Easy-to-Compute Tensors With Symmetric Inverse Approximating Hencky Finite Strain and Its Rate[1]

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It is shown that there exist approximations of the Hencky (logarithmic) finite strain tensor of various degrees of accuracy, having the following characteristics: (1) The tensors are close enough to the Hencky strain tensor for most practical purposes and coincide with it up to the quadratic term of the Taylor series expansion; (2) are easy to compute (the spectral representation being unnecessary); and (3) exhibit tension-compression symmetry (i.e., the strain tensor of the inverse transformation is minus the original strain tensor). Furthermore, an additive decomposition of the proposed strain tensor into volumetric and deviatoric (isochoric) parts is given. The deviatoric part depends on the volume change, but this dependence is negligible for materials that are incapable of large volume changes. A general relationship between the rate of the approximate Hencky strain tensor and the deformation rate tensor can be easily established.

Introduction

The Hencky strain tensor **H** (Hencky, 1925, 1928), which is also called the logarithmic strain tensor, the true strain tensor or the natural strain tensor (Nádai, 1937; Davis, 1937), is not the simplest finite strain measure to use. Many investigators nevertheless considered the Hencky strain measure to be attractive (e.g., Hill, 1970; Freed, 1995). Certain advantages, which overcome the shortcomings of the updated Lagrangian descriptions in finite-strain plasticity, have been pointed out by Heiduschke (1995a, b, c, 1996).

The Hencky strain measure has four advantageous properties:

1 The strain tensor for the inverse transformation is symmetric in the sense that it is equal to minus the strain tensor for the original transformation; and in particular the compression and tension are symmetric in the sense that the normal strain corresponding to principal stretch λ is equal to minus the normal strain component corresponding to principal stretch $1/\lambda$.
2 The trace of the strain tensor for isochoric deformations (i.e., deformations at constant volume) vanishes.
3 Subsequent co-axial strains are additive (which means that, after one deformation, the new configuration can be taken as the reference state for computing the additional strain for a further deformation.
4 In consequence of the additivity, the strain tensor can, in particular, be separated into volumetric and isochoric strain tensors that are additive and independent even if both the shear strain and the volume change are large.

The last property is very useful for generalizing to finite strain the existing small-strain complex constitutive laws for pressure-sensitive frictional dilatant materials such as concrete or soil. If the volume change is small, this property can be approximately attained for any finite strain tensor by introducing a certain special definition of the volumetric and deviatoric finite strain tensors (Bažant, 1996). But the error of the approximate additive volumetric-deviatoric split becomes significant if the volume change is large.

Although the Hencky strain tensor is used in some commercial finite element codes, it has, unfortunately, three serious computational disadvantages which have so far prevented widespread practical applications:

1 The conjugate stress tensor is in general very difficult to calculate.
2 The general relationship between the rate of Hencky strain tensor and the deformation rate tensor is very complicated (Hill, 1968; Stören and Rice, 1975; Gurtin and Spear, 1983; Hoger, 1987). A recent claim that a simple relation can be established (Freed, 1995) has turned out to be invalid.[2]
3 The computations of the Hencky strain tensor, which need to be based on the spectral representation (e.g., Malvern 1969; Ogden 1984) and require calculating the principal strains and principal directions, are still quite expensive in very large finite element programs in which the finite strain may have to be computed as many as 10^8 to 10^9 times.

The increments $\Delta\mathbf{H}$, instead of being calculated from the rate of **H**, can of course be calculated directly by taking the difference of two subsequent tensors **H** evaluated by spectral representation. But such an approach poses high demands on computer time. In the case that two or three principal strains are equal, a choice of the principal direction vectors among infinitely many possible such vectors must be made in a manner consistent between two successive states, and this causes further complications.

If the elastic part of strain is small, which is usually true for plastic and brittle materials, the aforementioned disadvantage 1

[1] *Dedicated to Franz Ziegler, Professor at the Technical University Vienna, on the Occasion of his 60th Birthday.*

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[2] Freed presented an elegant and powerful new approach, but his final expression for $\dot{\mathbf{H}}$ in Eq. (31), which is $\mathbf{F}^{-1}\mathbf{d}\,\mathbf{F}$, is not symmetric (the notations are defined later in this paper). This was pointed out and demonstrated by Rice (1996). Another demonstration: $\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{d}\,\mathbf{F}$; so $\mathbf{d} = \mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{T} = \mathbf{F}^{-T}(\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}})\mathbf{F}^{-1/2} = (\mathbf{F}^{-T}\dot{\mathbf{F}}^T + \dot{\mathbf{F}}\mathbf{F}^{-1})/2 = \text{sym}(\dot{\mathbf{F}}\mathbf{F}^{-1})$. Therefore, $\mathbf{F}^{-1}\mathbf{d}\,\mathbf{F} = \mathbf{F}^{-1}\,\mathbf{F}^{-T}\mathbf{X} = \mathbf{G}\mathbf{X}$ where $\mathbf{X} = \mathbf{F}^T\mathbf{d}\,\mathbf{F}$, which is symmetric, and $\mathbf{G} = \mathbf{F}^{-1}\mathbf{F}^{-T} = (\mathbf{F}^T\mathbf{F})^{-1}$, also symmetric. But tensor $\mathbf{G}\mathbf{X}$ is generally not symmetric.

can be circumvented by adopting a nonconjugate strain measure. This of course requires certain caution in order to ensure the nonnegativeness of dissipation (Bažant, 1997). The purpose of this article is to present a new class of easy-to-compute finite strain tensors (recently proposed by Bažant, 1995) which satisfy property 1 exactly and properties 2–4 approximately but closely enough, while at the same time avoiding the aforementioned disadvantages 2 and 3. As for disadvantage 1, the conjugate stress tensor will be much easier to calculate for the proposed class of tensors.

