Stability of Cohesive Crack Model: Part II—Eigenvalue Analysis of Size Effect on Strength and Ductility of Structures

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The preceding paper is extended to the analysis of size effect on strength and ductility of structures. For the case of geometrically similar structures of different sizes, the criterion of stability limit is transformed to an eigenvalue problem for a homogeneous Fredholm integral equation, with the structure size as the eigenvalue. Under the assumption of a linear softening stress-displacement relation for the cohesive crack, the eigenvalue problem is linear. The maximum load of structure under load control, as well as the maximum deflection under displacement control (which characterizes ductility of the structure), can be solved explicitly in terms of the eigenfunction of the aforementioned integral equation.

1 Introduction

As explained in the preceding paper (Bažant and Li, 1995), the cohesive crack model is a nonlinear theory of fracture mechanics in which the condition of stability limit is expressed in terms of the singularity condition of the second variation of the energy potential with respect to cohesive stresses or crack-opening displacements. Although the criterion of stability limit can also be formulated in terms of energy variation with respect to the crack length, the resulting equation is not very useful, since the energy release rate in the cohesive crack model depends on the cohesive stresses or crack-opening displacements.

For a given structure, the criterion of stability limit leads to a highly nonlinear equation for crack length. However, when a class of geometrically similar structures of different sizes is considered and the relative crack length is given, the criterion of stability limit can be treated as an equation for the structure size at which the stability limit occurs at the given relative crack length. In this manner, the criterion of the stability limit is transformed into an eigenvalue problem, with the structure size as the eigenvalue. In the special case of linear softening, the eigenvalue problem is linear. It can be solved independently of the solution of the cohesive crack model. Furthermore, the corresponding maximum value of the load or loading parameter can be expressed explicitly in terms of the eigenfunction. In this way, the size effect curve can be obtained readily, without having to calculate the load-deflection curves for structures of various sizes.

The eigenvalue problem of the cohesive crack model was studied by Li and Hong (1992), Li and Liang (1993) and Li and Bažant (1993). However, only the peak-load solution was discussed in these previous papers. In the present paper, the influence functions are used to formulate the condition of stability limit of a structure with a cohesive crack in the form of a homogenous Fredholm integral equation. The peak load, as well as the maximum displacement (which corresponds to snap-back instability), is obtained. In addition, the cases of a structure loaded through a spring coupled in series (i.e., the case of a soft loading device) and a structure restrained by a spring coupled in parallel are analyzed. Finally, some computational techniques are discussed and a numerical example of the size effect curves for maximum deflection is given.

2 Dimensionless Process Zone Equations

We consider a two-dimensional structure of a unit thickness and introduce the following dimensionless variables:

$$\bar{C} = CE, \quad \alpha = \frac{a}{D}, \quad \xi = \frac{x}{D}, \quad \bar{\sigma} = \frac{\sigma}{f_t},$$

$$\bar{w} = \frac{w}{w_c}, \quad \bar{P} = \frac{P}{Df_t}, \quad \bar{D} = \frac{D}{2L_0} \quad (1)$$

where $L_0 = EG_f/f_t^2$ = characteristic size of the process zone, f_t = direct tensile strength of the material, and w_c = threshold value of the crack-opening displacement. All the notations from the preceding paper (Bažant and Li, 1994) are retained. To simplify notations in the following text, we will drop the bars, with the understanding that all the variables are dimensionless unless specified otherwise.

For a generic elastic structure, the crack-opening displacement w, the load-line displacement u, the load P, and the crack-bridging stress σ must satisfy the compatibility equations:

$$\frac{w(\xi)}{D} = -\int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi, \xi')\sigma(\xi')d\xi' + C^{\sigma P}(\xi)P \quad (2)$$

$$\frac{u}{D} = -\int_{\alpha_0}^{\alpha} C^{P\sigma}(\xi')\sigma(\xi')d\xi' + C^{PP}P \quad (3)$$

which represent the special case of Eqs. (11) and (10) or (14) of the preceding paper for $C_f = 0$; $C^{\sigma\sigma}(\xi, \xi')$, $C^{\sigma P}(\xi)$, $C^{P\sigma}(\xi)$, C^{PP} are dimensionless compliance influence functions (Green's functions). The lower integration limit α_0 is the relative length

