creep. The additional creep due to drying may be determined in the manner described elsewhere [e.g. *Mathematical* (1988)].

GENERALIZED KELVIN CHAIN MODEL FOR NONAGING BASIC CREEP

The linear viscoelastic behavior may be completely characterized by the compliance function $J(t, t')$, representing the strain ε at age t of concrete caused by a uniaxial sustained (constant) stress $\sigma = 1$ applied at age t'. The response for any stress history $\sigma(t)$ then follows by the principle of superposition. For three-dimensional behavior one would generally need a similar compliance function for shear strain, but for concrete this is not needed because the Poisson's ratio for basic creep happens to be approximately constant and equal to its instantaneous (elastic) value (about 0.18). Denoting as $1/E_0 = q_1$ = instantaneous elastic strain caused by $\sigma = 1$, one may write

$$J(t, t') = q_1 + C(t, t') \tag{1}$$

where $C(t, t')$ = creep compliance function. After lengthy studies (*Mathematical* 1988) it transpired that the creep description is the simplest if q_1 is taken as the deformation corresponding to extremely fast loading, shorter than about 10^{-9} s, which of course cannot be directly measured but must be obtained by asymptotically extrapolating the measured creep curve in the logarithmic time scale to $-\infty$. The advantage of doing this is that q_1 (or E_0, called the asymptotic modulus) can be considered as age-independent because in log-time scale the short-time creep curve (stress duration 10^{-4} s to 1 hr) happens to be a smooth (leftward) extension of the long-time creep curve (*Mathematical* 1988). The normal static elastic modulus, which agrees quite well with the value recommended for design purposes, is then obtained as $E(t) = 1/J(t + \Delta, t)$, where $\Delta = 0.1$ day. This expression also gives a good description of the age dependence of $E(t)$. The dynamic modulus is obtained for $\Delta = 10^{-8}$ s.

For the nonaging Kelvin chain model with N Kelvin units, the creep compliance function $C(t, t')$ is given by the Dirichlet series

$$C(\xi) = \sum_{\mu=1}^{N} A_\mu [1 - e^{-\xi/\tau_\mu}]; \qquad \text{with } A_\mu = \frac{1}{E_\mu} \tag{2}$$

where $\xi = t - t'$; t = time (age of concrete); t' = time (age) at the moment of loading and E_μ = elastic moduli of Kelvin units. In (1), the retardation times τ_μ can be chosen but the choice must satisfy certain well-known restrictions [e.g. *Mathematical* (1988)]. The values of A_μ, which characterize the strain increment during the time leg corresponding to τ_μ, have to be determined by optimum fitting of the measured creep curves.

In the previous studies, certain semiempirical formulas have been derived to evaluate A_μ from the available creep response curves (Bažant and Prasannan 1989a,b). Those formulas have been calibrated by nonlinear curve-fitting of computer results, and their validity was restricted to the log-power law. In general, when a slightly different creep law is required, those formulas are not valid. Another problem of those formulas is that, although a certain set of the optimized A_μ values may suffice to accurately describe the given creep behavior, this set is not unique, depending on the choice of values τ_μ. To deal with general creep laws and to avoid the aforementioned weak points including the nonuniqueness, an effective approach is to introduce the continuous Kelvin chain model with infinitely many Kelvin units and retardation times spaced infinitely closely, for which A_μ becomes a continuous retardation spectrum. Such a spectrum may be evaluated by a certain method developed in the theory of viscoelasticity (Tschoegl 1989). Then the discrete spectrum A_μ, which is required for computer analysis, can be obtained simply by discretizing the known continuous retardation spectrum.

CONTINUOUS RETARDATION SPECTRUM AND INVERSE TRANSFORMATION METHOD

Eq. (1) may be approximated in a continuous form

$$C(\xi) \approx \int_0^\infty L^*(\tau)(1 - e^{-\xi/\tau})\, d\tau \tag{3a}$$

Setting $L^*(\tau) = L(\tau)/\tau$, we have

$$C(\xi) = \int_{-\infty}^{\infty} L(\tau)(1 - e^{-\xi/\tau})\, d(\ln \tau) \tag{3b}$$

where $L(\tau)$ is called the continuous retardation spectrum. It has the same meaning in the logarithmic time scale as A_μ in the actual time scale. Many studies have been undertaken to deduce $L(\tau)$ from the known compliance function of the material (Gross 1953; Eirich 1958; Aklonis 1983; Fredrickson 1964; Ferry 1980; Gittus 1975; Christensen 1982; Tschoegl 1971, 1989). We will adopt a very efficient general method developed by Tschoegl (1971, 1989).

