Stability of Cohesive Crack Model: Part I—Energy Principles

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The paper deals with a cohesive crack model in which the cohesive (crack-bridging) stress is a specified decreasing function of the crack-opening displacement. Under the assumption that no part of the crack undergoes unloading, the complementary energy and potential energy of an elastic structure which has a cohesive crack and is loaded by a flexible
representing compliances or stiffnesses relating various points along the crack. By variational analysis, in which the derivatives of the compliance or stiffness functions with respect to the crack length are related to the crack-tip stress intensity factors due to various unit loads, it is shown that the minimizing conditions reduce to the usual compatibility or equilibrium equations for the cohesive cracks. The variational equations obtained can be used as a basis for approximate solutions. Furthermore, the conditions of stability loss of a structure with a growing cohesive crack are obtained from the condition of vanishing of the second variation of the complementary energy or the potential energy. They have the form of a homogeneous Fredholm integral equation for the derivatives of the cohesive stresses or crack opening displacements with respect to the crack length. Loadings with displacement control, load control, or through a flexible loading frame are considered. Extension to the analysis of size effect on the maximum load or maximum displacement are left to a subsequent companion paper.

1 Introduction

Quasi-brittle materials, such as concrete, ice (especially sea ice), rocks, ceramics, and certain composites, exhibit a large fracture process zone in which the material undergoes progressive softening damage. The process zone may be approximated by the cohesive crack model, in which the fracture process zone is represented by crack-bridging tensile stresses (cohesive stresses) which decrease with crack opening. The basic concept originated with the work of Barenblatt (1962) and Dugdale (1960), who introduced two different versions of the cohesive crack model. While Dugdale considered the cohesive stresses to be constant in order to simulate plastic behavior of metals near the crack tip, Barenblatt, in an effort to model the reduction of interatomic bond forces, introduced cohesion as a gradual softening process. Barenblatt studied equilibrium but not stability and solved only problems in which the cohesive zone is infinitesimal and the distribution of the cohesive stresses can be assumed a priori, independent of the solution of the crack opening profile. Dugdale, on the other hand, considered a cohesive zone of finite length, in the context of plastic yielding. The problem of finding the peak load for the Dugdale model was later solved by Bilby, Cottrell, and Swinden (1963), who introduced for plastic cohesive stresses a critical crack-opening displacement at which the cohesive (crack-bridging) stress drops to zero.

Later studies have led to further diversification of the cohesive stress models. Some generalizations of Barenblatt's model do not have a uniquely defined stress-displacement law. Instead, the stress distribution along the cohesive portion of the crack is assumed a priori, with a process zone which may or may not be infinitesimal. Within this category, the models of Willis (1967), Smith (1974), and Reinhardt (1985) deserve mention. In another common type of cohesive crack models, there is a stress-displacement law but the effective energy release rate is calculated according to Rice's (1968) equation which was originally developed for small-scale yielding only. Nevertheless, this equation has often been extended beyond the small-scale yielding conditions, to situations in which the cohesive zone (or damage zone) is not negligible compared to the characteristic size of the structure (e.g., Suo et al., 1992). Although one basic characteristic of the cohesive crack models introduced by Barenblatt and Dugdale is that the stress intensity factor at the crack tip is zero, some recent modified forms of the cohesive crack model admit a positive stress intensity factor at the crack tip (Foote et al., 1986; Rice, 1992). There are also differences in the definition of stress-displacement law. Normally the cohesive stress is finite as soon as a crack starts to open, being equal to the tensile strength of the material, and subsequently the stress declines. But in some recent models, the cohesive stress starts to increase from zero as the crack begins to open, and only later softening takes place. This formulation, which is not considered in the present paper, has recently been used by Needleman (1990) and Tvergaard and Hutchinson (1992) in studies of the interface between plastic materials.

This study deals with the normal case of cohesive crack model, different from that in the aforementioned studies. The basic characteristics are (1) the length of the cohesive zone is finite, i.e., not negligible compared to the structure dimensions; (2) the stress intensity factor at the crack tip is zero (which means the stress at the tip is finite); (3) the cohesive stress depends on the crack-opening displacement according to a specified softening law; and (4) the material surrounding the crack behaves in a linearly elastic manner. Such a model (under the name "fictitious crack model") was first introduced for con-

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for presentation at the International Mechanical Engineering Congress and Exhibition, the Winter Annual Meeting of the ASME, San Francisco, CA, Nov. 12-17, 1995.

Discussion on this paper should be addressed to the Technical Editor, Professor Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the ASME JOURNAL OF APPLIED MECHANICS.