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of the initial traction-free crack (notch); α is the total relative crack length which includes both the process zone (crack-bridging zone) and the stress-free crack. The problem can also be formulated as equilibrium conditions written in terms of stiffness influence functions:

$$D\sigma(\xi) = \int_{\alpha_0}^{\alpha} R^{ww}(\xi, \xi')w(\xi')d\xi' + R^{wu}(\xi)u \quad (4)$$

$$DP = \int_{\alpha_0}^{\alpha} R^{uw}(\xi)w(\xi)d\xi + R^{uu}u. \quad (5)$$

These equations represent the special case of Eqs. (26) and (25). Equation (4) for prescribed load P ensues by solving u from Eq. (5) and substituting it into Eq. (4). The dimensionless stiffness functions are here defined with a unit value of Young's modulus.

In the cohesive crack model, the cohesive stress σ is related to the crack-opening displacement w by the stress-displacement relation, which can be described by either of the following forms

$$w = g(\sigma), \quad \sigma = f(w). \quad (6a, b)$$

Substituting (6a) into (2), we obtain what we call the crack compatibility equation in terms of compliance functions:

$$\frac{1}{D} g[\sigma(\xi)] = -\int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi, \xi')\sigma(\xi')d\xi' + C^{\sigma P}(\xi)P \quad (7)$$

Substituting (6b) into (4), we obtain the crack equilibrium equation in terms of stiffness functions:

$$Df[w(\xi)] = -\int_{\alpha_0}^{\alpha} R^{ww}(\xi, \xi')w(\xi')d\xi' + R^{wu}(\xi)u. \quad (8)$$

3 Peak-Load Solution by the Condition of Structural Stability Limit

As established in Bažant and Li (1995), the singularity condition for the compliance formulation under load control can be expressed as the condition of finding a nonzero solution $v(\xi)$ of the following homogenous equation:

$$D\int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi', \xi)v(\xi')d\xi' = -\frac{dg[\sigma(\xi)]}{d\sigma} v(\xi). \quad (9)$$

Since we are considering geometrically similar structures only, (9) can be regarded as an eigenvalue problem if the relative crack length α is given. The dimensionless quantity D plays the role of an eigenvalue. In the actual calculation, the singularity condition should be solved simultaneously with the basic equations to obtain the nominal strength as the maximum load parameter and the corresponding size for a given relative crack length. Calculation of size effect curves in this manner is very efficient. A discussion of the discrete form of the present formulation has been given by Li and Bažant (1994).

In the following, we restrict attention to the case of linear softening, which is defined as

$$w = g(\sigma) = 1 - \sigma, \quad \sigma = f(w) = 1 - w. \quad (10)$$

Since for linear softening $dg/d\sigma = -1$, the eigenvalue is no longer coupled with the basic equations of the cohesive crack. The eigenvalue problem can now be written as

$$D\int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi', \xi)v(\xi')d\xi' = v(\xi). \quad (11)$$

If the relative crack length is specified and geometrically similar structures are considered, Eq. (11) represents a linear homogeneous Fredholm integral equation (Tricomi, 1957) for function $v(\xi)$, with size D as the eigenvalue. The size D for which the given α corresponds to the maximum load is the largest eigenvalue of (11). This approach, proposed by Li and Bažant (1994), makes it possible to avoid solving the load-deflection curves for various sizes D. It represents an efficient method of calculating the size effect curve.

The dimensionless crack compatibility equation can be written as

$$\frac{1 - \sigma(\xi)}{D} = -\int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi, \xi')\sigma(\xi')d\xi' + C^{\sigma P}(\xi)P. \quad (12)$$

Multiplying this with the eigenfunction $v(\xi)$ and then integrating with respect to ξ, we obtain

$$\int_{\alpha_0}^{\alpha} \left[\frac{1}{D} - C^{\sigma P}(\xi)P\right]v(\xi)d\xi = \int_{\alpha_0}^{\alpha} \sigma(\xi) \int_{\alpha_0}^{\alpha} \left[\frac{\delta(\xi - \xi')}{D} - C^{\sigma\sigma}(\xi, \xi')\right]v(\xi')d\xi'd\xi. \quad (13)$$

If the singularity condition is satisfied, then the applied load is at its maximum. This maximum value is found to be

$$P = \frac{1}{D}\frac{\int_{\alpha_0}^{\alpha} v(\xi)d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma P}(\xi)v(\xi)d\xi}. \quad (14)$$

An equivalent expression for the peak load was obtained by Li and Hong (1992), and by Li and Bažant (1994). The eigenvalue problem (11) and the peak load solution (14) provide a powerful set of equations for solving the size-effect curve of the cohesive crack model directly, without any need to solve the load-deflection curve from the basic equations.