Using (3), and setting $\tau = 1/\zeta$ with $d(\ln \tau) = -d(\ln \zeta)$, we get

$$C(\xi) = \int_0^\infty L(\zeta^{-1})(1 - e^{-\zeta\xi})\zeta^{-1}\, d\zeta; \quad C(\xi) = \int_0^\infty L(\zeta^{-1})\zeta^{-1}\, d\zeta - \int_0^\infty \zeta^{-1}L(\zeta^{-1})e^{-\zeta\xi}\, d\zeta \tag{4, 5}$$

Then, denoting

$$f(\xi) = \int_0^\infty \zeta^{-1}L(\zeta^{-1})e^{-\zeta\xi}\, d\zeta \tag{6a}$$

we have

$$C(\xi) = -f(\xi) + f(0) \tag{6b}$$

$f(\xi)$ = Laplace transform of the function $\zeta^{-1}L(\zeta^{-1})$; and ξ = transform variable.

Now the important point is that the transform can be inverted by Widder's (1971) inversion formula, which is based on an asymptotic method. The inversion operator is

$$F_{k,\zeta}[f(\xi)] = \frac{(-1)^k}{k!}\left(\frac{k}{\zeta}\right)^{k+1} f^{(k)}\left(\frac{k}{\zeta}\right) \tag{7}$$

with the property

$$\lim_{k\to\infty} F_{k,\zeta}[f(\xi)] = \lim_{k\to\infty}\left[\frac{(-1)^k}{k!}\left(\frac{k}{\zeta}\right)^{k+1} f^{(k)}\left(\frac{k}{\zeta}\right)\right] = \zeta^{-1}L(\zeta^{-1}) \tag{8}$$

where $f^{(k)}$ = kth derivative of function f. Noting that $f(0)$ is a constant, we have ($k \geq 1$):

$$L(\tau) = -\lim_{k\to\infty} \frac{(-k\tau)^k}{(k-1)!} C^{(k)}(k\tau) \tag{9}$$

The approximate spectrum of order k is obtained by using a finite value of k ($k \geq 1$). The compliance data are entered through (6), and then derivative $C^k(k\tau)$ yields $L(\tau)$. But experimental data generally exhibit random scatter which does not allow taking higher derivatives except perhaps the second derivative. Therefore, certain methods have been developed to modify the creep data to allow taking higher-order derivatives (Akloni 1983; Ferry 1980). Instead of numerical differentiation of the test data, one must differentiate a smooth continuous compliance function, which matches the experimental data well enough.

At this point, the problem that remains is what kind of compliance function should be chosen. For concrete, the basic features that are exhibited by most sets of data [e.g. *Mathematical* (1988)] are that the short-term creep follows the power curve while the long-term creep follows the logarithmic curve. This means that the log-power law is a simple yet reasonable representation of the compliance function for concrete.

Nevertheless, the more complex functional expression in the solidification theory (Bažant and Prasannan 1989a) gives a better representation. The expressions and its derivatives are involved, but the derivatives can be evaluated numerically from finite-difference expressions.

EXAMPLE OF APPLICATION TO NONAGING LOG-POWER CREEP LAW

A simple expression for the creep compliance $C(t, t')$, which leads to a good approximation of the creep test data (Bažant and Prasannan 1989a) is the log-power law

$$C(\xi) = q_2 \ln\left[1 + \left(\frac{\xi}{\lambda_0}\right)^n\right] \tag{10}$$

with parameters q_2, λ_0, and n. The dimension q_2 as well as q_1 is MPa^{-1} (i.e., the same as $1/E$, where E = elastic modulus). Empirically, $\lambda_0 = 1$ day can be used for most concrete. The values of q_2 as well as q_1 has been expressed as empirical functions of concrete strength and composition (Bažant and Kim 1991).

