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Energetic-Statistical Size Effect in Quasibrittle Failure at Crack Initiation

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The size effect on the nominal strength of quasibrittle structures failing at crack initiation, and particularly on the modulus of rupture of plain concrete beams, is analyzed. First, an improved deterministic formula is derived from the energy release due to a boundary layer of cracking (initiating fracture process zone) whose thickness is not negligible compared to beam depth. To fit the test data, a rapidly converging iterative nonlinear optimization algorithm is developed. The formula is shown to give an excellent agreement with the existing test data on the size effect on the modulus of rupture of plain concrete beams. The data range, however, is much too limited; it does not cover the extreme sizes encountered in arch dams, foundations, and retaining walls. Therefore, it becomes necessary to extrapolate on the basis of a theory. For extreme sizes, the Weibull type statistical effect of random material strength must be incorporated into the theory. Based on structural analysis with the recently developed statistical nonlocal model, a generalized energetic-statistical size effect formula is developed. The formula represents asymptotic matching between the deterministic-energetic formula, which is approached for small sizes, and the power law size effect of the classical Weibull theory, which is approached for large sizes. In the limit of infinite Weibull modulus, the deterministic-energetic formula is recovered. Data fitting with the new formula reveals that, for concrete and mortar, the Weibull modulus $m \approx 24$ *rather than 12, the value widely accepted so far. This means that, for extreme sizes, the nominal strength (modulus of rupture) decreases, for two-dimensional similarity, as the -1/12 power of the structure size, and for three-dimensional similarity, as the -1/8 power (whereas the -1/4 power has been assumed thus far). The coefficient of variation characterizing the scatter of many test results for one shape and one size is shown not to give the correct value of Weibull modulus because the energetic size effect inevitably intervenes. The results imply that the size effect at fracture initiation must have been a significant contributing factor in many disasters (for example, those of Malpasset Dam, Saint Francis Dam, and Schoharie Creek Bridge.)*

INTRODUCTION

There are basically two simple types of the deterministic-energetic size effect in quasibrittle materials, obeying different laws (Bažant and Chen 1997; Bažant and Planas 1998; Bažant 1997a,b, 1999): 1) the size effect in structures with notches or large cracks formed before the maximum load (Bažant 1984), typical of reinforced concrete structures; and 2) the size effect in structures failing at the initiation of fracture from a smooth surface, typical of the modulus of rupture test (Hillerborg et al. 1976; Bažant and Li 1995). This study is concerned only with the latter, which is important; for example, for safe design of very large unreinforced concrete structures such as arch dams, foundations, and earth-retaining structures.

Prior to the 1990s, it was commonplace in design to assume the maximum load of such structures to be governed by the strength of the material, and sometimes the possibility of a purely statistical, classical size effect of Weibull (1939) was admitted, but no attention was paid to the possibility of a deterministic size effect. More than two decades ago, however, the finite element calculations with the cohesive (or fictitious) crack model by Hillerborg et al. (1976) revealed the necessity of a strong deterministic size effect engendered by stress redistribution within the cross section due to softening inelastic response of the material in a boundary layer of cracking near the tensile face. A detailed finite element analysis of the size effect on the modulus of rupture with the cohesive crack model was presented by Petersson (1981). He numerically demonstrated that the deterministic size effect curve terminates with a horizontal asymptote and also observed that, for very deep beams, for which the deterministic size effect asymptotically disappears, the classical Weibull-type statistical size effect must take over.

As test data accumulated, various empirical formulas were proposed (for example, Rokugo et al. 1995). A simple deterministic formula giving good agreement with test data was theoretically derived in Bažant and Li (1995) and refined in Bažant and Li (1996a). Bažant and Li (1996b) rederived this formula by energy arguments of fracture mechanics that made it possible to capture the structure geometry effect on the coefficients in terms of the energy release function.

Because concrete is a highly random material, the statistical size effect must, of course, get manifested in some way. An early study of the stress analysis in presence of random strength was published by Shinozuka (1972). Sophisticated numerical simulations by finite elements, discrete elements, and random lattice models followed (for example, Breysse 1990; Breysse and Fokwa 1992; Breysse et al. 1994; Breysse and Renaudin 1996; and Roelfstra et al. 1985). These simulations usually assumed random strength following the normal or lognormal probability distribution.

Prediction of failure and size effect, however, calls for extreme value statistics using the Weibull probability distribution that is the basis of Weibull's classical theory (1939). This theory has been extremely successful for fatigue-embrittled metals, but for quasibrittle materials characterized by significant stress redistribution with the consequent energy release before the maximum load, this theory is inapplicable (Bažant et al. 1991; Bažant and Planas 1998; Planas et al. 1995). A nonlocal generalization, which was originally developed only for specimens with notches or structures with large cracks formed before the maximum load, is required (Bažant and Xi 1991; Bažant and Planas 1998).

A recent study of Bažant and Novák (2000a,b) resulted in a statistical structural analysis model that takes into account the postpeak strain softening of the material and calculates the failure probability from the redistributed stress field using the nonlocal Weibull approach of Bažant and Xi (1991), representing an extension of deterministic nonlocal damage theory (Pijaudier and Bažant 1987; Bažant and Planas 1998). They

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demonstrated a good agreement with the existing test results on the modulus of rupture of concrete. Their model, however, is numerical and not reducible to a simple formula for the size effect on modulus of rupture incorporating both the deterministic-energetic and the statistical causes. Development and verification of such a formula is the principal objective of this study.

ENERGETIC SIZE EFFECT DUE TO LARGE FRACTURE PROCESS ZONE

The modulus of rupture of plain concrete beams of a rectangular cross section is defined as

$$f_r = \frac{6M_u}{bD^2} \tag{1}$$

where M_u = maximum (ultimate) bending moment; D = characteristic size of the structure, chosen to coincide with the beam depth; and b = beam width. f_r would represent the value of the actual maximum stress in the beam if the beam was elastic up to the maximum load. The beam is not elastic, however, and thus, f_r represents merely the nominal strength, $f_r = \sigma_N$, which is a parameter of the maximum load having the dimension of strength.

A fracture process zone, represented by a boundary layer of distributed cracking that has a certain non-negligible thickness l_f, may be assumed to develop at the tensile face of beam before the maximum load is attained. Under this assumption, and assuming further the cross sections to remain plane and the postpeak softening stress-strain diagram of a characteristic volume of the material to be linear, Bažant and Li (1995) calculated the stress redistribution in the cross section caused by this boundary layer. This led to the following approximate formula

$$\frac{f_r}{f_t'} = 1 + 2\frac{l_f}{D} \tag{2}$$

where D = beam depth; and f_t' = standard direct tensile strength, assumed to coincide with the modulus of rupture of very deep beams.

A more general and fundamental derivation of (2), which automatically gives also the structure geometry (shape) effect, can alternatively be given on the basis of energetic aspects of fracture mechanics. Using the approach of equivalent linear elastic fracture mechanics (LEFM), one can approximate a cracked structure with a large fracture process zone by a structure with a longer sharp crack whose tip is placed approximately in the middle of the fracture pressure zone (the exact location being determined by the condition of compliance equivalence).

At first, one might think that fracture mechanics cannot be applied when the actual crack length $a_0 = 0$. It can be applied, however, because the equivalent LEFM crack length $a = a_0 + c_f$, having its tip in the middle of the fracture process zone (boundary layer of cracking), is nonzero. Notations: a_0 = notch length or traction-free crack length (here, $a_0 = 0$); and c_f = effective length of fracture process zone (roughly 1/2 of the actual length).