Manuscript received by the ASME Applied Mechanics Division, Mar. 21, 1995; final revision, Apr. 29, 1995. Associate Technical Editor: G. J. Dvorak.

Paper No. 95-WA/APM-7.

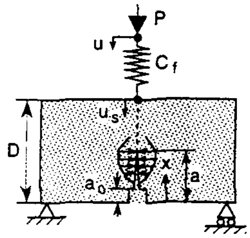

Fig. 1 Elastic structure with a cohesive crack, loaded through a spring

crete by Hillerborg, Modéer, and Petersson (1976) and extended by Petersson (1981). An equivalent crack band model was proposed by Bažant (1976, 1982, 1984), Bažant and Cedolin (1991), and Bažant and Oh (1983). All these studies were limited to the discrete finite element formulation. Although such a formulation is sufficient for numerical analysis of cracked structures, it is not well suited for characterizing the basic mathematical properties of the cohesive crack model. The aforementioned works were also limited to equilibrium analysis, but recently Li and Liang (1992, 1993) and Li and Bažant (1994) introduced a potential energy for the cohesive crack model and showed that the equilibrium conditions are the conditions of stationarity of this potential. However, the elastic strain energy of the structure and the energy of the cohesive stresses were treated in a combined manner, which does not reveal the basic mathematical properties and minimum principle.

This paper introduces continuous influence (Green's) functions along the crack in order to represent the elastic behavior of the structure separately from the cohesive stresses. The complementary energy and the potential energy of a general structure with cohesive crack is formulated and it is shown that the integral equations characterizing compatibility or equilibrium between the structure and the cohesive crack follow according to the well-known principles of minimum complementary energy or potential energy. The main objective of the paper is to present the variational derivation of these integral equations which reveals interesting relations to linear elastic fracture mechanics. The variational equations obtained could be used as a basis for approximate solutions. Finally, the conditions of stability of structures with cohesive cracks under load or displacement control and loading through a flexible frame are derived from the energy functionals in a general form. Further extensions to the analysis of size effect on the maximum load under load control and the maximum deflection under displacement control are relegated to a subsequent companion paper, and applications to sea ice will appear later.

2 Basic Energy Variables

Consider an elastic structure (or specimen) with a cohesive crack of length a (Fig. 1). For the sake of generality, we consider the structure to be loaded through an elastic loading frame of elastic compliance C_f, which is equivalent to a spring coupled in series, as in Fig. 1 (since the columns of a testing machine are placed parallel to the test specimen, a novice might think this is a parallel coupling, but it is a series coupling because the forces transmitted by the machine and by the specimen are equal and their displacements are additive). The special case $C_f = 0$ equivalent to a dead load applied directly on the structure. The problem can be formulated in terms of either stiffnesses or compliances. We first study the latter. In that case the basic variables are the forces, and the thermodynamic potential is the complementary energy Π^*, representing the Gibbs free energy in the case of isothermal conditions, or the enthalpy in the case of isentropic (or adiabatic) conditions.

According to the first law of thermodynamics (balance of energy), Π^* is additive, i.e., represents for sum of the complementary energies of all the parts of the structure-load system. Therefore,

$$\Pi^* = \Pi_s^* + \Pi_f^* + \Pi_l^*; \quad \Pi_s^* = U^* + \Pi_c^* \tag{1}$$

$$\Pi_c^* = \int_{a_0}^{a} \Gamma^*[\sigma(x)]dx \tag{2}$$

where a_0 = length of notch or initial traction-free crack; Π_s^*, Π_f^*, Π_l^*, U^*, Π_c^* = complementary energies of the structure (with the crack), the loading frame, the loading (Fig. 2(a)), the elastic structure (without crack), and the cohesive crack; and $\Gamma[\sigma(x)]$ = density of complementary energy of the crack at point of length coordinate x (Fig. 1). The elastic structure, as well as the loading frame, is assumed to be internally in equilibrium and internally compatible (which means their internal degrees of freedom are condensed out). Obviously, $\Pi_f^* = C_f P^2/2$ where C_f = compliance of the loading frame.

In the case of compliance formulation, the basic variables are the forces (e.g., Bažant and Cedolin, 1991, Sec. 10.1). So, if we want to obtain the load-point displacement u (displacement of the loading frame at the point where force P is applied), we must consider u to be constant and load P as variable (Fig. 2(c)). Thus, the complementary energy of the loading $\Pi_l^* = -W^* = -Pu$ (representing the complementary work W^* of the constant displacement u on the varying force P, which is equal to the area to the left of the vertical line in Fig. 2(c)).