According to (6) and (10)

$$f(\xi) = q_2 \ln(1 + \xi^n) - \int_0^{\infty} L(\zeta^{-1})\zeta^{-1}\, d\zeta \tag{11}$$

For $k = 3$, (9) yields the approximation

$$L(\tau) = \left[\frac{-2n^2(3\tau)^{2n-3}[n - 1 - (3\tau)^n]}{[1 + (3\tau)^n]^3}\right]\frac{(3\tau)^3}{2} q_2 + \left[\frac{n(n-2)(3\tau)^{n-3}[n - 1 - (3\tau)^n] - n^2(3\tau)^{2n-3}}{[1 + (3\tau)^n]^2}\right]\frac{(3\tau)^3}{2} q_2 \tag{12}$$

This is the approximate retardation spectrum of order 3, which appears to suffice for practical purposes. According to the data fitting in Bažant and Prasannan (1989b), n is a small constant. In the case of small n, the terms with n^3 in (12) may be neglected without significant loss of accuracy. Combining this with some other simplifications of (12), a simple approximation to the retardation spectrum can be obtained

$$L(\tau) \approx q_2 n(1 - n)\frac{(3\tau)^n}{1 + (3\tau)^n} \tag{13}$$

For very large τ, the spectrum $L(\tau)$ approaches a constant.

For simple practical problems, (13) is preferable because of its simple form. For computational analysis, such as finite-element analysis, or when n is large ($n > 0.45$), (12) ought to be used.

For the purpose of numerical computation, one can subdivide $\ln \tau$ into time intervals $\Delta(\ln \tau_\mu) = \ln 10\, \Delta(\ln \tau_\mu)$ and thus approximate the integral in (3*b*) by the finite sum

$$C(\xi) = \sum_{\mu=1}^{N} L(\tau_\mu)[1 - e^{-\xi/\tau_\mu}]\ln 10\Delta(\log \tau_\mu) \quad \text{or} \quad C(\xi) = \sum_{\mu=1}^{N} A_\mu[1 - e^{-\xi/\tau_\mu}] \tag{14a,b}$$

where

$$A_\mu = L(\tau_\mu)\ln 10\, \Delta(\log \tau_\mu) \tag{15}$$

where $L(\tau_\mu)$ is given by (12) or (13); and $\Delta(\log \tau_\mu)$ = time interval between two adjacent Kelvin units in the logarithmic scale. Computational experience shows that intervals $\Delta(\log \tau_\mu) = \log 10 = 1$ give sufficiently smooth creep curves (compliance function), while greater separations of τ_μ give creep curves of a bumpy appearance.

COMPARISON WITH PREVIOUS FORMULATION

Comparisons with some previous formulations are shown in Figs. 1–3, in which time is in days, and the value of q_2 is taken as 1, which makes J and A_μ dimensionless (this is possible because J as well as A_μ is proportional to q_2). Fig. 1 compares the previous formulae of Bažant and Prasannan (1989a, 1989b) with the simplified continuous spectrum obtained from (13). It can be seen that the discrete spectrum is not unique, but depends upon the different chosen values of τ_μ [in Fig. 1 only the retardation time for the second Kelvin unit, τ_2, is shown, others are spaced with intervals $\Delta(\log \tau_\mu) = \log 10 = 1$]. All of the discrete spectrum values are located close to the continuous spectrum. Fig. 2 compares $\ln(1 + \xi^n)$ with (14) and (15), in