Finite Strain Tensors With Symmetric Inverse

Most of the finite strain tensors practically used in the past belong to the class of Doyle-Ericksen tensors (e.g., Ogden, 1984; Bažant and Cedolin, 1991, Section 11.1) defined as:

$$\text{for} \quad m \neq 0: \quad \mathbf{E}^{(m)} = \frac{1}{m}(\mathbf{U}^m - \mathbf{I}),$$

$$\text{for} \quad m = 0: \quad \mathbf{E}^{(0)} = \mathbf{H} = \ln \mathbf{U} \tag{1}$$

Here m is a real parameter, and $\mathbf{U}$ is the right stretch tensor, defined by the polar decomposition $\mathbf{F} = \mathbf{RU}$ of the deformation gradient $\mathbf{F}$, with $\mathbf{R}$ being the rotation tensor. For $m = 2$ this expression yields the classical Green's Lagrangian finite strain tensor $\mathbf{E}$; for $m = 1$ the Biot strain tensor $\mathbf{e} = \mathbf{U} - \mathbf{I}$ ($\mathbf{I}$ being the unit tensor); and for $m = 0$ the Hencky (logarithmic) strain tensor $\mathbf{H}$. Incremental stability formulations and objective stress rates that are associated with the tensors for $m = -1$ and $m = -2$ have also been used (see Table 11.4.1 in Bažant and Cedolin, 1991). The dependence of $\mathbf{E}^{(m)}$ on parameter m is continuous because

$$\mathbf{H} = \lim_{m \to 0} \frac{1}{m}(\mathbf{U}^m - \mathbf{I}) \tag{2}$$

Let us now replace m by $-m$ in Eq. (1):

$$\mathbf{E}^{(-m)} = \frac{1}{m}(\mathbf{I} - \mathbf{U}^{-m}) \tag{3}$$

Evaluating (1) and (3) for various m, one may note that the deviations from $\ln \mathbf{U}$ are of opposite signs and similar magnitudes. Thus, the average of these two expressions, namely the tensors

$$\mathbf{B}^{(m)} = \frac{1}{2m}(\mathbf{U}^m - \mathbf{U}^{-m}) \tag{4}$$

(Bažant, 1995) should be much closer to $\mathbf{H}$. It is also obvious that

$$\ln \mathbf{U} = \lim_{m \to 0} \frac{1}{2m}(\mathbf{U}^m - \mathbf{U}^{-m}) \tag{5}$$

and the convergence to $\mathbf{H}$ should be much faster.

The replacement of $\mathbf{U}$ with $\mathbf{U}^{-1}$ in (4) merely changes the sign of $\mathbf{B}^{(m)}$, and so the compression-tension symmetry (property 1) is satisfied exactly. So the Hencky strain tensor is not the only one with this advantageous property.

The two simplest special forms of tensor $\mathbf{B}^{(m)}$ are:

$$\text{for} \quad m = 1: \quad \mathbf{B} = \tfrac{1}{2}(\mathbf{U} - \mathbf{U}^{-1}) \tag{6}$$

$$\text{for} \quad m = \tfrac{1}{2}: \quad \bar{\mathbf{B}} = (\mathbf{U}^{1/2} - \mathbf{U}^{-1/2}) \tag{7}$$

Noting the binomial series expansion:

$$\mathbf{U}^m = (\mathbf{I} + \mathbf{e})^m = \mathbf{I} + \binom{m}{1}\mathbf{e} + \binom{m}{2}\mathbf{e}^2 + \binom{m}{3}\mathbf{e}^3 + \ldots \tag{8}$$

we obtain the Taylor series expansion:

$$\mathbf{B}^{(m)} = \frac{1}{2m}\left\{\left[\binom{m}{1} - \binom{-m}{1}\right]\mathbf{e} + \left[\binom{m}{2} - \binom{-m}{2}\right]\mathbf{e}^2 + \left[\binom{m}{3} - \binom{-m}{3}\right]\mathbf{e}^3 + \ldots\right\}$$

$$= \mathbf{e} - \tfrac{1}{2}\mathbf{e}^2 + \frac{m^2+2}{6}\mathbf{e}^3 - \frac{m^2+1}{4}\mathbf{e}^4 + \ldots \tag{9}$$

For $m \to 0$, the expansion of $\mathbf{B}^{(m)}$ coincides with the expansion of $\mathbf{H}$, which reads:

$$\mathbf{H} = \ln \mathbf{U} = \ln(\mathbf{I} + \mathbf{e}) = \mathbf{e} - \tfrac{1}{2}\mathbf{e}^2 + \tfrac{1}{3}\mathbf{e}^3 - \tfrac{1}{4}\mathbf{e}^4 + \ldots \tag{10}$$

For any m, this expansion coincides with the Taylor series expansion (10) of Hencky strain up to the quadratic term. On the other hand, this expansion coincides with the Taylor series expansion of the Doyle-Ericksen strain tensors (1) for $m \neq 0$ only up to the linear term.