As shown previously (in detail, Bažant 1997a; Bažant and Planas 1998), equivalent LEFM, in general, yields for the nominal strength σ_N of the structure the general expression

$$\sigma_N = \sqrt{\frac{EG_f}{Dg(\alpha_0 + c_f/D)}}, \quad \alpha_0 = \frac{a_0}{D} \tag{3}$$

in which E = Young's modulus; G_f = fracture energy of the material; D = structure size (characteristic dimension); and g = nondimensionalized energy release function characterizing the structure geometry (shape). The function g should be sufficiently smooth to allow expansion into a Taylor series in terms of c_f/D, which represents an asymptotic expansion

$$\sigma_N = \sqrt{\frac{EG_f}{D\left[g(\alpha_0) + g'(\alpha_0)(c_f/D) + \frac{1}{2!}g''(\alpha_0)(c_f/D)^2 + \ldots\right]}} \tag{4}$$

It is important to realize that Eq. (4) describes not only the size effect, but also the shape effect. The shape effect is embedded in the LEFM function $g(\alpha)$; $g(\alpha) = [k(\alpha)]^2$ where $k(\alpha)$ is the dimensionless stress intensity factor that is available for many situations in handbooks (Tada et al. 1985; Murakami 1987) and textbooks (Bažant and Planas 1998), and can be easily obtained by linear elastic finite element analysis.

For failures at crack initiation, as is the case for the modulus of rupture test, $\alpha_0 = 0$. Because the energy release rate for a zero crack length is zero, that is, $g(0) = 0$, the first term of the series expansion in (4) vanishes and the series must be truncated no earlier than after the third, quadratic term. This yields the asymptotic expansion

in which the nominal strength σ_N is now represented by the

$$\sigma_N = f_r = \lim_{\alpha_0 \to 0} \sqrt{\frac{EG_f}{Dg(\alpha_0 + c_f/D)}} \tag{5}$$
$$= \sqrt{\frac{EG_f}{g'(0)c_f + \frac{1}{2!}g''(0)c_f^2 D^{-1} + \frac{1}{3!}g'''(0)c_f^3 D^{-2} + \ldots}}$$
$$= \frac{f_{r,\infty}}{\sqrt{1 - (q_1/D) + (q_2/D)^2 - (q_3/D)^3 + \ldots}}$$

modulus of rupture f_r, and

$$f_{r,\infty} = \sqrt{\frac{EG_f}{c_f g'(0)}}, \quad q_1 = c_f \frac{-g''(0)}{2!g'(0)}, \quad q_2^2 = c_f^2 \frac{g'''(0)}{3!g'(0)}, \quad \ldots \tag{6}$$

The interest herein is not merely in the large-size asymptotic approximation but also in a generally applicable approximate formula of the asymptotic matching type that has admissible behavior also at the opposite infinity (ln $D \to -\infty$, or $D \to 0$) and provides a smooth interpolation between the opposite infinities. The asymptotic behavior of (5) for $D \to 0$ is not acceptable because it yields an imaginary value. To get a proper asymptotic matching formula, (5) must be modified in such a manner that at least the first two terms of the asymptotic expansion of σ_N in terms of $1/D$ remain unchanged. This modification can be accomplished as follows.

Equation (5) may be rewritten as

$$f_r = f_{r,\infty}\left[(1-x)^{-r/2}\right]^{1/r} \tag{7}$$

where r is an arbitrary positive constant (that is related to the third term in the expansion of function $g(c_f/D)$), and

$$x = (q_1/D) - (q_2/D)^2 + (q_3/D)^3 - \ldots \tag{8}$$

Then, according to the binomial series expansion

$$f_r = f_{r,\infty}\left[1 + \binom{-r/2}{1}(-x) + \binom{-r/2}{2}(-x)^2 + \binom{-r/2}{3}(-x)^3 + \ldots\right]^{1/r} \tag{9}$$

$$= f_{r,\infty}\left[1 + \frac{r}{2}x + \frac{r(r+2)}{8}x^2 + \ldots\right]^{1/r} \tag{10}$$

$$= f_{r,\infty}\left[1 + \frac{r}{2}\frac{q_1}{D} + r\left(\frac{r+2}{8}q_1^2 - \frac{1}{2}q_2^2\right)\frac{1}{D^2} + \ldots\right]^{1/r} \tag{11}$$

In contrast to (5), this formula is admissible for $D \to 0$; it gives for f_r a real, rather than imaginary, limit value. The feature that $f_r \to \infty$ is shared by the widely used Petch-Hall formula for the strength of polycrystalline metals. One might prefer a finite limit for f_r, but this does not matter because, in practice, D cannot be less than approximately three maximum aggregate sizes (as the material could no longer be treated as a continuum). The limit $D \to 0$ is an abstract extrapolation.

Keeping only the first two terms, one obtains from (11) the final deterministic-energetic size effect formula

$$f_r = f_{r,\infty}\left(1 + \frac{rD_b}{D}\right)^{1/r} \tag{12}$$

in which D_b has the meaning of the thickness of the boundary layer of cracking

$$D_b = \left\langle \frac{-c_f g''(0)}{4g'(0)} \right\rangle \qquad (D_b > 0) \tag{13}$$

In the last expression, the signs $\langle \ldots \rangle$, denoting the positive part of the argument, have been inserted $[\langle X \rangle = \text{Max}(X, 0)]$. The reason is that $g''(0)/g'(0)$ can sometimes be positive, in which case there is no size effect, and this is automatically achieved by setting $D_b = 0$. In the modulus of rupture test, $g''(0)/g'(0) < 0$ and $D_b > 0$.

Note that for uniform tension (zero stress gradient, as in the direct tensile test), there is no deterministic size effect according to Eq. (13) because $g''(0) = 0$ or $D_b = 0$.

Formula (12) with (13) and (6) for $r = 1$ coincides with Eq. (2), but generalizes it by introducing, through function $g(\alpha)$, the effect of geometry. The special case of the present fracture mechanics derivation for $r = 1$ was presented first at a conference (Bažant 1995) and in more detail in Bažant (1997a). The general form with r was proposed without derivation in Bažant (1999), and the fracture mechanics derivation for $r = 2$ was given in Bažant (1998).

For $r = 1$, (12) yields as a special case formula (4). For $r = 2$,

$$f_r = \sigma_N = \sqrt{A_1 + \frac{A_2}{D}} \tag{14}$$

in which

$$A_1 = f_{r,\infty}^2 = \frac{EG_f}{\left[c_f g'(0)\right]^2} \tag{15}$$

$$A_2 = f_{r,\infty}^2 q_1 = 2f_{r,\infty}D_b = -\frac{EG_f g''(0)}{2c_f\left[g'(0)\right]^3}$$

Formula (14) was proposed and used to describe some size effect data by Carpinteri et al. (1994, 1995). These authors named this formula the multifractal scaling law (MFSL) and tried to justify it by fracture fractality using, however, strictly geometric (non-mechanical) arguments. This name, though, seems questionable because, as shown in Bažant (1997 b, c), the mechanical analysis of fractality leads to a formula different from (14) (this is the case whether one considers the invasive fractality of the crack surface or the lacunar fractality of microcrack distribution in the fracture process zone). No logical mechanical argument for the size effect on σ_N to be a consequence of the fractality of fracture has yet been offered.

EXPERIMENTAL VALIDATION OF ENERGETIC FORMULA

To check the validity of formula (12) and calibrate its coefficients, 10 data sets obtained in eight different laboratories (Lindner and Sprague 1956; Nielsen 1954; Reagel and Willis 1931; Rocco 1995 and 1997; Rokugo et al. 1995; Sabnis and Mirza 1979; Walker and Bloem 1957; Wright 1952) were used. These data, consisting of 42 values summarized in Table 1, represent all the relevant test data on modulus of rupture of plain concrete beams that could be found in the literature. The deterministic energetic character of formula (12) made it possible to adopt a simplified approach in which only the mean value of the measured f_r for each size was considered in checking the formula. This approach helped convergence and stability of the fitting algorithm; it also avoided the need of choosing different weights of data points to take into account different numbers of data points within various sets and different sizes, different numbers of sizes in each set, and different size ranges of various sets. The details of all the experiments were presented in Bažant and Novák's (2000b) study of a nonlocal Weibull theory.