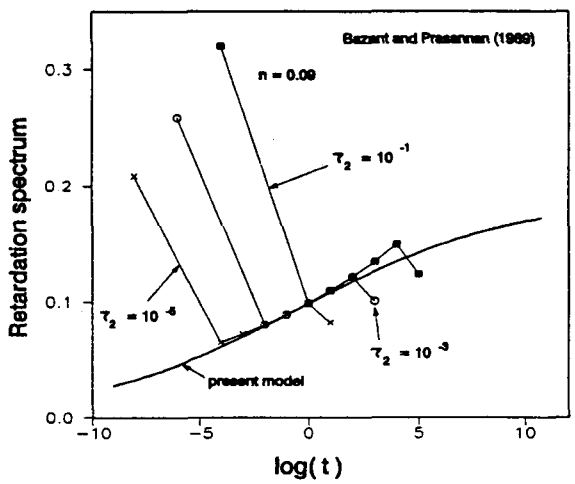

FIG. 1. Comparison of Retardation Spectra

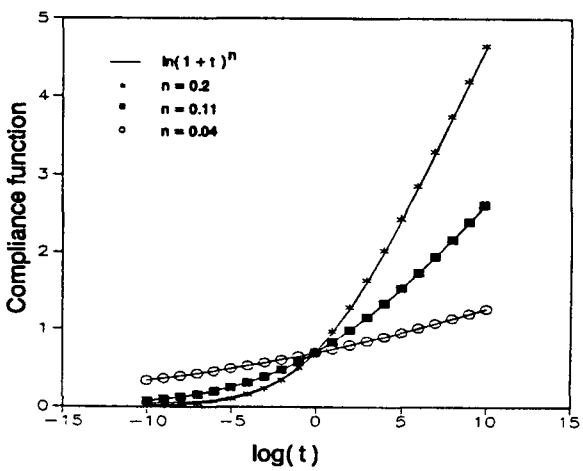

FIG. 2. Comparison of Compliance Functions

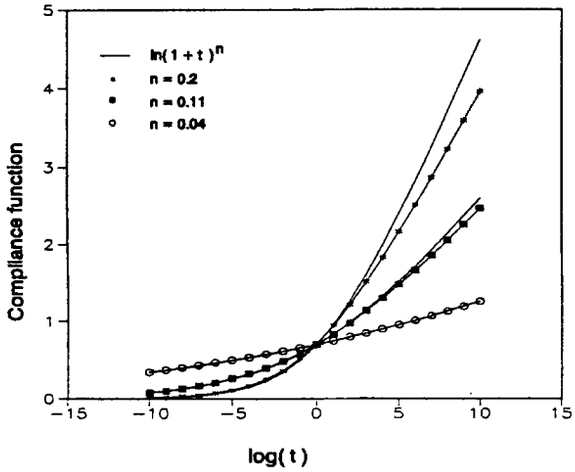

FIG. 3. Comparison of Log-Power Law with Simplified Compliance Function

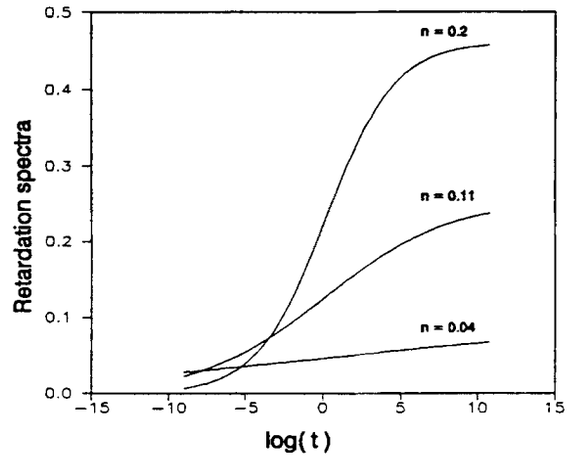

FIG. 4. Retardation Spectra with Various n

which $L(\tau)$ is approximated by the continuous spectrum (12). It can be seen that the compliance functions obtained from the continuous spectrum agree with the log-power curve very well, which proves that the approximation of order 3 is accurate enough. Fig. 3 shows the compliance functions obtained from the simplified continuous spectrum (13). It is clear that, for small n, (13) is also accurate enough.

Another advantageous feature of the continuous retardation spectrum is that some physical characteristics of creep can be obtained merely by comparison of the intensity of the spectrum within a certain retardation time range. For instance, Fig. 4 shows the retardation spectra for various values of n. The curve for $n = 0.04$ shows that the creep intensity (value of retardation spectrum) can be considered almost uniform within the time range of 10^{-5} to 10^{5} days. By checking the compliance function in Fig. 2 (curve with $n = 0.04$), it is seen that the creep indeed proceeds smoothly. However, the curve with $n = 0.2$ in Fig. 4 shows a relatively strong intensity in the time range of 10^{1} to 10^{10} days. This means that a significant part of the total creep will be delayed to the long-time range, and that the creep in the short-time range will be relatively small. By checking the compliance curve for $n = 0.2$ in Fig. 2, there is indeed a sharp increase of creep in the long-time range. So, in addition to computational advantages, the continuous retardation spectrum also reflects the creep intensity in various time ranges.