For the purpose of the analysis of critical load at the stability limit, only the quadratic term of the Taylor series expansion matters (see, e.g., Chapter 11 in Bažant and Cedolin, 1991). Therefore, the solutions of the critical loads of initially stressed bodies based on finite strain tensors $\mathbf{H}$ and $\mathbf{B}^{(m)}$ will be identical. The same will be true for the associated objective stress rates or increments, and for the associated tangential elastic moduli. However, the postcritical behavior and the stability conditions at the critical state will differ because they depend on the higher-than-quadratic terms of the potential energy expression.

Numerical Comparisons

According to the spectral representation, every finite strain tensor can be expressed (in cartesian components) in the form: $E_{kl} = \sum_{I=1}^{3} f(\lambda_I) n_k^I n_l^I$ where λ_I are the principal stretches (principal values of $\mathbf{U}$), n_k^I ($k = 1, 2, 3$) are the components of the unit vector of principal direction I, and $f(\lambda)$ (for $\lambda > 0$) is a smooth monotonically increasing function such that $f(1) = 0$ and $f'(1) = 1$. Since all the strain tensors are coaxial, the judgment of how close the tensors $\mathbf{B}^{(m)}$ approximate the Hencky tensor $\mathbf{H}$ can be made by comparing the values of $f(\lambda)$ for the maximum principal stretch λ_{max} and the minimum principal stretch λ_{min}.

For various values of $\lambda = \lambda_{max}$ or λ_{min}, Table 1 gives the corresponding principal values H_1, E_1, B_1 and $\bar{B}_1$ of tensors $\mathbf{H}$, $\mathbf{E}$, $\mathbf{B}$ and $\bar{\mathbf{B}}$ (where $H_1 = \ln \lambda$). Table 1 also gives (in percentages) the relative deviations of these values from the maximum or minimum principal Hencky strain, i.e. from $\ln \lambda = \ln \lambda_{max}$ or $\ln \lambda_{min}$, which are defined as

$$\delta_e = \frac{e_1}{\ln \lambda} - 1, \quad \delta_E = \frac{E_1}{\ln \lambda} - 1,$$

$$\delta_B = \frac{B_1}{\ln \lambda} - 1, \quad \delta_{\bar{B}} = \frac{\bar{B}_1}{\ln \lambda} - 1, \tag{11}$$

where $e_1 = \lambda - 1$. For $\mathbf{B}$ ($m = 1$) and principal stretches between $\frac{2}{3}$ and $\frac{3}{2}$, these deviations are seen to be at least an order of magnitude smaller than the deviations of $\mathbf{e}$, and even much smaller than those of $\mathbf{E}$. For $\bar{\mathbf{B}}$ ($m = \frac{1}{2}$), these deviations are at least two orders of magnitude smaller than the deviations of $\mathbf{e}$. The deviations from $\ln \lambda$ within this very broad range are seen to be under 2.8 percent for $\mathbf{B}$ and under 0.7 percent for $\bar{\mathbf{B}}$, which is less than the errors that inevitably arise from imperfect knowledge of the constitutive relation when the strains are calculated from the stresses.

Table 1 Principal strains corresponding to various principal stretch values, and their percentage deviations from the corresponding principal Hencky (logarithmic) strains

λ	H_1	E_1	B_1	$\bar{B}_1$	$\tilde{B}_1$	$\hat{B}_1$	δ_e	δ_E	δ_B	$\delta_{\bar{B}}$	$\delta_{\tilde{B}}$	$\delta_{\hat{B}}$
1.005	.0050	.0050	.0050	.0050	.0050	.0050	.2498%	.5004%	.0004%	.0001%	.0000%	.0000%
1.01	.0100	.0100	.0100	.0100	.0100	.0100	.4992%	1.002%	.0017%	.0004%	.0001%	.0000%
1.03	.0296	.0304	.0296	.0296	.0296	.0296	1.493%	3.015%	.0146%	.0036%	.0012%	.0001%
1.05	.0488	.0512	.0488	.0488	.0488	.0488	2.480%	5.042%	.0397%	.0099%	.0031%	.0002%
1.1	.0953	.1050	.0955	.0953	.0953	.0953	4.921%	10.17%	.1515%	.0379%	.0117%	.0008%
1.3	.2624	.3450	.2654	.2631	.2626	.2624	14.34%	31.50%	1.151%	.2871%	.0763%	.0053%
1.5	.4055	.6250	.4167	.4082	.4060	.4055	23.32%	54.14%	2.763%	.6864%	.1336%	.0096%
2	.6931	1.5000	.7500	.7071	.6924	.6931	44.27%	116.4%	8.202%	2.014%	−.1024%	−.0034%
4	1.386	7.500	1.875	1.500	1.227	1.378	116.4%	441.0%	35.25%	8.202%	−11.46%	−.6164%
8	2.079	31.50	3.938	2.475	.2355	1.998	236.6%	1415.%	89.35%	19.02%	−88.67%	−3.914%
1/1.005	−.0050	−.0050	−.0050	−.0050	−.0050	−.0050	−.2490%	−.4971%	.0004%	.0001%	.0000%	.0000%
1/1.01	−.0100	−.0099	−.0100	−.0100	−.0100	−.0100	−.4959%	−.9885%	.0017%	.0004%	.0001%	.0000%
1/1.03	−.0296	−.0287	−.0296	−.0296	−.0296	−.0296	−1.463%	−2.898%	.0146%	.0036%	.0012%	.0001%
1/1.05	−.0488	−.0465	−.0488	−.0488	−.0488	−.0488	−2.400%	−4.724%	.0397%	.0099%	.0031%	.0002%
1/1.1	−.0953	−.0868	−.0954	−.0953	−.0953	−.0953	−4.618%	−8.953%	.1515%	.0379%	.0117%	.0008%
1/1.3	−.2624	−.2041	−.2654	−.2631	−.2626	−.2624	−12.04%	−22.19%	1.151%	.2871%	.0763%	.0053%
1/1.5	−.4055	−.2778	−.4167	−.4082	−.4060	−.4055	−17.79%	−31.49%	2.763%	.6864%	.1336%	.0096%
1/2	−.6932	−.3750	−.7500	−.7071	−.6924	−.6931	−27.87%	−45.90%	8.20%	2.014%	−.1024%	−.0034%
1/4	−1.386	−.4688	−1.875	−1.500	−1.227	−1.378	−45.90%	−66.19%	35.25%	8.202%	−11.46%	−.6164%
1/8	−2.079	−.4922	−3.938	−2.475	−.2344	−1.998	−57.92%	−76.33%	89.35%	19.02%	−88.67%	−3.914%