The complementary energy Π^* as a potential exists, of course, if and only if the complementary energy potential exists for each part of the structure. For the crack, Π_c^* exists if we assume that no part of the crack ever undergoes unloading (i.e., the crack opens monotonically at all points x) and that the crack-opening displacement $w(x)$ at x (half the crack width) is a function of the cohesive (crack-bridging) stress $\sigma(x)$, i.e.,

$$w(x) = g[\sigma(x)] \tag{3}$$

(Fig. 3). This is the same as if the faces of the crack were held together by nonlinear continuously distributed springs following the law in Eq. (3). This law is softening (i.e., g is a decreasing function), as illustrated by the curve in Fig. 3.

The density of the potential energy of the cohesive crack is

$$\Gamma(w) = \int_0^w f(w')dw' \tag{4}$$

(Fig. 3), where $\sigma = f(\sigma)$ defines the given softening law and is inverse to function $w = g[\sigma(x)]$. The density of complementary energy is $\Pi^* = \sigma w - \Gamma$, that is

$$\Gamma^*(\sigma) = \int_{f_t}^{\sigma} g(\sigma')d\sigma' \tag{5}$$

(Fig. 3), where $w = g(\sigma)$ defines the softening law by function g that is inverse to function f; f_t' = tensile strength, which determines the σ-value at which the cohesive crack begins to open.

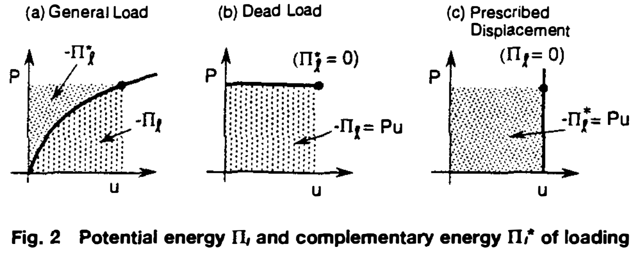

Fig. 2 Potential energy Π_l and complementary energy Π_l^* of loading

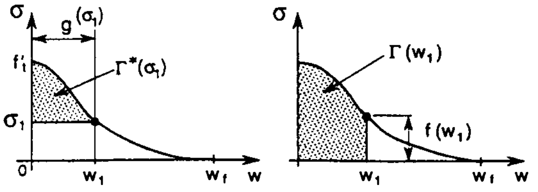

Fig. 3 Softening law relating crack-bridging (cohesive) stress σ and crack-opening displacement w

3 Complementary Energy of Structure With a Cohesive Crack

The complementary strain energy of the elastic structure, U^*, is a function of the load P as well as the surface tractions on the crack faces, which are equal to $-\sigma(x)$. Let C^{PP} = load-point compliance of the structure, $C^{\sigma\sigma}(x, x')$ = influence function representing a crack compliance function, such that the unit surface traction $\sigma(x) = \delta(x - x')$ applied at x' causes at x the opening $w = C^{\sigma\sigma}(x, x')$; $C^{\sigma P}(x)$ = influence function representing a cross-compliance function = crack-opening displacement at x caused by unit load P. Evidently, $C^{\sigma\sigma}(x, x') = C^{\sigma\sigma}(x', x)$, and $C^{\sigma P}(x)$ is equal to $C^{P\sigma}(x)$, defined so that a unit surface traction $\sigma = \delta(x - x')$ causes the load-point displacement $u = C^{P\sigma}(x)$. Expressing U^* in terms of the compliances and combining all the expressions for the complementary energies in (1) and (2), we have

$$\Pi^* = \Pi^*[\sigma(x), a, P] = \int_{a_0}^{a} \Gamma^*(x)dx + \tfrac{1}{2}\int_{a_0}^{a}\int_{a}^{a} C^{\sigma\sigma}(x, x')\sigma(x)\sigma(x')dx'dx - P\int_{a_0}^{a} C^{\sigma P}(x)\sigma(x)dx + \tfrac{1}{2}C^{PP}P^2 + \tfrac{1}{2}C_f P^2 - Pu \quad (6)$$

The reason for the first minus sign is that in our notation the positive directions of $\sigma(x)$ and $w(x)$ are opposite (the vector of w points toward the crack face, while the vector of σ points away from the crack face).

Equation (6) represents a function of P and $\sigma(x)$, giving the complementary energy for all the equilibrium states of the structure which are generally not compatible with the crack opening $w(x)$ and do not have the correct crack length a. The compatible crack opening is obtained by minimizing Π^* with respect to $\sigma(x)$, and the energetically correct crack length a is obtained by minimizing Π^* with respect to a. Although the equations that must result from this minimization are basically known, it will be instructive to carry out the variational procedure of minimization. Besides, the variational equations obtained by this procedure may be of interest for approximate solutions.