Once the retardation spectrum has been determined, an exact numerical transformation from the retardation spectrum to the relaxation spectrum can be made easily [see Gross (1953)]. Thus the time behavior of relaxation can be analyzed in a similar manner. In this regard it may be noted that a recent reformulation of the solidification theory revealed some advantages of the relaxation spectrum (Carol and Bažant 1993).

AMALGAMATION WITH SOLIDIFICATION THEORY FOR AGING AND RATE-TYPE FORMULATION

For the sake of completion of our formulation, we will briefly indicate how the foregoing formulation is combined with the solidification theory for basic creep, which was presented and justified in detail in Bažant and Prasannan (1989a,b). This theory assumes that the aging property of basic creep is caused by the processes of hydration and polymerization of cement (the hydration is also manifested by the increase of strength with age). Concrete is divided into three parts: the liquid part, which cannot bear load; and two load-bearing parts exhibiting viscous flow and viscoelastic deformation. Thus, the total creep strain is composed of two terms, the viscous flow strain, ε^{f}, and the viscoelastic strain, ε^{v}.

The key feature of the theory is that the aging aspect of basic creep of concrete is considered to be due to the growth of the volume fraction $v(t)$ (Fig. 5) of the effective load-bearing portion of solidified matter (i.e. hydrated cement), representing both the increase of the volume fraction of hydrated cement and the increase of the load-bearing solid fraction caused by formation of further bonds (or polymerization of calcium silicate hydrates). The advantage is that, in this theory, the properties of the load-bearing matter are age independent. Thus, the conventional viscoelastic (and viscoplastic) theories, as well as thermodynamic relations, can be applied. The creep strain rate corresponding to the viscoelastic solid part, $\dot{\varepsilon}^{v}$, can be expressed as the product of the age-independent strain rate of solid, $\dot{\gamma}$, and the increase of the volume fraction $v(t)$ of the solid (Bažant and Prasannan 1989a,b)

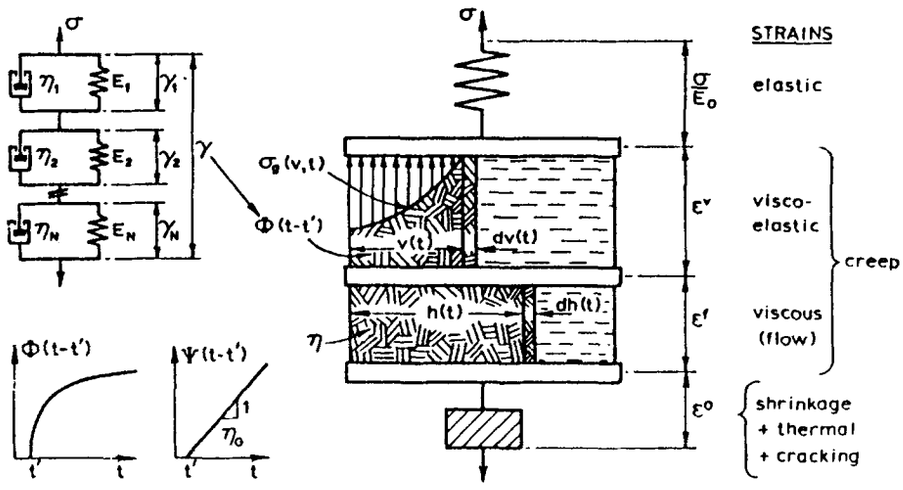

FIG. 5. Solidification Theory and Kelvin Chain Model

$$\dot{\varepsilon}^{v}(t) = \frac{F[\sigma(t)]}{v(t)}\,\dot{\gamma} \tag{16}$$

where function $F[\sigma(t)]$ is introduced to reflect nonlinear behavior at high stress (at low stress, $F[\sigma(t)] = 1$).