Improved Approximation by Linear Combination of $B^{(m)}$

It may be expected that a linear combination of tensors $\mathbf{B}^{(m)}$ for various m values should provide an even better approximation of $\mathbf{H}$. Let us consider the tensors

$$\mathbf{B}^{(m,n)} = k\mathbf{B}^{(m)} + (1-k)\mathbf{B}^{(n)} \quad (12)$$

where k is a constant. These tensors are almost as easy to calculate as $\mathbf{B}^{(m)}$ and $\mathbf{B}^{(n)}$.

At first thought, it might seem that the best linear combination is that which makes the fourth term in the Taylor series expansion the same as for the Hencky tensor, i.e., equal to $-\mathbf{e}^4/4$. This is achieved for $k = (n^2+2)/(n^2-m^2)$. For $m = \frac{1}{2}$ and $n = 1$, this yields $k = 4$ and $1 - k = -3$. However, it turns out that this makes the approximation of $\mathbf{H}$ better only for very small strains ($|e_1| <$ about 1 percent), for which the approximations by $\mathbf{B}$ and $\bar{\mathbf{B}}$ are already extremely close. For larger strains, the approximations become worse.

Therefore, the optimum approximation has been determined numerically from the condition that the magnitude of the maximum percentage deviation of the uniaxial strain from $\ln\lambda$ within the range $0.5 < \lambda < 2$ be minimized. The results are the following two tensors:

$$\tilde{\mathbf{B}} = \mathbf{B}^{(1,2)} = 1.307\mathbf{B}^{(1)} - 0.307\mathbf{B}^{(2)}$$
$$= 0.6535\,(\mathbf{U} - \mathbf{U}^{-1}) + 0.07675(\mathbf{U}^{-2} - \mathbf{U}^2) \quad (13)$$
$$\hat{\mathbf{B}} = \mathbf{B}^{(1/2,1)} = 1.326\mathbf{B}^{(1/2)} - 0.326\mathbf{B}^{(1)}$$
$$= 1.326(\mathbf{U}^{1/2} - \mathbf{U}^{-1/2}) + 0.163(\mathbf{U}^{-1} - \mathbf{U}) \quad (14)$$

in which $\mathbf{B}^{(2)} = \mathbf{E}$ = Green's Lagrangian strain tensor, $\mathbf{B}^{(1)} = \mathbf{e}$ = Biot strain tensor, and $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = \mathbf{C}$ = Cauchy-Green deformation tensor. Tensors $\tilde{\mathbf{B}}$ and $\hat{\mathbf{B}}$ are almost as easy to calculate as $\mathbf{B}^{(1)}$ or $\mathbf{B}^{(1/2)}$, respectively.

Table 1 gives the principal values $\tilde{B}_1$ and $\hat{B}_1$ of the tensors $\tilde{\mathbf{B}}$ and $\hat{\mathbf{B}}$. As we see, $\tilde{\mathbf{B}}$ is better than $\bar{\mathbf{B}}$ approximately for the range $\frac{1}{3} < \lambda < 3$, and $\hat{\mathbf{B}}$ much better than $\bar{\mathbf{B}}$ for the entire range calculated, i.e. $\frac{1}{8} < \lambda < 8$, and its deviation from the Hencky strain does not exceed about 2 percent within this large range. This should suffice for most imaginable practical applications.

However, the tensors $\tilde{\mathbf{B}}$ and $\hat{\mathbf{B}}$ are not monotonic, because of the negative sign in Eq. (13) and (14). Therefore, unlike all the other tensors we have considered, they are not usable as measures of finite deformations for an unbounded range. The tensor is monotonic (and thus usable) if every principal stretch λ lies within the following range:

$$\text{for } \tilde{\mathbf{B}}:\quad 0.2232 < \lambda < 4.4807 \quad (15)$$
$$\text{for } \hat{\mathbf{B}}:\quad 0.05453 < \lambda < 18.340 \quad (16)$$

The practical applicability range is somewhat narrower than this range.