The efficient Levenberg-Marquardt nonlinear optimization algorithm was used with all the strategies of fitting. First, direct fitting of all data provided the values of parameters $f_{r,\infty}$, r, and D_b of formula (12). The merit function to be minimized was considered in the form

$$\Phi = \sum_{i=1}^{N} \sum_{j=1}^{n_i} \left(\frac{f_{r,formula}^{i,j} - f_{r,data}^{i,j}}{\overline{f_{r,i}}}\right)^2 = \text{Min} \tag{16}$$

where N = number of all data sets ($N = 10$); n_i = number of all data points within data set number i; $\overline{f_{r,i}}$ = mean value of all the data points (the mean of means is considered herein for the sake of simplicity) of data set i. The total number of all the data points is $\Sigma_{i=1}^{N} n_i = 42$.

The result of this straightforward fitting is shown in Fig. 1. The optimum values of parameters obtained by this simultaneous fitting of all the data are $f_{r,\infty}$ = 3.27 MPa; r = 1.30; and D_b = 21.57 mm. Note that the optimum value of r differs from the value of 1 that resulted from the simplified analysis of Bažant and Li (1995), but is closer to 1 than to the value of 2 used in Carpinteri's formula (14).

The scatter of the data in Fig. 1, however, is high, with coefficient of variation ω = 0.2, and therefore, the result is

Table 1—Means of modulus of rupture for various test data used in study

Size D, mm	Mean, MPa	Size D, mm	Mean, MPa
Reagel and Willis (1931), 4-point bending		Walker and Bloem (1957), 4-point, d_a = 1 in.	
101.6	5.94	101.6	4.70
152.4	5.74	152.4	4.50
203.2	5.45	203.2	4.25
254	5.26	254	4.27
Wright (1952), 3-point bending		Walker and Bloem (1957), 4-point, d_a = 2 in.	
76.2	4.13	101.6	4.68
101.6	3.82	152.4	4.34
152.4	2.96	203.2	4.15
203.2	2.76	254	3.74
Wright (1952), 4-point bending		Sabnis and Mirza (1979), 4-point bending	
76.2	3.21	10	8.8
101.6	2.94	19.1	6.9
152.4	2.60	38.1	5.6
203.2	2.31	76.2	4.8
—	—	152.4	4.3
Nielsen (1954), 3-point bending		Rokugo (1995), 4-point bending	
100	3.57	50	4.35
150	3.16	100	4.04
200	3.30	200	3.66
—	—	300	3.46
—	—	400	3.30
Lindner and Sprague (1956), 4-point bending		Rocco (1997), 3-point bending	
152.4	4.48	17	7.04
228.6	4.07	37	6.52
304.8	3.93	75	5.60
457.2	3.79	150	5.12
—	—	300	4.67

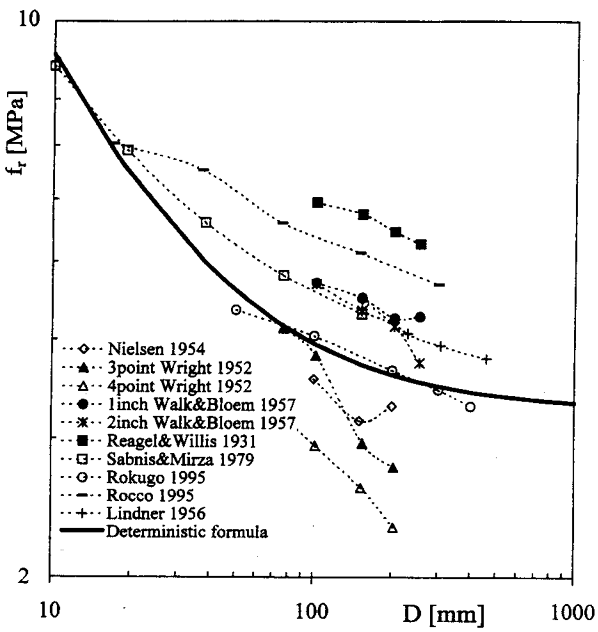

Fig. 1—Optimum fit of existing test data by various investigators on modulus of rupture f_r *versus beam size (depth)* D *by deterministic energetic formula (12).*

unconvincing. It must be realized that the individual test data sets are contaminated by different initial assumptions for size effect testing as well as other uncertainties. Consequently, a more suitable alternative approach to fitting should be adopted. Furthermore, because the scope and range of each individual data set is too limited, the data sets must be combined and analyzed jointly to extract more useful information from the data that exist.

It is reasonable to assume that what varies most from one concrete or one testing approach to another are the values of $f_{r,\infty}$ and D_b, while the exponent should be approximately the same for different concretes and test series. The following improved two-step iterative algorithm for optimizing the fit of the combined data sets, which considerably reduces the scatter by alternating the fitting of individual data sets with the fitting of overall data, has been devised:

- An initial value of r is chosen (typically, $r = 1$);
- Step 1—The individual data sets are fitted separately by Eq. (12) using the same constant parameter r, optimizing only parameters $f_{r,\infty}$ and D_b, allowed to have different values for each data set;
- Step 2—The combined set of all the data is then analyzed in one overall plot (Fig. 2) in which the logarithms of the normalized values, $\log(f_r/f_{r,\infty})$, are plotted versus the values of $\log(D/D_b)$ of each data point. Different normalizing factors, $f_{r,\infty}$ and D_b, as determined in Step 1, are used for the data points from each different set. With the help of the Levenberg-Marquardt algorithm, the fit of these normalized data is then in this plot optimized considering as unknown the overall values of three parameters $f_{r,\infty}$, D_b, and r for one overall size effect curve (Fig. 2). This yields the values of these three parameters, and especially an improved value of r, which is the whole purpose of the second step; and
- Steps 1 and 2 are then iterated always using, in Step 1, the last improved value of r as fixed, and optimizing only the values of $f_{r,\infty}$ and D_b separately for each data set. The iterations are terminated when the change of the r value from one iteration to the next becomes negligible (according to a chosen tolerance).

The iterative algorithm converged rapidly. In the fourth iteration, the change of r from the previous iteration was less than 0.001. The results are shown in Fig. 2 in which the data points of each set are plotted using the values $f_{r,\infty}$ of D_b obtained in Step 1 individually for that data set. The corresponding optimum overall parameter values are $f_{r,\infty}$ = 2.98 MPa, $r = 1.47$, and $D_b = 28.49$ mm. The normalized means of the individual data sets are now very close to the fitted curve. The coefficient of variation of the errors of the formula curve, compared to the data points, is very low; $\omega = 0.0269$.

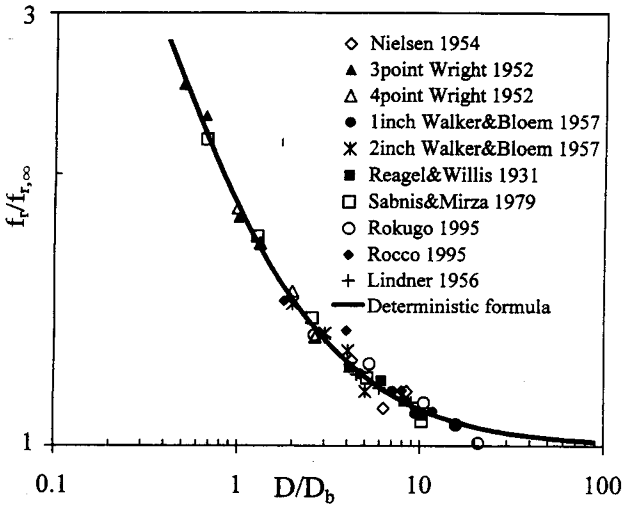

Fig. 2—Optimum fit of existing test data by various investigators on modulus of rupture f_r versus relative size D/D_b by deterministic energetic formula (12).