The solution can also be generalized to include the case of multiple (conservative) loads. They can vary arbitrarily but in such a manner that there is no crack closure. Then the relation among the load values at the stability limit of the structure is linear. For instance, when a beam is subjected to combined action of lateral load P and axial load N, as shown in Fig. 1, the crack compatibility equation can be written as

$$\frac{1 - \sigma(\xi)}{D} = -\int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi, \xi')\sigma(\xi')d\xi' + C^{\sigma P}(\xi)P + C^{\sigma N}(\xi)N \quad (15)$$

where the symbols are self-explanatory. Since the loading terms do not enter the criterion of stability limit, the equation for the structural stability limit remains the same. If the condition for the stability limit is satisfied, the relation between these two loads is found to be linear:

$$\frac{P}{P^*} + \frac{N}{N^*} = 1 \quad (16)$$

where the denominators, defined as

$$P^* = \frac{1}{D}\frac{\int_{\alpha_0}^{\alpha} v(\xi)d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma P}(\xi)v(\xi)d\xi}$$

$$N^* = \frac{1}{D}\frac{\int_{\alpha_0}^{\alpha} v(\xi)d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma N}(\xi)v(\xi)d\xi}, \quad (17)$$

represent the critical loads when P and N are applied to the structure separately. Equation (16) is the general interaction relation when the structure fails by tensile fracture and the

softening stress-displacement law is linear. A relation of this type was also reported by Li, Müller, and Wörner (1994) in a discrete (matrix) form. Generalization to an arbitrary number of applied loads is self-evident.

When the stress-displacement relation for a cohesive crack is nonlinear, one can use an iterative succession of linear approximations representing tangents of the stress-displacement curve according to the preceding approximation (this approach was formulated for the maximum load in Li and Bažant, 1994).

4 Solution of Maximum Deflection

If the structure is loaded by controlled displacement (i.e., with a rigid grip), the stability limit is reached when there is a snap back in the diagram of load P versus load-line displacement u. The crack equilibrium equation for this case is Eq. (30) of the preceding paper which, in the case of linear softening, yields

$$[1 - w(\xi)]D = -\int_{\alpha_0}^{\alpha} R^{ww}(\xi, \xi')w(\xi')d\xi' + R^{wu}(\xi)u. \quad (18)$$

The dimensionless condition of stability limit may now be written as

$$\frac{1}{D}\int_{\alpha_0}^{\alpha} R^{ww}(\xi, \xi')v(\xi)d\xi = v(\xi'). \quad (19)$$

Since α is constant for geometrically similar structures, (19) is a linear homogeneous Fredholm integral equation for the unknown cohesive stress $v(\xi)$ in the process zone. This represents an eigenvalue problem with $1/D$ as the eigenvalue. Only the smallest eigenvalue $1/D$ represents a stability limit. The maximum deflection, characterizing snap back, is found to be

$$u = \frac{D\int_{\alpha_0}^{\alpha} v(\xi)d\xi}{\int_{\alpha_0}^{\alpha} S^{wu}v(\xi)d\xi}. \quad (20)$$

However, the maximum deflection can also be solved in terms of the compliance functions. To this end, we eliminate the load parameter P from (3) and (7) and obtain the following crack compatibility equation under displacement control:

$$1 - \sigma(\xi) = -D\int_{\alpha_0}^{\alpha} \tilde{C}^{\sigma\sigma}(\xi, \xi')\sigma(\xi')d\xi' + \frac{C^{\sigma P}(\xi)}{C^{PP}}u \quad (21)$$

where

$$\tilde{C}^{\sigma\sigma}(\xi, \xi') = C^{\sigma\sigma}(\xi, \xi') - C^{\sigma P}(\xi)C^{P\sigma}(\xi')\frac{1}{C^{PP}}. \quad (22)$$