Trace of Strain Tensor at Isochoric Deformations

For constitutive modeling of complex material behavior, it is advantageous if the trace of the finite strain tensor for isochoric deformations (deformations at constant volume) is zero. This property is satisfied only by the Hencky strain tensor, i.e., $H_V = (\text{Tr}\,\mathbf{H})/3 = 0$ (Tr denotes the trace of a tensor, and subscript $_V$ denotes the volumetric component of the tensor).

The symmetric tensors proposed here have the advantage that their trace for isochoric deformations is negligibly small for most practical purposes. To check it consider the following two isochoric right stretch tensors, which represent the extreme cases between which other isochoric deformations lie:

$$\mathbf{U}^b = \begin{bmatrix} \lambda, & 0, & 0 \\ 0, & 1/\lambda, & 0 \\ 0, & 0, & 1 \end{bmatrix} \quad (17)$$

$$\mathbf{U}^t = \begin{bmatrix} \lambda, & 0, & 0 \\ 0, & 1/\sqrt{\lambda}, & 0 \\ 0, & 0, & 1/\sqrt{\lambda} \end{bmatrix} \quad (18)$$

$\mathbf{U}^b$ represents a biaxial isochoric strain, and $\mathbf{U}^t$ a triaxial isochoric strain.

The volumetric components E_V, e_V, B_V, $\bar{B}_V$, $\tilde{B}_V$, $\hat{B}_V$ defined as $\frac{1}{3}$ of the trace of the tensors $\mathbf{E}$, $\mathbf{e}$, $\mathbf{B}$, $\bar{\mathbf{B}}$, $\tilde{\mathbf{B}}$, $\hat{\mathbf{B}}$, are calculated in Tables 2 and 3 for the biaxial and triaxial isochoric strains. To indicate how close the volumetric components are to vanishing, Tables 2 and 3 also give the percentages of these volumetric components compared to the principal Biot strain $e_1 = \lambda - 1$, defined as follows:

$$r_e = \frac{e_V}{e_1} - 1,\quad r_E = \frac{E_V}{e_1} - 1,\quad r_B = \frac{B_V}{e_1} - 1,$$
$$r_{\bar{B}} = \frac{\bar{B}_V}{e_1} - 1,\quad r_{\tilde{B}} = \frac{\tilde{B}_V}{e_1} - 1,\quad r_{\hat{B}} = \frac{\hat{B}_V}{e_1} - 1 \quad (19)$$

Table 2 Volumetric strains corresponding to various biaxial isochoric stretches, and their ratios (in percentages) to the principal Biot strains

λ	E_V	e_V	B_V	$\bar{B}_V$	$\tilde{B}_V$	$\hat{B}_V$	r_E	r_e	r_B	$r_{\bar{B}}$	$r_{\tilde{B}}$	$r_{\hat{B}}$
1.005	.000008	.00002	0	0	0	0	.166%	.337%	0	0	0	0
1.01	.00168	.00170	0	0	0	0	16.7%	16.96%	0	0	0	0
1.03	.00834	.00850	0	0	0	0	27.8%	28.32%	0	0	0	0
1.05	.01501	.01543	0	0	0	0	30.02%	30.86%	0	0	0	0
1.1	.03168	.03335	0	0	0	0	31.67%	33.35%	0	0	0	0
1.3	.09834	.1133	0	0	0	0	32.78%	37.78%	0	0	0	0
1.5	.1650	.2067	0	0	0	0	33.00%	41.34%	0	0	0	0
2	.3317	.4983	0	0	0	0	33.17%	49.83%	0	0	0	0
1/1.005	.000008	.00002	0	0	0	0	.166%	.337%	0	0	0	0
1/1.01	.00168	.00170	0	0	0	0	16.7%	16.96%	0	0	0	0
1/1.03	.00834	.00850	0	0	0	0	27.8%	28.32%	0	0	0	0
1/1.05	.01501	.01543	0	0	0	0	30.02%	30.86%	0	0	0	0
1/1.1	.03168	.03335	0	0	0	0	31.67%	33.35%	0	0	0	0
1/1.3	.09834	.1133	0	0	0	0	32.78%	37.78%	0	0	0	0
1/1.5	.1650	.2067	0	0	0	0	33.00%	41.34%	0	0	0	0
1/2	.3317	.4983	0	0	0	0	33.17%	49.83%	0	0	0	0

For a close approximation of the Hencky strain tensor, these values should be as small as possible. In the range $\frac{1}{2} < \lambda < 2$, B_V at isochoric deformations does not exceed 2.9 percent of the maximum principal stretch λ; $\bar{B}_V$ does not exceed 0.7 percent of λ; $\tilde{B}_V$ does not exceed 0.1 percent of λ; and $\hat{B}_V$ does not exceed 0.05 percent of λ. In the range $\frac{1}{4} < \lambda < 4$, $\hat{B}_V$ does not exceed 0.4 percent of λ. In the range $\frac{1}{8} < \lambda < 8$, $\hat{B}_V$ does not exceed 3 percent of λ.

Efficient Computation of Increments of Proposed Tensors

The use of tensor **B** in large finite element programs requires efficient computation of the right stretch tensor **U** and of its inverse $\mathbf{U}^{-1}$. This can be achieved by calculating first in each load step or time step the increments $\Delta\mathbf{R}$ of the rotation tensor **R** according to the Hughes-Winget (1980) algorithm (used, e.g., in ABAQUS; Hibbitt et al., 1995) or another similar algorithm by Rashid (1993). Then **U** can be effectively evaluated as

$$\mathbf{U} = \mathbf{R}^T\mathbf{F} \qquad (20)$$

where $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ = deformation gradient, and **X** and **x** are the initial and final coordinate vectors of material points. This procedure is computationally much more efficient than calculating $\mathbf{U} = \sqrt{\mathbf{C}}$ as a matrix square root by spectral representation; $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Besides, **R** often needs to be calculated anyway, even if **U** is not needed. The dots in singly contracted tensorial products are omitted in this paper, as in (20).