Figure 3 further shows the plots of each individual data set using the overall optimum exponent $r = 1.47$, but the values of parameters $f_{r,\infty}$ and D_b optimized separately for each data set. It is this figure, rather than Fig. 2, that should be seen as a visual check on the goodness of fit of the present formula. The optimization of the fit in Fig. 2 is necessary to obtain the overall optimum value of r, although visually, this figure conveys an exaggerated impression of the quality of fit.

AMALGAMATION OF ENERGETIC AND STATISTICAL SIZE EFFECTS

The large-size asymptote of the deterministic energetic size effect formula (12) is horizontal; $f_r/f_{r,\infty} = 1$. The same is true of all the existing formulas for the modulus of rupture; refer to, for example, Bažant and Planas (1998). But this is not in agreement with the results of Bažant and Novák's (2000) nonlocal Weibull theory as applied to modulus of rupture in which the large-size asymptote in the logarithmic plot has the slope $-n/m$ corresponding to the power law of the classical Weibull statistical theory (Weibull 1939).

In view of this theoretical evidence, there is a need to amalgamate the energetic and statistical theories, despite the fact that the agreement in Fig. 2 is excellent and looks very convincing. Such amalgamation will be important, for example,

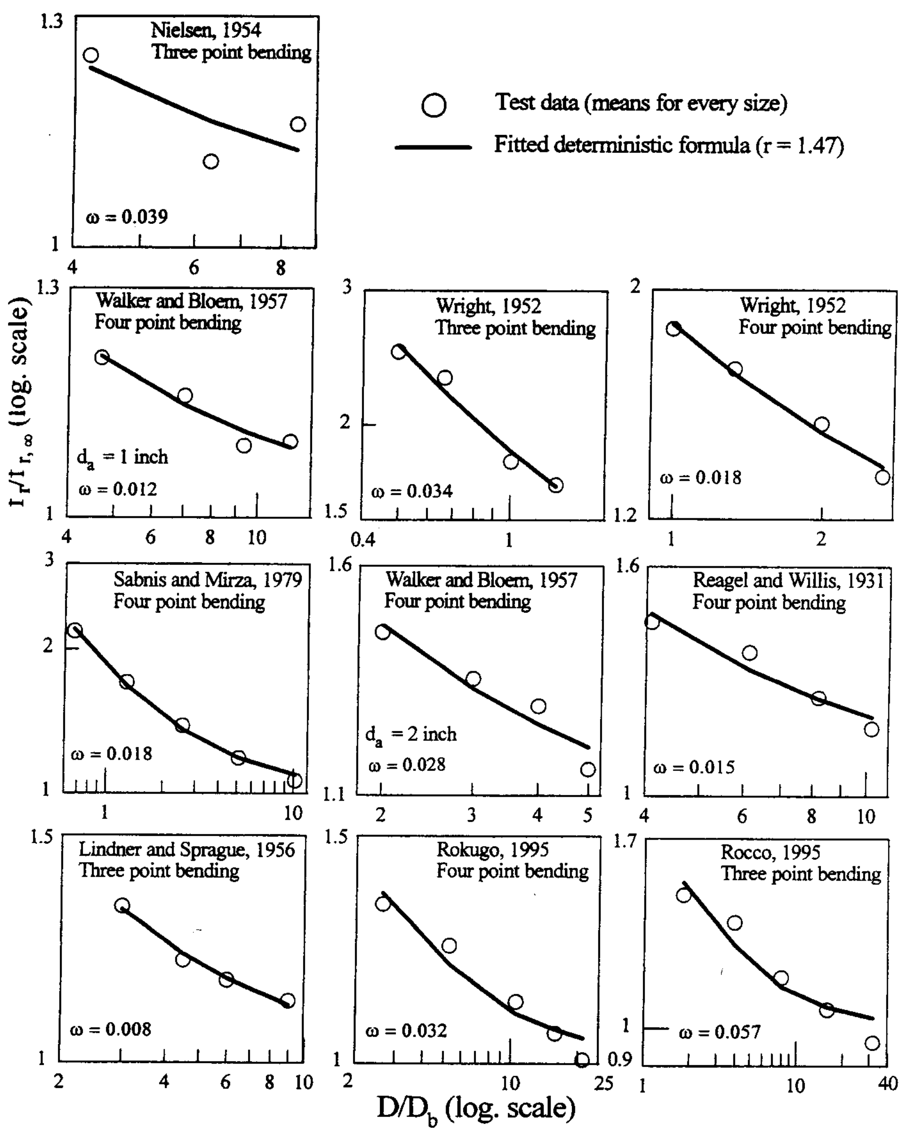

Fig. 3—Optimum fits of individual data sets by deterministic formula (12).

for analyzing the size effect in vertical bending fracture of arch dams, foundation plinths, or retaining walls.

A statistical generalization of formula (12) may be deduced as follows. According to the deterministic-energetic model, $\Delta^r = (f_r/f_{r,\infty})^r - rD_b/D = 1$, which is the value of the large-size horizontal asymptote. From the statistical viewpoint, this difference, characterizing the deviation of the nominal strength from the asymptotic energetic size effect for a relatively small fracture process zone (large D), should conform to the size effect of Weibull theory $D^{-n/m}$ where m = Weibull modulus, and n = number of spatial dimensions (n = 1, 2, or 3; in the present calculations, 2). Therefore, instead of $\Delta = 1$, one should set $\Delta = (D/D_b)^{-n/m}$. This leads to the following Weibull-type statistical generalization of the energetic size effect formula (12)

$$f_r = f_{r,\infty}\left[\left(\frac{D_b}{D}\right)^{rn/m} + \frac{rD_b}{D}\right]^{1/r} \quad (17)$$

or

$$f_r = f_{r,\infty}\left(\frac{D_b}{D}\right)^{n/m}\left[1 + r\left(\frac{D_b}{D}\right)^{1-rn/m}\right]^{1/r} \quad (18)$$

where $f_{r,\infty}$, D_b, and r are positive constants representing the unknown empirical parameters to be determined by experiments. Because in all practical cases, $rn/m < 1$ (in fact, $<< 1$), formula (17) satisfies three asymptotic conditions:

1. For small sizes, $D \rightarrow 0$, it asymptotically approaches the deterministic energetic formula (12)

$$f_r = f_{r,\infty} r^{1/r}\left(\frac{D_b}{D}\right)^{1/r} \propto D^{-1/r} \quad (19)$$

2. For large sizes, $D \rightarrow \infty$, it asymptotically approaches the Weibull size effect

$$f_r = f_{r,\infty}\left(\frac{D_b}{D}\right)^{n/m} \propto D^{-n/m} \quad (20)$$

3. For $m \rightarrow \infty$, the limit of (17) is the deterministic energetic formula (12).

Equation (17) is, in fact, the simplest formula with these three asymptotic properties. It may be regarded as the asymptotic matching of the small-size deterministic and the large-size statistical size effects.

Based on the conclusions of Zech and Wittmann (1977), the value of Weibull modulus was, at first, fixed as $m = 12$, which implies the final asymptote to have the slope $-n/m = -1/6$ (because $n = 2$ for most of the data). The same iterative algorithm of nonlinear optimization, as already described for the deterministic formula, was used, although the convergence was very slow this time. The optimized parameters are $f_{r,\infty} = 3.8$ MPa, $D_b = 8.2$ mm, and $r = 0.9$. The corresponding optimized data fit with the energetic-statistical formula (17) is shown in Fig. 4. The coefficient of variation of errors of this fit is $\omega = 0.0275$, which is low and only slightly higher than before.

Zech and Wittmann (1977), however, based their conclusions on a very limited data set, and therefore, the question of the value of m for concrete has been reopened. These authors obtained the value $m = 12$ in the standard way, which was from the coefficient of variation of strength values measured on specimens of one size and one shape. Numerical calculations, however, show that for $m = 12$, the size effect for larger sizes is unrealistically strong. The value of m is crucial, for it has a large effect on the size effect plot. Taking higher values of m increases the curvature of the logarithmic plot of (17) and decreases the downward slope of the large-size asymptote, which improves the fit of data.