The corresponding eigenvalue problem now becomes

$$v(\xi) = D\int_{\alpha_0}^{\alpha} \tilde{C}^{\sigma\sigma}(\xi', \xi)v(\xi')d\xi'. \quad (23)$$

This is equivalent to the eigenvalue problem (19) of stiffness formulation, because the modified compliance function is the inverse of the stiffness function R^{ww}. The maximum deflection can be expressed as

$$u = C^{PP}\frac{\int_{\alpha_0}^{\alpha} v(\xi)d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma P}v(\xi)d\xi}. \quad (24)$$

The compliance formulation is of course equivalent to the stiffness formulation. In a similar way, we can also express the maximum load in terms of the stiffness influence functions. The details will not be given because the derivation is analogous.

5 Stability Limit of Structure Loaded Through a Spring

If the device that controls loading (e.g., the testing machine) has finite compliance C_f, the device can be represented as a spring connected to the structure in series. In such a connection, the device and the structure share the same force. Denote as u the total deflection that is controlled, which is the sum of the deflection u_s of the structure and the deflection of the device $u - u_s$. Using (3), we can solve load P in terms of u as

$$P = (C^{PP} + C^f)^{-1}\left[\frac{u}{D} - \int_{\alpha_0}^{\alpha} C^{P\sigma}(\xi)\sigma(\xi)d\xi\right] \quad (25)$$

In the dimensionless form, the process zone equation is

$$1 - \sigma(\xi) = -D\int_{\alpha_0}^{\alpha} \tilde{C}^{\sigma\sigma}(\xi, \xi')\sigma(\xi')d\xi' + C^{\sigma P}(\xi)(C^{PP} + C_f)^{-1}u \quad (26)$$

where

$$\tilde{C}^{\sigma\sigma}(\xi, \xi') = C^{\sigma\sigma}(\xi, \xi') - C^{\sigma P}(\xi)(C^{PP} + C_f)^{-1}C^{P\sigma}(\xi'). \quad (27)$$

The form of the eigenvalue problem is the same as (23) except that the modified compliance function is defined by (27). The maximum deflection is found to be

$$u = (C^{PP} + C_f)\frac{\int_{\alpha_0}^{\alpha} v(\xi)d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma P}v(\xi)d\xi}. \quad (28)$$

This formula reduces to (24) when compliance C_f approaches zero.

On the other hand, if the spring is connected to the structure in parallel, it shares the same deflection with the structure. Denote by P the total load applied to the structure-spring system, which is the sum of the load P_s which acts on the structure and $S_f u$ where $S_f = 1/C_f$. Using (5) we can express u in terms of P as

$$u = \left[\int_{\alpha_0}^{\alpha} R^{uw}(\xi)w(\xi)d\xi + DP\right](R^{uu} + R_f)^{-1}. \quad (29)$$

Substituting (29) into (8), one obtains the following crack compatibility equation:

$$1 - w(\xi) = -\frac{1}{D}\int_{\alpha_0}^{\alpha} \tilde{R}^{ww}(\xi, \xi')w(\xi')d\xi' + R^{wu}(\xi)P(R^{uu} + R_f)^{-1} \quad (30)$$

where the modified stiffness function is defined as

$$\tilde{R}^{ww}(\xi, \xi') = R^{ww}(\xi, \xi') - R^{wu}(\xi)R^{uw}(\xi')(R^{uu} + R_f)^{-1}. \quad (31)$$

The eigenvalue problem is to find a nonzero eigenfunction satisfying

$$v(\xi) = \frac{1}{D}\int_{\alpha_0}^{\alpha} \tilde{R}^{ww}(\xi, \xi')v(\xi')d\xi'. \quad (32)$$

The maximum total applied load can be calculated from the following equation:

$$P = (R^{uu} + R_f)\frac{\int_{\alpha_0}^{\alpha} v(\xi)d\xi}{\int_{\alpha_0}^{\alpha} R^{wu}v(\xi)d\xi}. \quad (33)$$

Of course when the spring constant of the connected spring

approaches zero, (33) becomes the peak-load solution in the stiffness formulation without a spring.