Since the Hughes-Winget algorithm is only approximate, an error may accumulate after many steps and $\mathbf{U}^2$ might not represent **C** with sufficient accuracy, that is, the difference of the norms $\Delta = |\mathbf{C}| - |\mathbf{U}^2|$ might exceed a certain small tolerance Δ_0. The value of **U** obtained from (20) may then be improved by adding a small correction $\Delta\mathbf{U}$, such that $(\mathbf{U} + \Delta\mathbf{U})^2 = \mathbf{C}$ or $\mathbf{U}^2 + 2\mathbf{U}\Delta\mathbf{U} + (\Delta\mathbf{U})^2 = \mathbf{C}$. The term $(\Delta\mathbf{U})^2$ is second-order small and can be neglected. This yields the correction:

$$\Delta\mathbf{U} = \tfrac{1}{2}\mathbf{U}^{-1}(\mathbf{C} - \mathbf{U}^2) \qquad (21)$$

If the corrected value $\mathbf{U} \leftarrow \mathbf{U} + \Delta\mathbf{U}$ still does not satisfy the given tolerance, one may again substitute this corrected **U** into (21) and calculate a second correction. However, this is usually unnecessary because the convergence is very fast.

An alternative algorithm in which the use of **R** is unnecessary is also possible if the loading steps are very small. The known old value of **U** for the beginning of the loading step can be substituted into (21), along with the current new value of **F**.

Table 3 Volumetric strains corresponding to various biaxial triaxial stretches, and their ratios (in percentages) to the corresponding principal Biot strains

λ	B_E	B_e	B_V	$\bar{B}_V$	$\tilde{B}_V$	$\hat{B}_V$	r_E	r_e	r_B	$r_{\bar{B}}$	$r_{\tilde{B}}$	$r_{\hat{B}}$
1.005	.0000	.0000	.0000	.0000	.000000	.000000	.124%	.249%	.0001%	.00003%	.00001%	.00000%
1.01	.0000	.0000	.0000	.0000	.000000	.000000	.248%	.4967%	.0004%	.00010%	.00003%	.00000%
1.03	.0002	.0004	.0000	.0000	.000000	.000000	.732%	1.471%	.0036%	.00090%	.00028%	.00002%
1.05	.0006	.0012	.0000	.0000	.000000	.000000	1.200%	2.421%	.0097%	.00242%	.00076%	.00005%
1.1	.0023	.0047	.0000	.0000	.000003	.000000	2.308%	4.697%	.0361%	.00902%	.00278%	.00019%
1.3	.0180	.0381	.0008	.0002	.000048	.000003	6.013%	12.69%	.2519%	.06278%	.01589%	.00112%
1.5	.0443	.0972	.0028	.0007	.000114	.000008	8.866%	19.44%	.5612%	.13923%	.02287%	.00166%
2	.1381	.3333	.0143	.0035	−.000500	−.000027	13.81%	33.33%	1.430%	.34951%	−.05004%	−.00265%
4	.6667	2.2500	.1250	.0286	−.052484	−.002832	22.22%	75.00%	4.167%	.95318%	−1.74948%	−.09441%
8	1.9024	10.2083	.4875	.1002	−.596779	−.002612	27.18%	145.83%	6.965%	1.4309%	−8.52541%	−.37314%
1/1.005	.0000	.0000	.0000	.0000	.000000	.000000	−.125%	−.250%	.0001%	.00003%	.0000%	.00000%
1/1.01	.0000	.0000	.0000	.0000	.000000	.000000	−.250%	−.498%	.0004%	.0001%	.0000%	.000000%
1/1.03	.0002	.0004	.0000	.0000	.000000	.000000	−.746%	−1.485%	.0037%	.0009%	.0003%	.000002%
1/1.05	.0006	.0012	.0000	.0000	.000000	.000000	−1.240%	−2.460%	.0102%	.0025%	.0008%	.000056%
1/1.1	.0022	.0044	.0000	.0000	−.000003	.000000	−2.460%	−4.848%	.0397%	.0099%	.0031%	.000213%
1/1.3	.0165	.0320	−.0008	−.0002	−.000048	−.000003	−7.162%	−13.85%	.3275%	.0816%	.0207%	.001452%
1/1.5	.0387	.0741	−.0028	−.0007	−.000114	−.000008	−11.62%	−22.22%	−.8418%	.2089%	.0343%	.002490%
1/2	.1095	.2083	−.0143	−.0035	.000500	.000027	−21.90%	−41.67%	2.860%	.6990%	−.1001%	−.005305%
1/4	.4167	.8438	−.1250	−.0286	.052484	.002832	−55.56%	−112.50%	16.667%	3.8127%	−6.9979%	−.377653%
1/8	.9273	2.1693	−.4875	−1.002	.596779	.026120	−105.98%	−247.92%	55.719%	11.4474%	−68.2032%	−2.985126%

Equation (21) thus yields the first estimate of the increment $\Delta\mathbf{U}$ for the current loading step. The first estimate of $\mathbf{U}$ for the end of the current step is $\mathbf{U} \leftarrow \mathbf{U} + \Delta\mathbf{U}$. To further improve the estimate, the updated $\mathbf{U}$ may be substituted again into (21). Further updates are usually not needed if the loading step is small.