Figure 5 shows the best individual data set available to authors at present, and the optimized size effect curves for different choices of m. It is readily noticed that $m = 12$ is certainly not a good choice for this data set. To get the optimum fit of this individual data set, m needs to be approximately doubled.

The optimum value of m, however, will differ for each individual data set. Therefore, it is appropriate to analyze all the data sets again jointly to determine the optimum common value of m. The same optimization algorithm as already described was used for various chosen m values, particularly m = 12, 16, 20, 25, 30, 40 and ∞. The convergence improved significantly as m was increased and was excellent for $m > 20$. The coefficient of variation ω of the optimized fits is plotted as a function of Weibull modulus m in Fig. 6. The lowest values of ω are between 0.0226 and 0.0230, and occur in the range of $m \in (20, 25)$. The horizontal line in the figure represents the deterministic formula, for which $\omega = 0.0269$.

Even though the changes of the coefficient of variation of errors seen in Fig. 6 are rather small, and the test data sets are contaminated by different uncertainties, a better assessment of Weibull modulus can be made than in previous works. From the joint analysis of all the data sets, and more clearly from the best existing individual data set (namely, that of Rocco (1995)), it transpires that the overall optimum value of the Weibull exponent is approximately

$$m = 24 \quad (21)$$

Accordingly, the Weibull size effect for two-dimensional geometrical similarity is, in the logarithmic plot, a straight line of slope $-n/m = -1/12$ instead of the slope $-1/6$, generally considered in most previous studies and shown in Fig. 4.

In view of this conclusion, the nonlinear iterative optimization of data fits has been repeated using $m = 24$. The result is shown in Fig. 7. The corresponding coefficient of variation is $\omega = 0.023$, and the optimum values of the parameters are $f_{r,\infty} = 3.68$ MPa, $D_b = 15.53$ mm, and $r = 1.14$. The figure shows that the decrease of modulus of rupture with size is, for large sizes, much less than that seen in Fig. 4 for $m = 12$. The individual test data sets, fit with the energetic-statistical size effect formula for $m = 24$, are given in Fig. 8. The fitting of the energetic-statistical formula resulted in a smaller coefficient of variation in most cases, compared to sets with the deterministic size effect in Fig. 3; for seven of those data, the coefficient of variation is less than or equal to the coefficient of variation of deterministic formula, and for three, is greater. In the case of the data spanning a broad range of sizes, namely, those of Rocco (1995), Rokugo (1995), and Sabnis and Mirza (1979), the result of fitting is much better, which is the main evidence that the proposed energetic-statistical formula (17) works well.

CORROBORATION BY NONLOCAL WEIBULL MATERIAL MODEL

The existing experimental results are of a rather limited range and do not include extreme sizes that are of the greatest practical interest. The present theory, for example, has serious

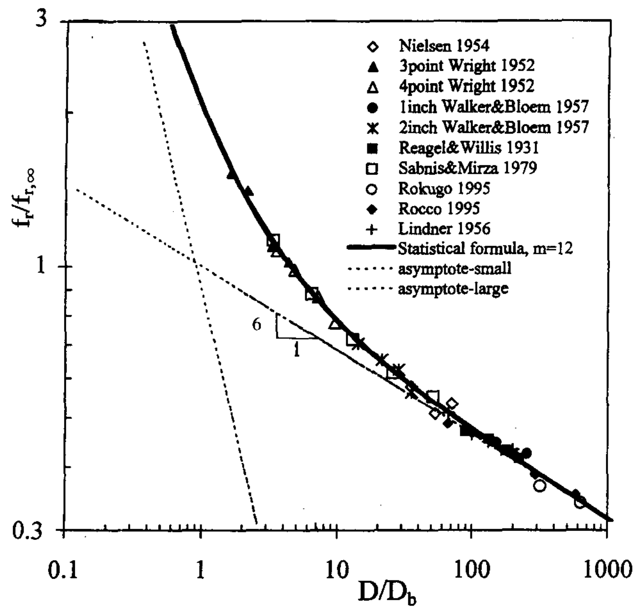

Fig. 4—Optimum fit of existing test data on modulus of rupture f_r *versus relative size* D/D_b *by energetic-statistical formula (17) with* m = *12.*

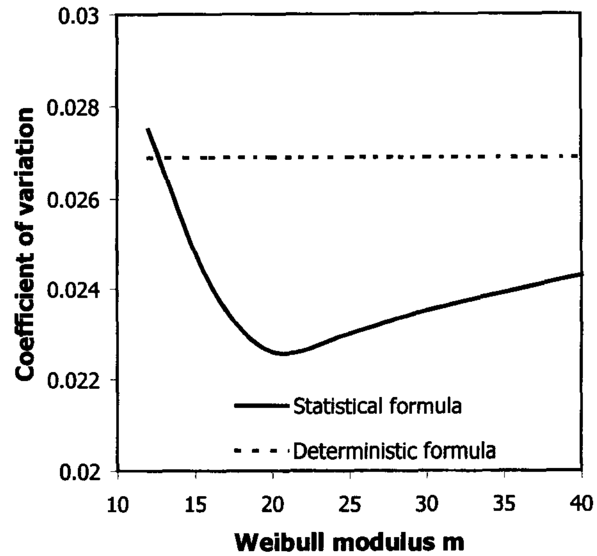

Fig. 6—Coefficient of variation of errors of formula as function of Weibull modulus m.

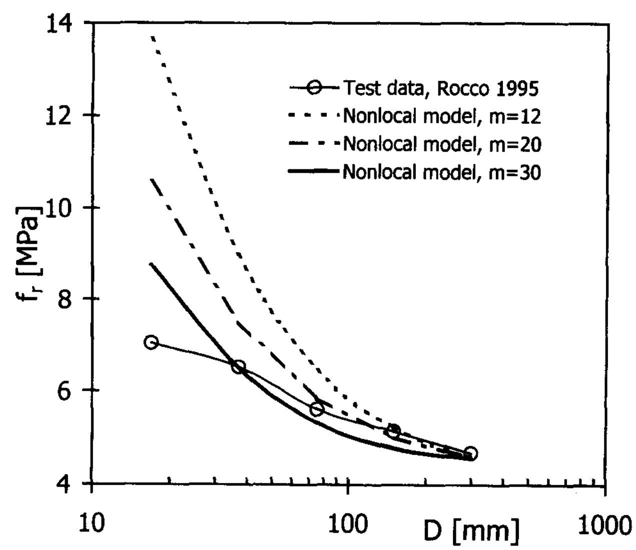

Fig. 5—Influence of Weibull modulus m *on optimum fit, shown for Rocco's (1995) data.*

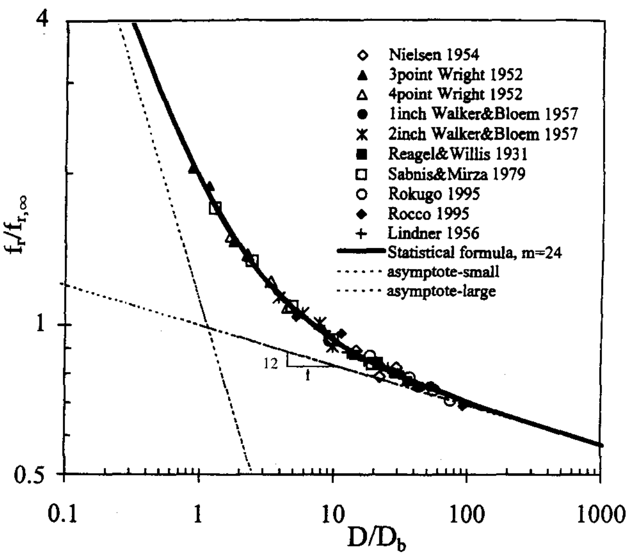

Fig. 7—Optimum fit of existing test data on modulus of rupture f_r *versus relative size of* D/D_b *by energetic-statistical formula (17) with* m = *24.*

implications for the nominal strength in bending failure of an arch dam, but the typical thickness of such dams, which is approximately 5 to 10 m on top, is way beyond the range explored experimentally. Therefore, it is important to check and verify the theory by other means.