6 Numerical Implementation

As a numerical example, a three-point bent fracture specimen (Fig. 1) is analyzed. The finite element method is used to obtain the compliance functions in a discretized form (although other methods, such as the boundary element method, might also be suitable). The four-node finite element, which is the simplest, is chosen to discretize one half of the beam. To determine the nodal compliance matrix, the displacement solutions are obtained for one unit load applied successively at each node along the potential crack path or at the load point.

Each column of matrix $C^{\sigma\sigma}$ represents nodal displacements on the crack line when a unit load is applied to one node in the process zone, $C^{\sigma P}$ represents the nodal displacements in the process zone when a unit load is applied at the load point, and C^{PP} represents the load-line displacement when a unit load is applied at the load point. During the calculation, the total relative crack length α is first taken to correspond to the node that is farthest from the crack mouth as allowed by the compliance matrix, and then cracks reaching successively to nodes closer and closer to the crack mouth are considered. In each case the nodal displacements that lie in the uncracked ligament are eliminated. In this way, the dependence of the compliance function on the crack length is reflected by the sizes of the compliance matrices.

Starting with Hillerborg (1976), the zero-K condition has been approximated by the condition that the elastic stress ahead of the cohesive crack tip be equal to the tensile strength. So in our dimensionless definition, $\sigma_{tip} = 1$. In the space of continuous functions, this condition is mathematically equivalent to the condition that the stress intensity factor K at the crack tip be zero (Barenblatt, 1962). After discretization, however, these two conditions are equivalent only approximately. Thus the use of the condition $\sigma_{tip} = 1$ inevitably introduces additional numerical error into the discrete solution. But this small price is quite justifiable, because we do not need the corresponding stress intensity factors, which are not easy to calculate anyway.

However, numerical results (Li and Bažant, 1994) show that, in order to obtain good accuracy for large (dimensionless) structural sizes, it seems important to assume the cohesive stress to vary linearly from node to node in the process zone, rather than a piece-wise constant manner. The assumption of linear variation of cohesive stress between the nodes leads to a tridiagonal matrix connecting the nodal values of cohesive stresses to the cohesive nodal forces (in detail, see Li and Bažant, 1994). Numerically, the differences in the maximum load values calculated by the eigenvalue analysis and by the load-deflection curves are usually in the fifth or sixth digit for linear (or nearly linear) softening laws.

7 Numerical Solution of the Maximum Deflection

Numerical examples for the peak load solution using the eigenvalue approach have been given in previous papers (e.g., Li and Hong, 1992; Li and Bažant, 1994). Therefore, we will discuss only the numerical solution of maximum deflection, which characterizes ductility of a structure. Although the maximum load solution and the maximum deflection solution are

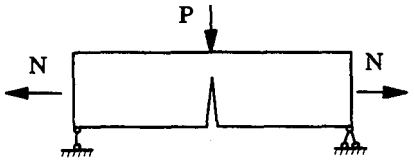

Fig. 1 Beam under combined lateral load and axial load

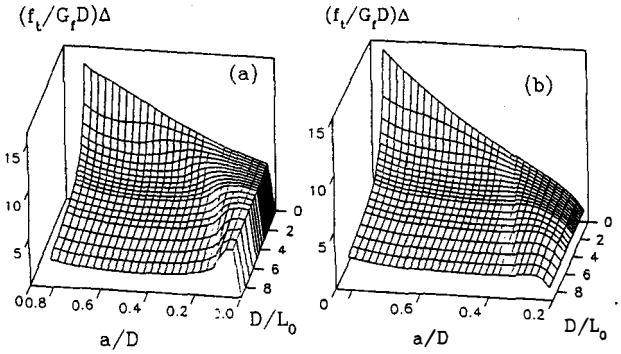

Fig. 2 Dimensionless deflection for (*a*) $\alpha_0 = 0$ and (*b*) $\alpha_0 = 0.2$

mathematically similar, there exists one important difference. For three-point-bent beams, the maximum load always exists, no matter how large the relative process zone length $\alpha - \alpha_0$ is, or how small the dimensionless size D/L_0 is. However, for maximum deflections, the situation is different. As shown in Fig. 2, there is no maximum deflection if the relative length α of the cohesive crack is large enough. The smallest dimensionless size D below which there is no snap back will be called the critical size of the structure. The critical size is a function of relative notch depth α_0 as well as the slenderness ratio (span-to-depth ratio of the beam).

