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Reinterpretation of Karihaloo's size effect analysis for notched quasibrittle structures

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Abstract. Karihaloo recently published an analytical study of the size effect in concrete based on large-size asymptotic approximations of the cohesive crack model. From this analysis, he concluded that the nominal strength can be determined only for sizes above a certain lower bound, large enough to invalidate, at least for concrete, all the existing experimental methods based on size effect measurements, such as the size effect method of Bažant or the general bilinear fit method of Planas, Guinea and Elices. The purpose of this paper is to show that this conclusion is misleading, and to explain why.

Key words: Asymptotic analysis, cohesive crack, size effect, softening curve.

1. Introduction

Karihaloo (1999) recently presented a very interesting higher-order asymptotic analysis of the size effect in notched three-point bend fracture specimens of concrete and other quasibrittle materials, based on the cohesive crack model. His approach is appealing because it is fully analytical, and leads to the correct form for the asymptotic size effect for large sizes.

From this asymptotic expression, Karihaloo draws the conclusion that there is a lower bound of the specimen size below which *it is not possible* to determine the nominal strength (or peak load). This lower bound is so large, at least for concrete, that in practice it would preclude determining fracture parameters based on measurement of the peak load, as used in various experimental methods (Bažant, 1984; Bažant et al., 1986; Bažant and Kazemi, 1990; RILEM, 1990; Guinea et al., 1994; Planas et al., 1999a).

In an effort to carefully examine the basis of this intriguing conclusion, this paper analyzes in some detail the main points of Karihaloo's arguments. The present analysis strictly adhers to the framework used by Karihaloo (1999): a cohesive zone extends in mode I straight ahead of a preexisting crack in an otherwise elastic body. Same as in that paper, the word 'notch' is taken as a synonym for 'preexisting crack'. This means that the effect of a notch having, in practice, a finite notch-tip radius is neglected (which is normally justified for concrete since the notch width is usually much less than the maximum aggregate size as well as the length of the fracture process zone at peak load).

The first part of the present paper (Section 2) reinterprets Karihaloo's conclusion that a lower bound exists for the specimen size below which the peak load cannot be determined. There is no doubt that if the specimen is too small compared to the size and spacing of major inhomogeneities (e.g., aggregates in concrete), the experimental results are no longer representative this is also manifested by the large scatter often observed for for small specimens. However, the present analysis shows that the critical size below which Karihaloo's asymptotic expression yields imaginary values is more a limitation of the simplifications involved in his approach rather than a true physical lower bound. In reality, the specimen must have some positive strength, no matter how small it is. The imaginary values of nominal strength obtained from Karihaloo's expression merely indicate that this expression cannot cover the full range of specimen sizes.

The second part of this paper (Section 3) discusses more technical and quantitative aspects. Although the *form* of the asymptotic size effect formula obtained by Karihaloo seems at first to be correct, a deeper analysis and comparison with the *exact* expressions deduced from the work of Planas and Elices (1991, 1992, 1993) and Smith (1994a, b, 1995) reveals that Karihaloo's result is *not exact quantitatively*, which may be surprising for a fully analytical procedure. A reappraisal of Karihaloo's work shows that, although his final two-term expansion is nearly correct (in the large-size-range), certain important intermediate steps are physically an mathematically unrealistic, owing to some apparently innocuous simplifications. Among the side effects of the simplifications made, it appears that the softening curve (the stress versus the opening of cohesive crack) implied by this approach is size dependent and exhibits physically impossible shapes for small sizes.

The main conclusions of the present paper are stated in the closing Section 4.