For this purpose, structural analysis based on the nonlocal Weibull material model recently developed by Bažant and Novák (2000) is suitable because a good agreement with the existing limited test data, the same data as used herein, has been demonstrated. Numerical solutions of beams according to this model have now been used to obtain both the energetic-deterministic and the statistical asymptotic behaviors of size effect in the modulus of rupture tests of plain concrete beams.

The Weibull integral for probability P_f of structural failure (Bažant and Planas 1998) was reformulated by Bažant and Novák in a nonlocal form. In this reformulation, the local stresses are replaced by the nonlocal (spatially averaged) strains multiplied by the modulus of elasticity, as proposed by Bažant and Xi (1991). Then the multidimensional generalization of the Weibull integral may be written as

$$P_f = 1 - \exp\left\{ -\int_V \sum_{i=1}^{n} \left\langle \frac{\overline{\sigma}_i(x)}{\sigma_0} \right\rangle^m \frac{dV(x)}{V_r} \right\} \tag{22}$$

where n = number of dimensions (1, 2 or 3); σ_0 = Weibull scaling parameter; V_r = representative volume of material (having the dimension of material length); σ_i = principal stresses (i = 1,...n); and an overbar denotes nonlocal averaging. The failure probability now does not depend on local stresses $\sigma_i(x)$, but on the nonlocal stresses $\overline{\sigma}_i(x)$ that are the results of some form of spatial averaging of strains; for details, refer to Bažant and Xi (1991), Bažant and Planas (1998), and Bažant and Novák (2000). In the case of an unreinforced, simply supported, symmetric beam with a symmetric uniaxial stress field treated as two-dimensional, (22) becomes

$$P_f = 1 - \exp\left\{ -\frac{2}{V_r} \int_0^{L/2} \int_{-s}^{D/2} \left[\frac{\bar{\sigma}(x,y)}{\sigma_0} \right]^m dxdy \right\} \tag{23}$$

where L = span of the beam, and s = shift of the neutral axis of beam caused by distributed cracking.

Because the most meaningful numerical simulations are those representing a combination of all the data sets rather than one particular data set, certain average values of material parameters, representing a certain average concrete, need to be adopted. They have been chosen as follows: tensile strength $f_t' = 3.3$ MPa; modulus of elasticity $E = 35$ GPa; postpeak softening modulus $E_t = 25$ GPa coupled with material characteristic length $l = 3d_a$ and maximum aggregate size $d_a = 23$ mm, Weibull scaling parameter $\sigma_0 = 0.9f_t'$ and, as established herein, Weibull modulus $m = 24$.

Three-point symmetric bending of a beam with a span-depth ratio $L/D = 3$ is considered. The modulus of rupture is calculated for beam depths D spanning a very broad size range from $D = 0.01$ m (which is a hypothetical value, smaller than the maximum aggregate size assumed) to $D = 10$ m (which is a size of practical interest for arch dams, and not much larger than the thickness of some massive unreinforced foundation plinths or unreinforced retaining walls). The nonlocal averaging is carried out in the form found by Bažant and Novák (2000) as the most reasonable among several alternatives. It consists in spatial averaging of the inelastic strains over a characteristic neighborhood of the given point. The calculation of the median values (values corresponding to failure probability 0.5) of the nominal strength (modulus of rupture) of the beam has been programmed and then used to fit the present energetic-statistical size effect formula (17). The medians of the modulus of rupture of the beam were obtained by iterating the solutions of the beam so as to obtain failure probability 0.5 with a prescribed accuracy. For details, refer to Bažant and Novák (2000).

The result of nonlinear fitting of formula (17) by the Levenberg-Marquardt algorithm using the nonlocal solutions of failure probability (medians of modulus of rupture) of the beam is presented in Fig. 9. The corresponding parameters are $f_{r,\infty} = 3.76$ MPa, $r = 1.28$, and $D_b = 48.66$ mm. To make the comparison visually clear, the bilogarithmic size effect curves from Fig. 7 and Fig. 9 are plotted together in Fig. 10. As can be seen, both curves are very close.

This favorable comparison supports (but of course does not prove) the correctness of the present energetic-statistical size effect formula (17) as well as the nonlocal Weibull material model of Bažant and Novák (2000).

COMMENTS ON ZECH AND WITTMANN'S ANALYSIS

Zech and Wittmann (1977) used two methods to estimate Weibull modulus m: first, using the measured strength values for one shape and one size, they fit Weibull probability distri-

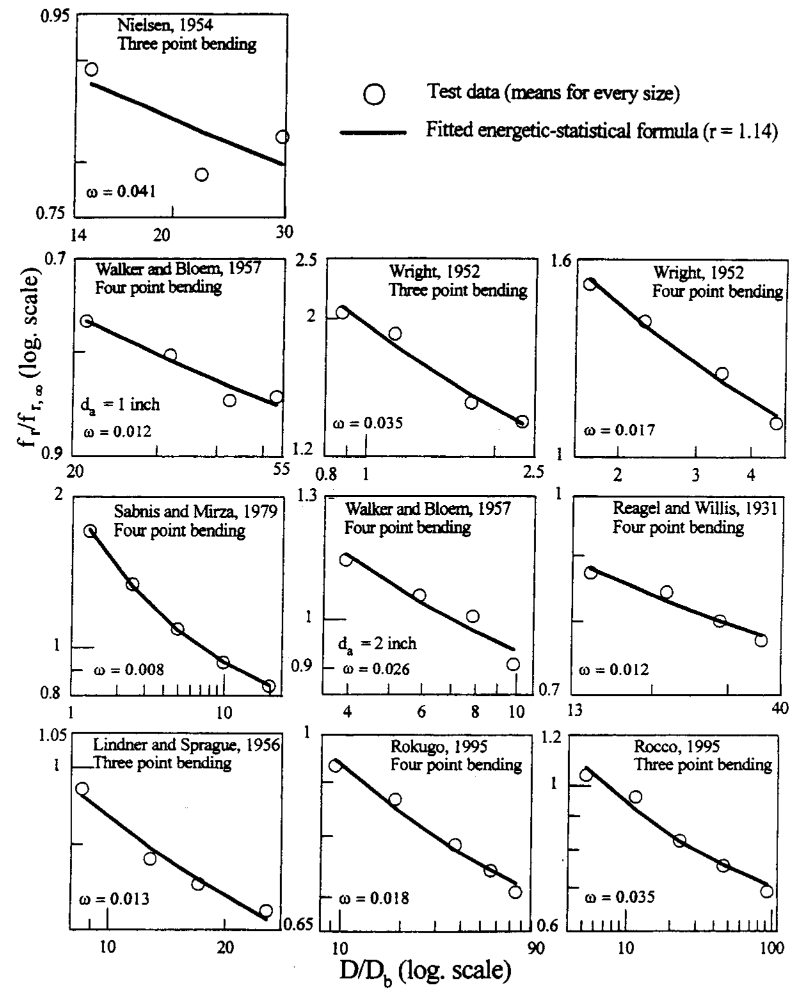

Fig. 8—Optimum fits of individual data sets by energetic-statistical formula (17).

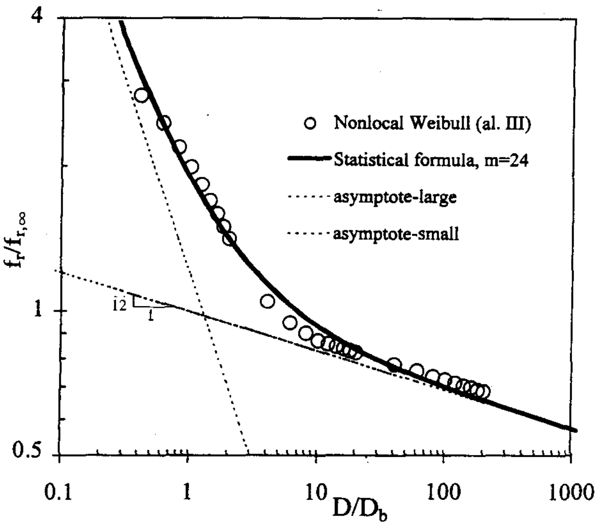

Fig. 9—Optimum fit of the energetic-statistical formula (17) to results of computer simulation with nonlocal Weibull theory.