A similar procedure may also be used for efficient computation of $\mathbf{U}^{1/2}$ from $\mathbf{U}$, which is needed for the tensor $\bar{\mathbf{B}} = \mathbf{B}^{(1/2)}$. Let $\mathbf{U}_{old}$ and $\mathbf{U}_{new}$ be the values of $\mathbf{U}$ for the beginning and the end of the current loading step or time step. Let $\mathbf{U}_{old}^{1/2} = \mathbf{A}$, whose value is known. We need to find the increment $\Delta\mathbf{A}$ such that $(\mathbf{A} + \Delta\mathbf{A})^2 = \mathbf{U}_{new}$ or $\mathbf{A}^2 + 2\mathbf{A}\Delta\mathbf{A} + \Delta\mathbf{A}^2 = \mathbf{U}_{new}$. If the step is small, $\Delta\mathbf{A}^2$ may be neglected, and this yields the approximation:

$$\Delta\mathbf{A} = \tfrac{1}{2}\mathbf{A}^{-1}\Delta\mathbf{U}, \quad \Delta\mathbf{U} = \mathbf{U}_{new} - \mathbf{U}_{old} \tag{22}$$

This can be further improved by iterations. To this end, $\mathbf{U}_{old}$ and $\mathbf{U}_{new}$ are redefined as the values for the end of step before and after the current iteration, and $\mathbf{A}$ is the redefined value of $\mathbf{U}_{old}^{1/2}$. Substituting into (22) the updated value $\mathbf{A} \leftarrow \mathbf{A} + \Delta\mathbf{A}$ and using the redefined $\Delta\mathbf{U}$, one can obtain an improved approximation.

Additive Volumetric–Deviatoric Split

Many types of constitutive equations require that the strain be decomposed into its volumetric and deviatoric parts. This decomposition has traditionally been expressed in the multiplicative form $\mathbf{U} = \mathbf{F}_D\mathbf{U}_V$ where $\mathbf{U}_V = J^{1/3}\mathbf{I}$ = isotropic tensor, describing the volumetric deformation, and $\mathbf{F}_D = J^{-1/3}\mathbf{U}$ = strain tensor describing the deviatoric deformation, which is isochoric (causes no change of volume) (Flory, 1961; Sidoroff, 1974; see Bažant, 1996), and $J = \det \mathbf{F} = \det \mathbf{U} = \det \mathbf{U}_V$ = Jacobian of the transformation. However, the available small-strain constitutive models for concrete or soils use an additive volumetric-deviatoric decomposition. They cannot be easily generalized to finite strain using the multiplicative volumetric-deviatoric decomposition.

As recently shown (Bažant, 1996), the finite strain tensors of the Doyle-Ericksen class can be decomposed into volumetric and deviatoric parts also in an additive manner. The volume change vanishes for purely deviatoric deformations and the deviatoric part vanishes for purely isotropic deformations. The additive decomposition was successfully used in a generalization of the microplane model for concrete to moderately large strains (Bažant et al., 1996a, 1996b). It will now be shown that the additive decomposition is also possible for the tensors $\mathbf{B}^{(m)}$ proposed here.

According to the definition of tensor $\mathbf{B}^{(m)}$ in Eq. (4), the volumetric strain is characterized by the following isotropic tensor:

$$\mathbf{B}_V^{(m)} = \frac{1}{2m}(\mathbf{U}_V^m - \mathbf{U}_V^{-m}) = \frac{1}{2m}(J^{m/3} - J^{-m/3})\mathbf{I} \tag{23}$$

Subtracting now this tensor from the total strain tensor $\mathbf{B}^{(m)}$, we obtain the additive deviatoric part:

$$\begin{aligned}\mathbf{B}_D^{(m)} &= \mathbf{B}^{(m)} - \mathbf{B}_V^{(m)}\\ &= \frac{1}{2m}(\mathbf{U}^m - \mathbf{U}^{-m}) - \frac{1}{2m}(J^{m/3} - J^{-m/3})\mathbf{I}\\ &= \frac{1}{2m}(J^{m/3}\mathbf{F}_D^{m/3} - J^{-m/3}\mathbf{F}_D^{-m/3}) - \frac{1}{2m}(J^{m/3} - J^{-m/3})\mathbf{I}\\ &= \frac{1}{2m}[J^{m/3}(\mathbf{F}_D^{m/3} - \mathbf{I}) - J^{-m/3}(\mathbf{F}_D^{-m/3} - \mathbf{I})] \end{aligned}\tag{24}$$

This strain tensor has the property that it vanishes for purely volumetric deformation, for which $\mathbf{F}_D = \mathbf{I}$, and therefore it can be regarded as a measure of the deviatoric (isochoric) deformation. The tensor $\mathbf{B}_V^{(m)}$ vanishes for a purely deviatoric (isochoric) deformation, and therefore it can be regarded as a measure of purely volumetric deformation.