2. Reinterpretation of Karihaloo's lower bound for size

2.1. Background

Consider a family of geometrically similar specimens of various sizes D, for example a three-point-bend beam of depth D, thickness B, loading span S proportional to D and initial crack (or notch) of length $a_0 = \alpha_0 D$, with α_0 = notch-to-depth ratio = constant. The expression for the stress intensity factor as a function of the applied nominal stress, size and notch-to-depth ratio can in general be written as

$$K_I = \sigma_N D^{1/2} k(\alpha) \,, \tag{1}$$

where σ_N = nominal stress, which is a load parameter proportional to the load and inversely proportional to the cross-section area. For a three-point-bend specimen, the usual definition is $\sigma_N = 3PS/(2BD^2)$, which reduces to $\sigma_N = 6P/(BD)$ for $S/D = 4$, a typical ratio in fracture mechanics testing. $k(\alpha)$ is a dimensionless shape-dependent function depending only on the crack-to-depth ratio for bodies that have a fixed shape and are characterized by boundary conditions in dimensionless form.

The nominal strength σ_{Nu} of a specimen of size D is equal to the nominal stress at peak load. If the material behaves plastically, then σ_{Nu} = constant and there is no size effect and if linear elastic fracture mechanics (LEFM) is applicable, then, from Equation (1) it follows that $\sigma_{Nu} \propto D^{-1/2}$ (for details, see e.g., Bažant, and Planas, 1998). When nonlinear fracture models apply, then the size effect deviates from the LEFM size effect, but it is generally admitted that for very large specimens, i.e., for $D \to \infty$, the LEFM behavior must be approached because in that limit the fracture process zone is negligibly small compared to the dimensions of the body. More specifically, the so-called nominal stress intensity factor K_{IN}, defined as the stress intensity factor evaluated from Equation (1) for the actual nominal stress and the initial crack or notch length, must be equal to K_{Ic} at the peak load, and so

$$\sigma_{Nu} \to \sigma_{Nu\infty} = \frac{K_{Ic}}{k(\alpha_0)} D^{-1/2} \quad \text{for } D \to \infty . \tag{2}$$

The foregoing equations can be reformulated in terms of the energy release rate G which is related to K_I by Irwin's relationship $G = K_I^2/E'$, where E' is the effective elastic modulus for the type of problem under consideration (plane stress or plane strain). Therefore, Equations (1) and (2) can be recast as

$$G = \frac{\sigma_N^2}{E'} D g(\alpha) \quad \text{with } g(\alpha) = [k(\alpha)]^2 , \tag{3}$$

$$\sigma_{Nu} \to \sigma_{Nu\infty} = \sqrt{\frac{E' G_F}{g(\alpha_0) D}} \quad \text{for } D \to \infty , \tag{4}$$

where $G_F = K_{Ic}^2/E'$ is the specific fracture energy, and the function $g(\alpha)$ is introduced for convenience.

2.2. Karihaloo's size effect equation

Karihaloo assumes the material nonlinearity to be taken into account by the cohesive crack model and handles the corresponding problem by means of an integral equation formulation. The kernels for the integral equations are those corresponding to infinite size (semi-infinite crack in an infinite body). The only deviation from the standard small-scale yielding analysis (i.e., analysis conducted for truly infinite size), is that a first-order approximation is considered for the stress field ahead of an elastic crack in a body of finite size, i.e., the distribution of normal stress σ is approximated as

$$\sigma = \frac{K_I}{\sqrt{2\pi r}} \left[1 + \frac{r}{\beta} + o\left(\frac{r}{\beta}\right) \right] \approx \frac{K_I}{\sqrt{2\pi r}} \left(1 + \frac{r}{\beta} \right) , \tag{5}$$

where K_I = stress intensity factor, r = distance to the crack tip (radial coordinate), β = parameter proportional to the size of the body and depending on the geometry, and $o(x)$ = function that approaches zero faster than x when $x \to 0$ (typically as x^2). This truncation implies the accuracy of the solution (in strictly mathematical terms) to be at best of the first order, i.e., one can retain at most two terms in the final solution.