4 Compatibility and Minimization of Complementary Energy

A necessary condition of minimum of Π^* is that the first variation $\delta\Pi^* = 0$. We have

$$\delta\Pi^* = \left\{(C^{PP} + C_f)P - u - \int_{a_0}^{a} C^{\sigma P}(x)\sigma(x)dx\right\}\delta P + \int_{a_0}^{a}\left\{g[\sigma(x)] + \int_{a_0}^{a} C^{\sigma\sigma}(x, x')\sigma(x')dx' - C^{\sigma P}(x)P\right\}\delta\sigma(x)dx$$

$$+ \left\{\frac{1}{2}\int_{a_0}^{a}\int_{a_0} \text{———} - P\int_{a_0}^{a} \frac{}{\partial a}\sigma(x)dx + \frac{1}{2}\frac{\partial C^{PP}}{\partial a}P^2 + \int_{a_0}^{a} C^{\sigma\sigma}(x, a)\sigma(x)\sigma(a)dx - PC^{\sigma P}(a)\sigma(a) + \Gamma^*[\sigma(a)]\right\}\delta a = 0. \quad (7)$$

This may be simplified by noting that we may substitute $C^{\sigma\sigma}(x, a) = C^{\sigma P}(a) = 0$ because $w = 0$ at the crack tip, and $\Gamma^*[\sigma(a)] = \Gamma^*(f_t') = 0$.

It is now necessary to relate the compliance derivatives to the energy release rate and the stress intensity factors. The energy release rate is $\partial\Pi^*/\partial a = K^2/E'$ where E' = Young's modulus E for the case of plane stress or $E' = E/(1 - \nu^2)$ for the case of plane strain, with ν = Poisson ratio. According to the principle of superposition, the total stress intensity factor is

$$K = K_P + K_\sigma = Pk_P + \int_{a_0}^{a} k_\sigma(x)\sigma(x)dx \quad (8)$$

where K_P = stress intensity factor due to load P alone and K_σ = stress intensity factor due to surface tractions $-\sigma(x)$; $k_\sigma(x)$, k_P = stress intensity factor at the crack tip ($x = a$) caused by a unit surface traction, i.e., stress $-\sigma = \delta(x - x')$, or by a unit load $P = 1$. With these notations, we have

$$\frac{\partial C^{\sigma\sigma}(x, x')}{\partial a} = \frac{\partial}{\partial a}\frac{\partial^2\Pi^*}{\partial\sigma(x)\partial\sigma(x')} = \frac{\partial^2}{\partial\sigma(x)\partial\sigma(x')}\frac{\partial\Pi^*}{\partial a} = \frac{1}{E'}\frac{\partial^2}{\partial\sigma(x)\partial\sigma(x')}\left[Pk_P + \int_{a_0}^{a} k_\sigma(x)\sigma(x)dx\right]^2 = \frac{2}{E'}k_\sigma(x)k_\sigma(x') \quad (9a)$$

$$\frac{\partial C^{\sigma P}(x)}{\partial a} = -\frac{1}{E'}\frac{\partial^2}{\partial\sigma(x)\partial P}\left[Pk_P + \int_{a_0}^{a} k_\sigma(x)\sigma(x)dx\right. = -\frac{2}{E'}k_\sigma(x)k_P \quad (9b)$$

$$\frac{\partial C^{PP}}{\partial a} = \frac{2}{E'}k_P^2. \quad (9c)$$

Equations (9a, b) represent a continuous generalization of a similar well-known relation for concentrated loads given, e.g., by Tada et al. (1985).

Equation (7) must be satisfied for any variations δP, $\delta\sigma(x)$ and δa (such that $\delta\sigma \gtrsim 0$). This requires that the expressions in brackets {. . .} vanish. With the aforementioned substitutions, the vanishing of the variations with respect to δP, $\delta\sigma(x)$ and δa requires that

$$\frac{\partial\Pi^*}{\partial P} = -\int_{a_0}^{a} C^{P\sigma}(x)\sigma(x)dx + (C^{PP} + C_f)P - u = 0 \quad (10)$$

$$\frac{\partial\Pi^*}{\partial\sigma(x)} = \int_{a_0}^{a} C^{\sigma\sigma}(x, x')\sigma(x')dx' - C^{\sigma P}(x)P + g[\sigma(x)] = 0 \quad (11)$$

$$E'\frac{\partial\Pi^*}{\partial a} = \int_{a_0}^{a} k_\sigma(x)k_\sigma(x')\sigma(x)\sigma(x')dx'dx + 2Pk_P\int_{a_0}^{a} k_\sigma(x)\sigma(x)dx + P^2k_P^2 = 0. \quad (12)$$

Noting that Eq. (12) may be written as

$$E' \frac{\partial \Pi^*}{\partial a} = \left[Pk_P + \int_{a_0}^{a} k_\sigma(x)\sigma(x)dx \right]^2 = 0, \qquad (13)$$

we obtain the relation $K = K_P + K_\sigma = 0$.