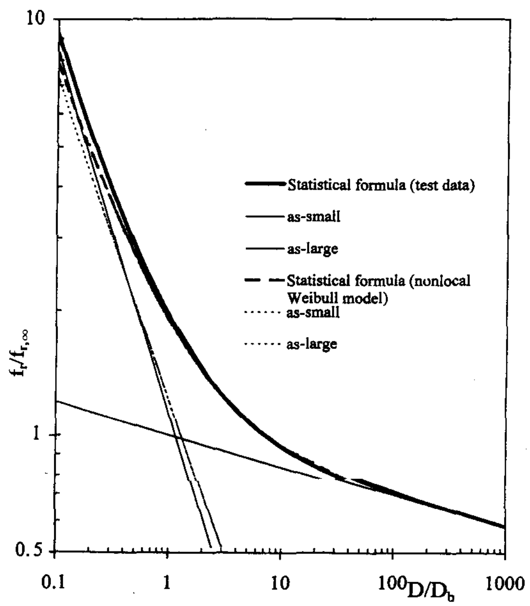

Fig. 10—Comparison of optimum fits of test data by energetic-statistical formula (17) with computer simulation results for nonlocal Weibull theory.

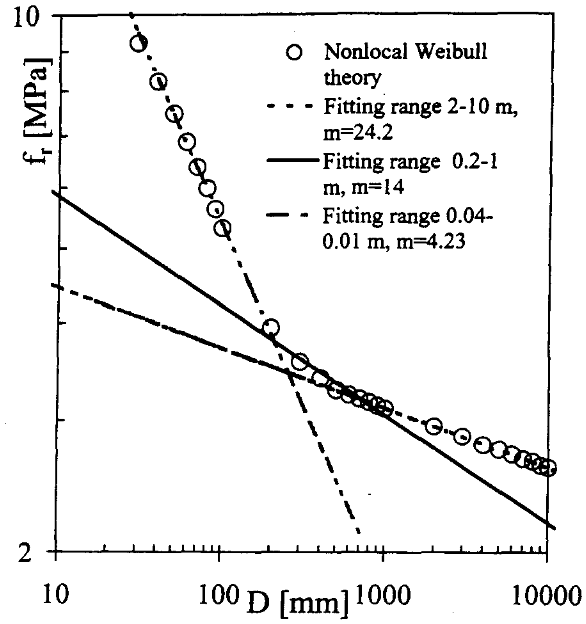

Fig. 11—Incorrect determination of Weibull modulus m *from modulus of rupture data using classical Weibull theory for size ranges in which minimum size is not sufficiently large.*

bution to the data (although they apparently performed no statistical test to check whether the data sample has this distribution). The fitting yielded the value of m as well as the scaling parameter; and second, using the strength values for different sizes, they fit to the data the Weibull size effect in terms of specimen volume V that again yielded the value of m.

When the energetic size effect is also present but is neglected, however, the result of the first method must be very different for different specimen sizes, and the result of the second method must be very different for different limited size ranges. These limitations (which, of course, could not have been understood in 1977) are illustrated in Fig. 11, in which the numerical results of the statistical nonlocal model (Bažant and Novák 2000b), spanning the size range almost 1:1000, are divided into three groups for three different size ranges of breadth approximately 1:10. Method 2 gives m = 24.2, 14.0, and 4.23 for the size ranges 2 to 10 m, 0.2 to 1 m, and 0.04 to 0.10 m, respectively, as is seen from the slopes of the lines in the logarithmic plot of Fig. 11. The middle range is the normal laboratory testing range, and the value m = 14, found to fit the numerical results best, is quite close to Zech and Wittmann's value m = 12.

As already mentioned, the Weibull modulus m has a significant influence on the shape of the size effect curve and the location of its asymptote. To illustrate this, the cases m = 12 and m = 24 may be compared. For small enough sizes, the case m = 12 yields a higher values of modulus of rupture, but the large-size asymptote lies lower. With an increase of m, the overall slope of size effect curve generally decreases.

This behavior can be explained by plotting what is called, in Weibull theory, the concentration function (for example, Bažant and Planas 1998), $c(\sigma) = (\sigma/\sigma_0)^m$ (Fig. 12). Scale parameter σ_0 = 4 MPa is assumed for this plot. Obviously, for m = 12, the concentration function has higher values for $\sigma \le \sigma_0$, and lower values for $\sigma \ge \sigma_0$. Consequently, in the small-size range, higher stresses are achieved, and thus, the effect of nonlocal strain averaging is stronger. Thus, the values of the concentration function are smaller compared to the case m = 24 (that corresponds to a smaller concentration of defects). This naturally leads to higher values of modulus of rupture. For large sizes, the situation is just the opposite. In particular, the influence of nonlocal strain averaging becomes negligible for very large sizes. Higher values of the concentration function for m = 12 lead to a steeper large-size asymptote than for m = 24.

According to (20), the revised value m = 24 (instead of m = 12) means that the large-size asymptotic size effect is, for two-dimensional similarity, $\sigma_N \propto D^{-1/12}$ (rather than $D^{-1/6}$), and for three-dimensional similarity, $\sigma_N \propto D^{-1/8}$ (rather than $D^{-1/4}$).

COMPARISON WITH DETERMINISTIC COHESIVE CRACK MODEL

For the sake of comparison, the maximum loads for beams of various sizes have also been calculated with the cohesive crack model, in a similar way as Petersson (1991). The results, with model parameters set so as to optimize the fit of the pres-

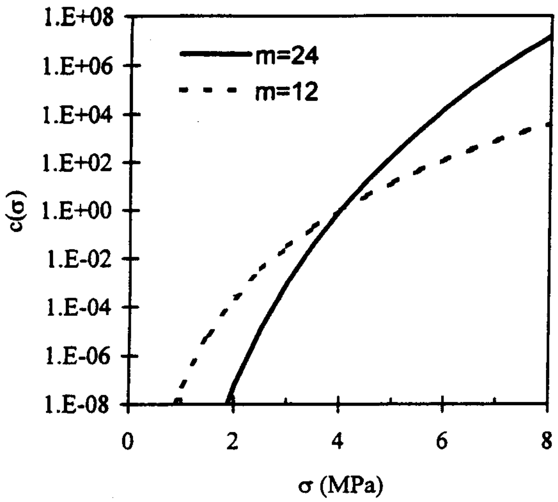

Fig. 12—Defect concentration functions of Weibull theory m = *12 and* m = *24.*

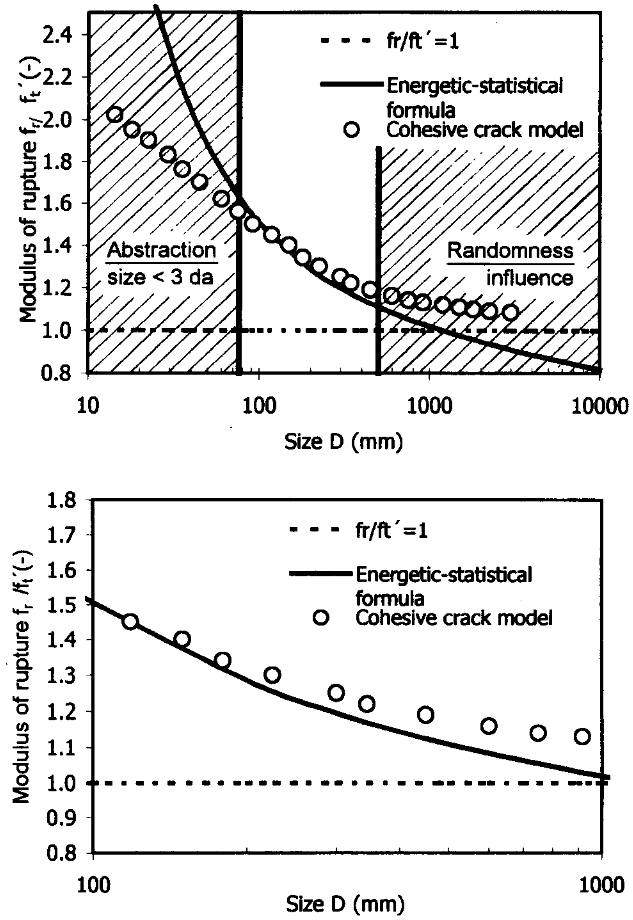

Fig. 13—Results of deterministic cohesive crack model, compared with present energetic-statistical formula: Top—theoretical size range of three decades, cross-hatched outside practical range; and bottom—zoomed practical range.