In this formulation, all of the procedures just developed for nonaging basic creep are applicable to the viscoelastic microstrain, γ. In the one-dimensional case, we may apply (14) at constant stress σ

$$\gamma = \sigma \sum_{\mu=1}^{N} A_{\mu}(1 - e^{-\xi/\tau_{\mu}}) \tag{17}$$

In analogy to (16), we have for the flow term

$$\dot{\varepsilon}^{f}(t) = q_3 \frac{F[\sigma(t)]}{v(t)}\,\sigma(t) \tag{18}$$

where q_3 = an empirical coefficient, which depends on the composition of concrete, same as does q_2 in (10).

Numerical computation, normally based on the finite-element method, proceeds in small time steps $\Delta t = t_{i+1} - t_i$ $(i = 1, 2, \ldots)$ and in an incremental form, which we give for the general case of three dimensions. We can always assume that, for a sufficiently short time step Δt, the stresses change linearly with t. Then, solving the differential equations for each Kelvin unit, we get (Bažant and Prasannan 1989b)

$$\Delta\boldsymbol{\sigma} = \mathbf{D}_c(\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}_c'') \tag{19}$$

where $\mathbf{D}_c = E\mathbf{D}$; and

$$E = \left[\frac{F(\sigma_{1,i+1/2})}{v_{i+1/2}} \sum_{\mu=1}^{N} \frac{1 - \lambda_{\mu}}{E_{\mu}}\right]^{-1}; \quad \lambda_{\mu} = \frac{(1 - e^{-\Delta y_{\mu}})}{\Delta y_{\mu}}; \quad \frac{1}{v_{i+1/2}} = \left(\frac{1}{t_{i+1/2}}\right)^{m_v} + \alpha_v \tag{20, 21a,b}$$

$$F(\sigma_{1,i+1/2}) = \frac{1 + \beta_{\sigma}^2}{1 - \omega}; \quad \text{with } \beta_{\sigma} = \left(\frac{\sigma_{1,i+1/2}}{f_c'}\right) \text{ and } \omega = (\beta_{\sigma})^{10} \tag{22}$$

Here $\Delta y_{\mu} = \Delta t/\tau_{\mu}$; $\Delta\boldsymbol{\varepsilon}_c$ = column matrix of strain increments of basic creep; $\Delta\boldsymbol{\sigma}$, $\Delta\boldsymbol{\varepsilon}$ = column matrices of stress and strain increments; $\mathbf{D}$ = square elasticity matrix divided by the conventional elastic modulus of concrete; f_c' = standard 28-day cylinder strength (in psi); $\Delta\boldsymbol{\varepsilon}_c''$ = column matrix of inelastic strain increments; and subscript $i + 1/2$ refers to the midstep values. In (20), $E_{\mu} = 1/A_{\mu}$ and A_{μ} is given by (15). In (21), $m_v = 0.5$, $\alpha_v = 1$ (Bažant and Prasannan 1989b). In (22), $\sigma_{1,i}$ is the maximum principal stress at time step i; for low stress level, $F(\sigma_{1,i}) = 1$. The matrix of inelastic strain increments, $\Delta\dot{\boldsymbol{\varepsilon}}_c''$, can be computed as $\Delta\boldsymbol{\varepsilon}_c'' = \Delta\boldsymbol{\varepsilon}_c^{v''} + \Delta\boldsymbol{\varepsilon}_c^{d''}$ in which

$$\Delta\boldsymbol{\varepsilon}_c^{v''} = F(\sigma_{1,i+1/2})\left(\frac{\Delta\boldsymbol{\gamma}^{v''}}{v_{i+1/2}} + \frac{q_3}{t_{i+1/2}} + \boldsymbol{\sigma}_i^{v}\Delta t\right); \quad \Delta\boldsymbol{\varepsilon}_c^{d''} = F(\sigma_{1,i+1/2})\left(\frac{\Delta\boldsymbol{\gamma}^{d''}}{v_{i+1/2}} + \frac{q_3}{t_{i+1/2}}\,\boldsymbol{\sigma}_i^{d}\Delta t\right) \tag{23, 24}$$