Equation (11) represents the crack compatibility condition, i.e., the condition that the crack openings calculated from the deformation of the elastic body match the openings obtained from the given softening law $g(\sigma)$. Equation (10) gives the load-point deflection and, in the sense of the complementary energy approach, has the meaning of the condition of compatibility of the elastic deformation of the structure with the given value of u (it is not an equilibrium condition). Equation (8) with $K = 0$, called the zero-K condition, is the basic condition of any cohesive crack model, proposed by Barenblatt (1962) and Dugdale (1960). This condition means that the stress at the crack tip must be finite, i.e., that there is no singularity. From the variational viewpoint, the condition $K = 0$ is a condition of energy rate balance between the structure and the crack. If K were positive, the rate of energy released from the structure would be larger than that dissipated in the cohesive crack (and then the propagation would be dynamic). If K were negative, propagation would be impossible.

To calculate structure displacement u_s we substitute $u = u_s + C_f P$ into (10), considering P to be positive when it puts the loading spring (Fig. 1) into compression. Then, solving for P, we get

$$u_s = C^{PP}P - \int_{a_0}^{a} C^{P\sigma}(x)\sigma(x)dx. \qquad (14)$$

5 Stability Loss in Terms of Compliance

According to the second law of thermodynamics, stability requires that the second variation $\delta^2\Pi^*$ be a positive definite functional of $\sigma(x)$, P and a (e.g., Bažant and Cedolin, 1991, Sec. 10.1). The limit of stability occurs when $\delta^2\Pi^* = 0$ for some variation $\delta\sigma(x)$, δP and δa. The variation of u cannot be considered arbitrary because displacements cannot be the variables in the complementary energy functional. Therefore, we consider loading under displacement control conditions, which is, of course, required by the form of the work term in our expression for Π^*. $\delta^2\Pi^* = 0$ means that

$$\delta(\delta_P\Pi^*) = 0, \quad \delta(\delta_\sigma\Pi^*) = 0, \quad \delta(\delta_P\Pi^*) = 0 \quad (15a, b, c)$$

for some variation. The conditions of stationary Π^*, which are necessary for min Π^*, are represented by the vanishing of (10), (11), and (12); they must be imposed only after variations δ are taken. According to the expressions for $\delta_P\Pi^*$, $\delta_\sigma\Pi^*$, and $\delta_a\Pi^*$ implied by (10), (11), and (12), with (13), Eq. (15a, b, c) read:

$$-\delta(\delta_P\Pi^*) = \left[\int_{a}^{a} C^{P\sigma}(x)\delta\sigma(x)dx + \int_{a_0}^{a} \frac{\partial C^{P\sigma}(x)}{\partial a}\sigma(x)dx\delta a + C^{P\sigma}(a)\sigma(a)\delta a - (C^{PP} + C_f)\delta P - \frac{\partial C^{PP}}{\partial a} P\delta a \right] \delta P = 0 \quad (16a)$$

$$-\delta(\delta_\sigma\Pi^*) = \int_{a_0}^{a}$$

$$+ \int_{a}^{a} \frac{\partial C^{\sigma\sigma}(x,}{\partial a} \sigma(x')dx'\delta a - C^{\sigma\sigma}(x, a)\sigma(a)\delta a - \frac{\partial C^{\sigma P}(x)}{\partial a} P\delta a - C^{\sigma P}(x)\delta P$$

$$+ \frac{dg[\sigma(x)]}{d\sigma}\delta\sigma(x) \Bigg] \delta\sigma(x)dx = 0 \quad (16b)$$

$$E'\delta(\delta_a\Pi^*) = 2(K_P + K_\sigma)(\delta K_P + \delta K_\sigma)\delta_a = 0. \quad (16c)$$