One feature, shared with the additive deviatoric tensor proposed by Bažant (1996), should now be noted. The additive deviatoric tensor is not independent of the volume change, characterized by J, except if $m = 0$ (i.e., in the case of Hencky strain). This would, of course, be an undesirable feature for constitutive modeling. However, for dilatant pressure-sensitive materials such as concrete or soil, this feature has only a negligible effect.

These materials can exhibit only very small volume changes while being capable of very large shear (or isochoric) strain if the hydrostatic pressure is very high. For pressures equal to about 10× the uniaxial compressive strength, recent, yet unpublished test results at Northwestern University show that Portland cement concrete can sustain normal Biot strain of 35 percent while still remaining compact and retaining about $\frac{1}{3}$ of its original uniaxial compressive strength. However, the accompanying volumetric strain is small, of the order of −1 percent. In the compressive uniaxial strain tests of concrete, a pressure as high as −2070 MPa (−300,000 psi) has been achieved, but the corresponding volumetric strain was only about −3 percent and the porosity was reduced to only about $\frac{1}{2}$ (Bažant et al., 1986). In most practical applications, the volume changes of concrete are much less, well below 1 percent in magnitude. So we can reckon that the value of $J^{1/3} - 1$ is practically always less than 0.003. The change of the additive volumetric parts of Green's Lagrangian strain tensor proposed by Bažant (1996) is then less than about 0.6 percent in magnitude, which is quite negligible compared to the uncertainty in the constitutive equation. For the additive volumetric part of the Biot strain tensor proposed by Bažant (1996), the correction is less than 0.3 percent, which is even more negligible.

For the improved linear combination tensors $B^{(n,n)}$ given by Eq. (12), their volumetric and deviatoric parts are likewise defined as:

$$\mathbf{B}_V^{(m,n)} = \frac{k}{2m}(J^{m/3} - J^{-m/3})\mathbf{I} + \frac{1-k}{2n}(J^{n/3} - J^{-n/3})\mathbf{I}.$$

$$\mathbf{B}_D^{(m,n)} = \mathbf{B}^{(m,n)} - \mathbf{B}_V^{(m,n)} \tag{25}$$

For $\mathbf{B}_D^{(m,n)}$, the dependence on J is much weaker than it is for $\mathbf{B}_D^{(m)}$, and is negligible (within the aforementioned ranges of λ) for all practical purposes, even for highly compressible materials.

Rate of Proposed Approximate Hencky Tensor

In contrast to the Hencky strain tensor, a general relationship between the rate of one of its aforementioned approximations and the deformation rate tensor $\mathbf{d}$ (velocity strain) or the rate of the right stretch tensor can be easily established. The increment $\Delta\mathbf{U}$ for a given time interval Δt may be calculated from $\mathbf{d}$ using the Hughes-Winget algorithm, as already explained. Then $\dot{\mathbf{U}} \approx \Delta\mathbf{U}/\Delta t$ where the superior dot denotes the time rate. The rate of one of the approximate Hencky tensors then follows by using the following relations:

$$\frac{d}{dt}(\mathbf{U}^2) = \mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U} \tag{26}$$

$$\frac{d}{dt}(\mathbf{U}^{-1}) = -\mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{U}^{-1} \tag{27}$$

$$\frac{d}{dt}(\mathbf{U}^{-2}) = -\mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{U}^{-2} - \mathbf{U}^{-2}\dot{\mathbf{U}}\mathbf{U}^{-1} = -\mathbf{C}^{-1}\dot{\mathbf{C}}\mathbf{C}^{-1} \tag{28}$$

Note also that $d(\mathbf{U}^2)/dt = \dot{\mathbf{C}} = 2\dot{\mathbf{E}}$ and $\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{d}\,\mathbf{F}$. By inversion, $\mathbf{d} = \mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}$. Consequently,

$$\mathbf{d} = \tfrac{1}{2}\,\mathbf{F}^{-T}(\dot{\mathbf{U}}\mathbf{U} + \mathbf{U}\dot{\mathbf{U}})\mathbf{F}^{-1} \tag{29}$$

So the deformation rate tensor $\mathbf{d}$ corresponding to any given $\dot{\mathbf{U}}$ may be evaluated from (29) and, at the same time, the rates of the approximate Hencky tensors may be evaluated from (26)–(28).

If $\mathbf{d}$ is given, then (29) represents a system of three linear algebraic equations for the components of $\dot{\mathbf{U}}$, which may replace the use of the Hughes-Winget algorithm.

Conclusions

1 There exist approximations of the Hencky (logarithmic) finite strain tensor that

(*a*) exhibit tension compression symmetry (i.e., the strain tensor of the inverse transformation is minus the original strain tensor),
(*b*) coincide with the Hencky strain tensor up to the quadratic term of the Taylor series expansion,
(*c*) are close enough to the Hencky strain tensor for most practical purposes, and
(*d*) are easy to calculate, and in particular the spectral representation of tensor is not needed.

Approximations of various accuracy are given by Eqs. (6), (7), (13), and (14).

2 The more accurate approximations (13) and (14) are not monotonic and thus the ranges of their usability as measures of finite strain are not unbounded. However, the range is sufficiently broad for most practical purposes, especially for tensor (14).

3 A general relationship between the rate of the approximate Hencky strain tensor and the deformation rate tensor can be easily established.

4 The proposed strain tensor can be decomposed into volumetric and deviatoric (isochoric) parts in an additive manner. The deviatoric part depends on the volume change but this dependence is negligible for materials that are incapable of large volume changes.

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