$$\Delta\boldsymbol{\gamma}^{v''} = \sum_{\mu=1}^{N}\left(\frac{\boldsymbol{\sigma}_i^{v}}{3E_{v_{\mu}}} - \boldsymbol{\gamma}_{\mu i}^{v}\right)(1 - e^{-\Delta y_{\mu}}); \quad \Delta\boldsymbol{\gamma}^{d''} = \sum_{\mu=1}^{N}\left(\frac{\boldsymbol{\sigma}_i^{d}}{3G_{\mu}} - \boldsymbol{\gamma}_{\mu i}^{d}\right)(1 - e^{-\Delta y_{\mu}}) \tag{25, 26}$$

where $E_{v_\mu} = E_\mu/3(1 - 2\nu)$; $G_\mu = E_\mu/2(1 + \nu)$; σ_i^v, $\boldsymbol{\sigma}_i^d$ = volumetric and deviatoric stress components at time step i, respectively; ν = Poisson's ratio, which is assumed to be constant for basic creep; $\gamma_{\mu i}^v$ and $\boldsymbol{\gamma}_{\mu i}^d$ = volumetric and deviatoric strain components for the μth Kelvin unit at time step i, respectively; and $\gamma_{\mu i}^v$ and $\boldsymbol{\gamma}_{\mu i}^d$ = internal variables, whose latest values need to be stored for each integration point of each finite element and each output point; $\gamma_{\mu i}^v$ and $\boldsymbol{\gamma}_{\mu i}^d$ can be evaluated on the basis of the information from the end of the previous time step

$$\gamma_{\mu i+1}^v = \gamma_{\mu i}^v e^{-\Delta\nu_\mu} + \frac{\sigma_i^v}{3F_{v_\mu}}(1 - e^{-\Delta\nu_\mu}) + \frac{1 - \lambda_\mu}{3E_{v_\mu}}\Delta\sigma_i^v \tag{27}$$

$$\boldsymbol{\gamma}_{\mu i+1}^d = \boldsymbol{\gamma}_{\mu i}^d e^{-\Delta\nu_\mu} + \frac{\boldsymbol{\sigma}_i^d}{2G_\mu}(1 - e^{-\Delta\nu_\mu}) + \frac{1 - \lambda_\mu}{2G_\mu}\Delta\boldsymbol{\sigma}_i^d \tag{28}$$

Subscript $i + 1/2$ refers to the midstep in log-time, i.e. $t_{i+1/2} = t_0 + [(t_i - t_0)(t_{i+1} - t_0)]^{1/2}$, where t_0 = time of first loading.

Eq. (19) reduces the solution of the basic creep problem to a sequence of elastic solutions with initial strains. In the case of high stress level, nonlinearity due to $F[\sigma]$ requires iterations of each time step to achieve good accuracy (Bažant and Chern 1985a,b, 1987).

Note that, in the present model, instantaneous deformation is not included but can be considered by adding additional term $1/E_0$, where E_0 is the asymptotic instantaneous elastic modulus.

The present model for basic creep, complemented by models for shrinkage, stress-induced shrinkage and drying creep, has been implemented in a nonlinear finite-element program and verified by comparisons with test results (Bažant and Xi 1994).

CONCLUSIONS

1. Despite the aging of concrete, it is possible and advantageous to use a continuous retardation spectrum for the Kelvin chain model in the solidification theory. This spectrum can be determined by the asymptotic transformation method, which is applicable for any creep law. By this method, a unique retardation spectrum can be obtained from the given compliance function.
2. Application to the log-power creep law reveals that for concrete the asymptotic retardation spectrum of order 3 is sufficiently accurate in practice. For small values of the time exponent, n, which are typical of concrete, the spectrum can be simplified and still describe the compliance function satisfactorily.
3. Based on the continuous retardation spectrum, the intensity of Kelvin units corresponding to the discrete retardation times chosen for numerical purposes can be determined, and then the time dependence of the nonaging creep component can be calculated. Furthermore, by discretization of the continuous retardation spectrum, a rate-type formulation for concrete creep can be obtained and combined with the solidification theory for aging. This yields a complete model for basic creep.