The last equation is automatically satisfied because $K_P + K_\sigma = 0$ at min Π^*. The other equations may again be simplified using Eq. (9a, b, c) and, noting that,

$$C^{\sigma\sigma}(x, a) = C^{P\sigma}(a) = 0, k_P P = K_P,$$

$$-\int_{a_0}^{a} k_\sigma(x)\sigma(x)dx = K_\sigma$$

we get

$$-\delta(\delta_P\Pi^*) = \left[\int_{a_0}^{a} C^{P\sigma}(x)\delta\sigma(x)dx - \frac{2}{E'} k_P(K_P + K_\sigma)\delta a - (C^{PP} + C_f)\delta P \right] \delta P = 0 \quad (17a)$$

$$\delta(\delta_\sigma\Pi^*) = \int_{a_0}^{a} \left[\int_{a_0}^{a} C^{\sigma\sigma}(x, x')\delta\sigma(x')dx' + \frac{2}{E'} k_\sigma(x)(K_P + K_\sigma)\delta a - C^{\sigma P}(x)\delta P + \frac{dg[\sigma(x)]}{d\sigma}\delta\sigma(x) \right] \delta\sigma(x)dx = 0. \quad (17b)$$

Here again we may set $K_P + K_\sigma = 0$. Eliminating δP from the last two equations, we obtain

$$-\frac{dg[\sigma(x)]}{d\sigma} v(x) = \int_{a_0}^{a}$$

in which we introduced the notations $\delta\sigma(x)/\delta a = v(x)$ and

$$\bar{C}^{\sigma\sigma}(x, x') = C^{\sigma\sigma}(x, x') - \frac{C^{\sigma P}(x)C^{\sigma P}(x')}{C^{PP} + C_f}. \qquad (19)$$

Equation (18) is a homogeneous Fredholm integral equation for function $v(x)$. It is linear if function $g(\sigma)$ is linear. It characterizes the loss of stability under displacement control with a flexible loading frame. Note that $\bar{C}^{\sigma\sigma}(x, x')$ represents the crack compliances of the system consisting of the structure and the loading frame combined.

The special case $C_f = 0$ is equivalent to loading the structure under displacement control at the point of transmission of load P into the structure (rather than at the loading frame). In this case, the stability limit determines the maximum deflection of the structure, corresponding to the onset of snapback instability. Of course, the condition of maximum deflection could have been obtained more directly, simply by considering a structure without any loading frame from the outset ($C_f = 0$).

The special case $C_f \to \infty$ (with $u \to \infty$), for which $\bar{C}^{\sigma\sigma}(x, x') = C^{\sigma\sigma}(x, x')$, is equivalent to loading the structure under conditions of load control (dead load). In that case, the structure and the crack become unstable at maximum load, and so Eq. (18) becomes the condition of maximum load, which is obtained from Eq. (14).

The condition of maximum load ($C_f \to \infty$) could have been, of course, also obtained more directly, namely by considering the structure without any loading frame, loaded directly by P. In that case, we would have had to discard the term $-Pu$ from Eq. (6) defining Π^*. This term represents the complementary energy for load P varying at constant displacement u. For displacement u varying at constant P (i.e., dead load), the complementary energy of load is zero (Fig. 2(b)).

6 Potential Energy of Structure With Cohesive Crack

Second, we study the stiffness formulation. In that case the thermodynamic potential is the potential energy Π, representing the Helmholtz free energy in the case of isothermal conditions or the total energy in the case of isotropic (adiabatic) conditions. According to the first law of thermodynamics,

$$\Pi = \Pi_s + \Pi_f + \Pi_l, \quad \Pi_s = U + \Pi \tag{20}$$

where

$$\Pi = \int_{a_0}^{a} \qquad\qquad {}_0 \tag{21}$$

The notations Π_s, Π_f, Π_l, and U are analogous to Π_s^*, Π_f^*, Π_l^*, and U^* (U = strain energy of structure, $-\Pi_l = W$ = work of load). Obviously, $\Pi_f = R_f(u - u_s)^2/2$ where $R_f = 1/C_f$ = stiffness of the loading frame, and u_s = displacement of structure at point of contact with the loading frame.

In the case of stiffness formulation, the basic variables are the displacements. If we consider load-control, then P is fixed (dead load) and u is variable, in which case the potential energy of the loading is $\Pi_l = -W = -Pu$. This happens to be the same expression as before but W represents the work of load P on displacement u, which is the area below the horizontal line in Fig. 1 (b). However, we will be more interested in displacement control, in which case $\Pi_l = 0$, because u is fixed ($du = 0$) and P is varied (in which case $dW = Pdu = 0$); Fig. 2(c).