The dependence of the critical size on the relative notch depth can also be seen in Fig. 2. Fig. 2(*a*) gives the deflection for beams without a notch ($\alpha_0 = 0$), and Fig. 2(*b*) for beams with relative notch depth $\alpha_0 = 0.2$. For $\alpha_0 = 0$, the critical size is found to be approximately 0.43 and for $\alpha_0 = 0.2$ approximately 1.4.

According to Eqs. (23) and (24), we can obtain the size effect curves for maximum deflection for any given relative length α of the cohesive crack. Figure 3 shows the size effect curves for different initial notch ratios. Note that, paradoxically, the curves extend even to the left of the critical sizes (dashed lines); these portions of the curves are of course physically meaningless since there exists no maximum deflection at all. The explanation is that these portions correspond to cases with negative σ, whereas our analytical expressions are valid only when the crack-opening displacement in the process zone is less than the crack-opening threshold w_c (at which the stress is reduced to zero). With careful observation, one finds that, when the condition of stability limit is satisfied, the critical size D is actually the size at which the crack-mouth-opening displacement becomes equal to the threshold w_c. Above the critical size (i.e., on right portions of the curves in Fig. 3), the obtained maximum deflections are exactly what one would obtain if the load-deflection curve were solved by the conventional method, that is, by solving the basic equations step by step for each different cohesive crack lengths.

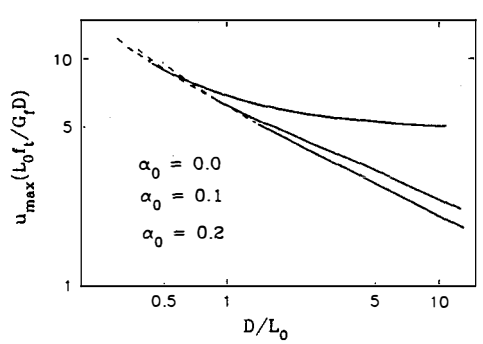

Fig. 3 Size effect curve for maximum deflection

As a check, we select, from the size effect curve, a maximum deflection value with its corresponding relative crack length α and its dimensionless size D. Then we use this dimensionless size as the input and solve the process zone equation together with the crack-tip equation ($\sigma_{tip} = 1$) for different crack lengths. In all the cases examined, the maximum deflection is found to be the same (within the numerical precision of the calculation) and to occur at the same relative crack length.

8 Final Remarks and Conclusions

The cohesive crack model can be effectively analyzed in terms of continuous influence functions. Under the assumption of a linear softening stress-displacement law, the criterion of stability limit, which has been analyzed by Bažant and Li (1995), becomes a linear eigenvalue problem when geometrically similar structures are considered. The peak value of the load parameter can be determined by solving the eigenvalue problem. In this manner, the size effect of the cohesive crack model becomes intimately related to the solutions of the eigenvalue problem. There are some similarities between the eigenvalue problem studied here and the eigenvalue problem for the buckling load of a structure. Both eigenvalue problems are derived from the criterion of structural stability limit. Whereas, in the buckling problem, the eigenvalue is Euler's critical load, in the cohesive crack model the eigenvalue is the structure size for which the loading parameter is maximized at a given relative cohesive crack length. The maximum load or load parameter can be calculated from the eigenfunctions. The following conclusions can be drawn:

1 When geometrically similar structures are considered, the criterion of stability limit becomes an eigenvalue problem. The size for which a given relative crack length corresponds to either the maximum load or the maximum displacement is the first eigenvalue of a homogeneous Fredholm integral equation. The size effect curve can thus be calculated efficiently.

2 If the softening stress-displacement law for the cohesive crack is linear, the eigenvalue problem becomes linearized and can be solved independently. The critical value of the loading parameter (either the maximum load or the maximum load-line deflection), can be determined through the eigenfunction obtained.

3 Numerical examples of the solution of the maximum deflection as a function of the dimensionless beam depth demonstrate that the maximum deflection solution ceases to be valid if the structure dimension (e.g., beam depth) becomes smaller than a certain critical value. This critical value is characterized by the condition that the crack opening at the stability limit reaches the threshold value at which the cohesive stress gets reduced to zero.

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