ACKNOWLEDGMENTS

Partial financial support under NSF grant MSS-9114476 to Northwestern University is gratefully acknowledged. Further support for studying materials science aspects has been received from the NSF Center for Advanced Cement-Based Materials at Northwestern University.

APPENDIX. REFERENCES

Aklonis, J. J., and MacKnight, W. J. (1983). *Introduction to polymer viscoelasticity*. John Wiley & Sons, Inc., New York, N.Y.

Bažant, Z. P., and Chern, J. C. (1985a). "Concrete creep at variable humidity: constitutive law and mechanism." *Mat. and Struct.*, Paris, France, 18(103), 1–20.

Bažant, Z. P., and Chern, J. C. (1985b). "Strain-softening with creep and exponential algorithm." *J. Engrg. Mech.*, ASCE, 111(3), 391–415.

Bažant, Z. P., and Chern, J. C. (1987). "Stress-induced thermal and shrinkage strains in concrete." *J. Engrg. Mech.*, ASCE, 113(10), 1493–1511.

Bažant, Z. P., and Kim, J.-K. (1991). "Improved prediction model for time-dependent deformations of concrete: Part 2: basic creep." *Mat. and Struct.*, Paris, France, 24, 409–421.

Bažant, Z. P., and Prasannan, S. (1989a). "Solidification theory of concrete creep. I: formulation." *J. Engrg. Mech.*, ASCE, 115(8), 1691–1703.

Bažant, Z. P., and Prasannan, S. (1989b). "Solidification theory of concrete creep. II: verification and application." *J. Engrg. Mech.*, ASCE, 115(8), 1704–1725.

Bažant, Z. P., and Wu, S. T. (1974a). "Rate-type creep law of aging concrete based on Maxwell chain." *Mat. and Struct.*, Paris, France, 7(37), 45–60.

Bažant, Z. P., and Wu, S. T. (1974b). "Thermoviscoelasticity of aging concrete." *J. Engrg. Mech. Div.*, ASCE, Vol. 100, 575–597.

Bažant, Z. P., and Xi, Y. (1993). "Continuous retardation spectrum for solidification theory of concrete creep." *Proc., 5th Int. Symp. on Creep and Shrinkage of Concrete*, Z. P. Bažant and J. Carol, eds., E&FN Spon, London, England, 225–230.

Bažant, Z. P., and Xi, Y. (1994). "Drying creep of concrete: constitutive model and new experiments separating its mechanisms." *Mat. and Struct.*, Paris, France.
Carol, I., and Bažant, Z. P. (1993). "Viscoelasticity with aging caused by solidification of a nonaging constituent." *J. Engrg. Mech.*, ASCE.
Christensen, R. M. (1982). *Theory of viscoelasticity*. Academic Press, Inc., San Diego, Calif.
Eirich, F. R. (1958). *Rheology, theory and applications*; Vol. 2, Academic Press, New York, N.Y.
Ferry, J. D. (1980). *Viscoelastic Properties of Polymers*. John Wiley & Sons, Inc., New York, N.Y.
Fredrickson, A. G. (1964). *Principles and applications of rheology*. Prentice-Hall, Inc., Englewood Cliffs, N.J.
Gittus, J. (1975). *Creep, viscoelasticity and creep fracture in solids*. John Wiley & Sons, Inc., New York, N.Y.
Gross, B. (1953). *Mathematical structure of the theories of viscoelasticity*. Herman, Paris, France.
Mathematical modeling of creep and shrinkage of concrete. (1988). Z. P. Bažant, ed., John Wiley and Sons, New York, N.Y.
Roscoe, R. (1950). "Mechanical models for the representation of viscoelastic properties." *British J. of Appl. Physics*, Vol. 1, 171–173.
Tschoegl, N. W. (1971). "A general method for the determination of approximations to the spectral distributions from the transient response functions." *Rheol. Acta*, Vol. 10, 595–600.
Tschoegl, N. W. (1989). *The phenomenological theory of linear viscoelastic behavior*. Springer-Verlag, Berlin, Germany.
Widder, D. V. (1971). *An introduction to transform theory*. Academic Press, New York, N.Y.