7 Equilibrium and Minimization of Potential Energy

Under our assumptions, the potential exists separately for the load, the frame, the structure, and the cohesive crack. Therefore, it must also exist for the entire system, as given by Eq. (21). The potential energy of the elastic structure (i.e., strain energy) is a function of u_s and $w(x)$ (all the internal displacements of the structure are assumed to be condensed out). Let $R^{ww}(x, x')$ = influence function = crack stiffness function representing the stress at x caused by a unit displacement $w = \delta(x - x')$; $R^{wu}(x)$ = influence function (or stiffness function) representing the stress at x caused by a unit displacement $u_s = 1$; and R^{uu} = load-point stiffness representing the force at the point of contact with the loading frame caused by $u_s = 1$. The potential energy expression for the case of displacement control may now be rewritten as

$$\Pi = \int_{a_0}^{a} \qquad + \frac{1}{2}\int_{a}^{a}$$
$$- u_s \int_{a_0}^{a}$$

(Note that $\Pi_l = 0$ because the load varies at prescribed displacement.) The first variation of this expression is

$$\delta\Pi = \left\{ (R^{uu} + R_f)u_s - R_f u - \int_{a_0}^{a} R^{uw}(x)w(x)dx \right\} \delta u_s$$
$$+ \int_{a_0}^{a} \left\{ f[w(x)] + \int_{a_0}^{a} R^{ww}(x, x') \times w(x')dx' - u_s R^{uw}(x) \right\} \delta w(x)dx$$
$$+ \left\{ \frac{1}{2} \quad {}_{a_0} \quad {}_{a_0} \quad \frac{\qquad}{\partial a} - u_s \int_{a_0}^{a} \frac{\qquad}{\partial a} w(x)dx + \frac{1}{2}\frac{\partial R^{uu}}{\partial a} u_s^2 \right.$$
$$+ \int_{a}^{a} R^{ww}(x, a)w(a)dx - u_s R^{uw}(a)w(a)$$
$$\left. + \Gamma[w(a)] \right\} \delta a = 0. \tag{23}$$

Now we note that $w(a) = \Gamma(0) = 0$ and, in analogy to Eq. (9a, b, c), we could prove that

$$\frac{\partial R^{ww}(x, x')}{\partial a} = -\frac{2}{E'} k_w(x)k_w(x'),$$
$$\frac{\partial R^{wu}(x)}{\partial a} = \frac{2}{E'} k_w(x)k_u, \quad \frac{\partial R^{uu}}{\partial a} = -\frac{2}{E'} k_u^2 \tag{24}$$

where $k_w(x)$ and k_u are the stress intensity factors at $x = a$ caused by unit displacement $w(x') = \delta(x' - x)$ or by unit displacement $u_s = 1$. The condition of minimum potential energy requires that the expression in the brackets $\{\ldots\}$ multiplied by δu_s, $\delta w(x)$, and δa vanish. This yields three equations:

$$u_s = \frac{1}{R^{uu} + R_f}\left[R_f u + \int_{a_0}^{a} \right.$$

$$f[w(x)] = -\int_{a_0}^{a} R^{ww}(x, x')w(x')dx' + R^{uw}(x)u_s \tag{26}$$

$$-\frac{\partial \Pi}{\partial a} = \frac{1}{E'}\int_{a_0}^{a} \quad {}_a \quad k_w(x)k_w(x')w(x)w(x')dx'dx$$
$$+ \frac{u_s}{E'}\int_{a_0}^{a} k_w(x)k_u dx + \frac{1}{E'} k_u^2 u_s^2$$
$$= \frac{1}{E'}\left[k_u u_s + \int_{a_0}^{a} k_w(x)w(x)dx \right]^2 = 0.$$

The last equation may be rewritten as

$$K = K_u + K_w = 0, \quad K_u = k_u u_s,$$
$$K_w = \int_{a_0}^{a} k_w(x)w(x)dx. \tag{28a, b, c}$$

This means that the total stress intensity factor K caused by load-point displacement and crack opening must vanish. This is the same well-known condition as we obtained before, but expressed in terms of the displacement-caused stress intensity factors k_u and k_w.

To calculate load P, we substitute $u = u_s + (P/R_f)$ into (25), and solve the equation for P:

$$P = R^{uu}u_s - \int_{a}^{a} R^{uw}(x)w(x)dx. \tag{29}$$

Equation (26) gives the equilibrium value of displacement of the structure at the point of loading by the frame. Equations (27) or (28) give the condition of equilibrium (static) propagation, at which the energy release and energy dissipation rates are equal.