ent formula in the practical range, are shown in Fig. 13, in which the zone outside the practical size range is cross-hatched. The size effect curve of the cohesive crack model terminates with a horizontal asymptote and has a smaller slope than the present formula. The reason is that the statistical size effect is not captured by the cohesive crack model (capturing would not be easy—one would have to allow the cohesive crack to form at various random locations and, especially for small sizes, allow simultaneous formation of several cohesive cracks until localization instability leads to a single crack).

REINTERPRETATION OF CAUSES OF SOME WELL-KNOWN STRUCTURAL CATASTROPHES

The Malpasset Dam in French Maritime Alps, an arch dam of record-breaking slenderness built in 1954, failed catastrophically on its first complete filling in 1959, causing a flood that wiped out the town of Fréjus founded by the Romans (for example, Levy and Salvadori 1992). Almost 400 lives were lost. The failure, which started from vertical cracks due to flexural action in the horizontal plane, was attributed to the movement of rock in the left abutment, magnified by a thin, clay-filled seam. There can be no disputing that this explanation was correct, but it was incomplete.

From the perspective of this study, the size effect must have been a significant contributing factor. The energetic size effect was unknown in 1959, and the Weibull statistical size effect was not yet established for concrete. Considering that the wall thickness was $D = 7$ m (the minimum thickness of the dam, on its top) and that the tensile strength was estimated from the standard compression strength measured on specimens with $D \approx 15$ cm, and assuming that $r = 1.14$ and $D_b = 10$ cm for dam concrete, today one may conclude from (17) that the nominal tensile strength for flexural analysis of tolerable abutment movement must have been reduced to approximately 45% of the value considered. (If only the energetic size effect were considered, it would be 64%, and if only the statistical one were considered, 73%.) The abutment movement that could have been tolerated to prevent the maximum flexural stress multiplied by safety factor from attaining the tensile strength limit must have been correspondingly smaller than that estimated at that time by the investigating committee, which was unaware of size effect.

Similar observations can be made about Saint-Francis Dam near Los Angeles, an arch-gravity dam that failed in 1928, causing a loss of over 500 lives. The primary cause also was an excessive displacement in the rock abutment (Pattison 1998), but the size effect must have reduced the maximum tolerable displacement to only 40% of the strength theory prediction based on the tensile strength of normal laboratory specimens.

Another example—an exception in the history of structural engineering disasters in the sense that, in this case, the investigating committee did, in fact, recognize the size effect as a significant contributing factor—was the 1987 failure of the Schoharie Creek Bridge on New York Thruway, built in 1952 (Levy and Salvadori 1992). A flood scoured the river bed to a depth of 5.5 m (18 ft) and bared approximately 1/2 of the length of an unreinforced foundation plinth of 6.7 m (22 ft) in depth, forcing it to act as a cantilever. Fracture of the plinth (analyzed by finite elements by Swenson and Ingraffea in 1991) caused the pier to sway, which in turn caused the precast prestressed beams to slip out of their bearings (five cars went down and 10 people drowned). Assuming $D_b = 0.05$ m and $r = 1.14$, today one may conclude from (17) that the nominal bending tensile strength must have been reduced to 54% compared to the standard modulus of rupture f_r measured for $D \approx 0.15$ m. (If only the energetic size effect was considered, it would be 77%, and if only the statistical one, it would be 73%.)

Many other examples exist. Further catastrophes in which the size effect must have been a significant contributing factor could be cited for reinforced concrete structures (for example, Sleipner oil platform 1991; Hanshin viaduct, Kobe 1995; Cypress viaduct, Oakland 1989; bridge columns in Los Angeles earthquake 1994), although it was not concluded in the official evaluations. The size effect in these structures is generally stronger because the load capacity of the structure is reached only after the development of a large crack. It obeys, however, a different law (Bažant 1984; Bažant and Chen 1997; Bažant and Planas 1998) with a negligible statistical contribution to the mean. Analysis of these disasters is planned for a separate paper.

SUMMARY AND CONCLUSIONS

1. A deterministic formula (12) for the size effect on the modulus of rupture in beam bending, or generally the size effect for failures at fracture initiation, has been derived from energy release caused by a large fracture process zone. Its special cases are the formulas previously proposed by Bažant and Li (1995, 1996b) and Carpinteri et al. (1994, 1995) (the latter called by Carpinteri the multifractal scaling law, MFSL).

2. A rapidly converging iterative nonlinear optimization algorithm for fitting the formula to test data has been developed.

3. The new energetic formula gives excellent agreement with the existing test data on the modulus of rupture of beams of various sizes.

4. The range of these data, however, is much too limited. It does not (and hardly ever could) cover the extreme sizes encountered in arch dams, foundations, and earth-retaining structures, for which the size effect is of primary importance. Therefore, extrapolation to such sizes must be based on theory.

5. As confirmed by recent structural analyses based on a new statistical nonlocal material model, the energetic formula is inadequate for extrapolation to very large sizes because it terminates with a horizontal asymptote in the size effect plot. The theory must take into account the Weibull statistical size effect, which causes that the large-size asymptote in the logarithmic size effect plot must he inclined.

6. For extrapolation to very large sizes, a new generalized formula (17) that amalgamates the energetic and statistical size effects for failures at crack initiation is developed. This new formula is of asymptotic matching type. Its asymptotic behaviors for small and large sizes conform to the energetic and statistical theories, respectively, and its limit for an infinite value of Weibull modulus is the energetic formula.

7. The correctness of the new energetic-statistical size effect formula (17) is supported by good agreement with structural analysis according to the recently developed statistical nonlocal material model.

8. Minimization of the coefficient of variation of errors of the energetic-statistical formula compared to the bulk of the existing data indicates that, for concrete, the Weibull modulus $m \approx 24$, rather than $m = 12$, which has so far been generally accepted on the basis of the limited small-size test results of Zech and Wittmann (1977). This means that the size effect on the modulus of rupture at very large sizes is proportional, for two-dimensional similarity, to $D^{-1/12}$ (rather than $D^{-1/6}$), and for three-dimensional similarity, to $D^{-1/8}$ (rather than $D^{-1/4}$).

9. When the size effect has both statistical and energetic sources, which is the case for concrete, Weibull modulus m could be determined in beam tests by the classical method (that is, from the coefficient of variation characterizing the scatter of many test results for one shape and one size) only if extremely large beams (several meters deep), which are not practically feasible, were used. For feasible beam dimensions, the smaller the size, the lower the m value obtained by the classical method. Likewise, estimation of m from size effect tests cannot be based on the classical Weibull formula for size effect unless such enormous beam dimensions could be used. For feasible dimensions of laboratory beams, the Weibull modulus must be determined by fitting the present energetic-statistical size effect law (17) to the measured nominal strength values.

10. The results imply that the size effect at fracture initiation must have been a significant contributing factor in many catastrophic structural failures; for example, those of Malpasset Dam, Saint Francis Dam, and Schoharie Creek Bridge.

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