8 Stability Loss in Terms of Stiffness

Let us now consider stability at prescribed displacement u. According to the Lagrange-Dirichlet theorem (which is a consequence of the second law of thermodynamics under the hypothesis of a tangentially equivalent elastic structure, Bažant and Cedolin, 1991, Sec. 10.1), stability requires that $\delta^2\Pi$ be positive definite. The limit of stability is reached when $\delta^2\Pi = 0$ becomes possible. To simplify the calculations, let us eliminate displacement u_s by substituting Eq. (26) into (27) and (28). This yields

$$f[w(x)] = -\int_{a_0}^{a} \bar{R}^{ww}(x, x')w(x')dx' + \bar{R}^{uw}(x)u \quad (30)$$

$$\bar{K}_u + \bar{K}_w = 0, \quad \bar{K}_u = \bar{k}_u u, \quad \bar{K}_w = \int_{a_0}^{a} \bar{k}_w(x)w(x)dx. \quad (31)$$

Here we introduced the notations

$$\bar{R}^{ww}(x, x') = R^{ww}(x, x') - \frac{R^{uw}(x)R^{uw}(x')}{R^{uu} + R_f}, \quad (32)$$

$$\bar{R}^{uw}(x) = \frac{R_f}{R^{uu} + R_f} R^{uw}(x) \quad (32)$$

$$\bar{k}_w = k_w(x) + \frac{R^{uw}(x)k_u}{R^{uu} + R_f}, \quad \bar{k}_u = \frac{R_f k_u}{R^{uu} + R_f} \quad (33)$$

which have the meaning of stiffnesses and unit stress intensity factors of the system of the structure and the loading frame combined. The stability limit is reached when $\delta^2\Pi = 0$ for some variation $\delta w(x)$ and δa. This means that

$$\delta(\delta_w \Pi) = \int_{a_0}^{a} \frac{df[w(x)]}{dw}[\delta w(x)]^2 dx + \int_{a_0}^{a}\int_{a_0}^{a} \bar{R}^{ww}(x, x')\delta w(x)\delta w(x')dx'dx + \frac{\bar{K}_u + \bar{K}_w}{E'}\int_{a_0}^{a} \bar{k}_w(x)\delta w(x)dx\delta a = 0 \quad (34a)$$

$$\delta(\delta_a \Pi) = \frac{\bar{K}_u + \bar{K}_w}{E'}\int_{a_0}^{a} \bar{k}_w(x)\delta w(x)dx\delta a + \frac{\bar{K}_u + \bar{K}_w}{E'}\frac{\partial(\bar{K}_u + \bar{K}_w)}{\partial a}(\delta a)^2 = 0. \quad (34b)$$

We find that Eq. (34b) is automatically satisfied because $\bar{K}_u + \bar{K}_w = 0$. Furthermore, denoting $\mu(x) = \delta w(x)/\delta a$, we get from Eq. (34$a$) the condition for the loss of stability under displacement control occurs:

$$-\frac{df[w(x)]}{dw}\mu(x) = \int_{a_0}^{a} \bar{R}^{ww}(x, x')\mu(x')dx'. \quad (35)$$

Similar to Eq. (18), this is again a homogeneous Fredholm integral equation for function $\mu(x)$. It is linear if function $f(w)$ is linear.

The special case for $R_f \to \infty$ (with $u = u_s$) represents direct displacement control of the structure (no flexible loading frame). In that case Eq. (35) decides the maximum deflection of the structure, i.e., the onset of snapback instability.

The special case $R_f \to 0$ (with $u \to \infty$) represents loading of the structure under load control. Of course, the case of load control can also be obtained directly. To that end, one must set $R_f = 0$, $u_s = u$ and add to Eq. (22) the term $\Pi_l = -Pu$ representing the potential energy of dead load.

9 Conclusion

Under the assumption of no unloading anywhere within the crack, the cohesive crack model can be formulated in terms of minimization of either the complementary energy or the potential energy of the system. Using the relation of the compliance or stiffness derivatives with respect to the crack length to the unit stress intensity factors, the minimum condition yields the usual compatibility or equilibrium relations for the opening displacements and the cohesive (crack-bridging) stresses in the cohesive crack, and the condition of zero stress intensity factor at the crack tip. The energy formulation also provides the conditions for the loss of stability of a structure with a growing cohesive crack. They have the form of a homogeneous Fredholm integral equation for the derivative of the cohesive stresses or crack opening displacements with respect to the crack length. The variational equations obtained can be used for formulation or approximate solutions.

Acknowledgment

Partial financial support under ONR grant N0014-91-J-1109 to Northwestern University, dealing with applications to sea ice, is gratefully acknowledged. Further support for applications to concrete has been obtained from the Center of Advanced Cement Based Materials at Northwestern University.

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