Karihaloo's main result, his Equation (36), is that the size effect is given by

$$\frac{\sigma_{Nu}}{\sigma_{Nu\infty}} = \left[1 - \frac{g'(\alpha_0)}{g(\alpha_0)} \frac{l_{p\infty}}{D} \right]^{1/2} , \tag{6}$$

where $l_{p\infty}$ = size of the cohesive zone at peak load for an infinitely large specimen, and $g'(\alpha_0)$ is the first derivative of the function $g(\alpha)$ with respect to α for $\alpha = \alpha_0$ = initial crack-to-depth ratio (note that in Karihaloo's paper α is used to denote the initial crack length).

2.3. Comparison with exact first-order size effect

The asymptotic form of Karihaloo's result (6) is correct. Indeed, Planas and Elices (1991, 1992, 1993) proved in complete generality that, for a cohesive crack, the asymptotic size effect, exact up to and including all first-order terms, can be written as

$$\frac{E'G_F}{K_{INu}^2} = 1 + \frac{g'(\alpha_0)}{g(\alpha_0)}\frac{\Delta a_{c\infty}}{D} + o\left(\frac{\Delta a_{c\infty}}{D}\right) \tag{7}$$

where, as before, E' is the effective elastic modulus, G_F the fracture energy (which for a cohesive crack coincides with the area under the softening curve), K_{INu} the nominal stress intensity factor at peak load, and $\Delta a_{c\infty}$ the effective critical equivalent crack extension for infinite size, for which there is a closed form expression based on the solution for infinite size (Planas and Elices, 1991, 1992, 1993; see Section 3).

Setting in the preceding equation $K_{INu}^2 = \sigma_{Nu}^2 D g(\alpha_0)$ (which follows from inserting Equation (3) for $\alpha = \alpha_0$ and $\sigma_N = \sigma_{Nu}$ in Irwin's relation), solving for σ_{Nu}, and dividing the result by $\sigma_{Nu\infty}$ as written in Equation (4), one can transform Equation (7) into

$$\frac{\sigma_{Nu}}{\sigma_{Nu\infty}} = \left(1 + \frac{g'(\alpha_0)}{g(\alpha_0)}\frac{\Delta a_{c\infty}}{D}\right)^{-1/2} + o\left(\frac{\Delta a_{c\infty}}{D}\right) . \tag{8}$$

Since $(1+x)^{-1} = 1 - x + o(x)$, this expression coincides with Karihaloo's formula (6) up to and including the first-order terms in $l_{p\infty}/D$ if $\Delta a_{c\infty}$ is taken to be equal to $l_{p\infty}/2$.

2.4. Karihaloo's lower bound for specimen size

Deferring until Section 3 the more technical and quantitative considerations, let us first discuss the most striking property of Karihaloo's formula – namely that it is impossible to determine the nominal strength for specimen sizes D smaller than a certain lower bound.

$$D_{lb} = l_{p\infty}\frac{g'(\alpha_0)}{2g(\alpha_0)} . \tag{9}$$

This conclusion comes from the fact that when $D < D_{lb}$ the right-hand side of (6) becomes imaginary. But this is, in the writers' opinion, a lower bound that limits the applicability of Karihaloo's formula. It would be a misconception to claim that this bound limits the validity of experimental results.

Indeed, as already stated, Karihaloo's solution is, from its very start, at best a first-order (two-term) approximation which will be close to the exact solution only for large enough sizes. The problem is to define how large is large enough. Karihaloo intends to apply his formula to any size, which is, in principle, impossible, as the radius of convergence of the expansion is not known. The only way to ascertain the range over which a two-term approximation coincides, within a certain tolerance, with the exact solution is to make a direct comparison. Analytically, this is not feasible, although very accurate numerical solutions can of course be obtained. But practical common sense dictates that if the third and subsequent terms are neglected (thus being assumed small compared to the second term), then the two-term expansion can be used provided that the second term is also small compared to the first. This means that the result is expected a priori to be accurate for

$$\frac{g'(\alpha_0)}{2g(\alpha_0)}\frac{l_{p\infty}}{D} = \frac{D_{lb}}{D} \ll 1 . \tag{10}$$

Here the exact meaning of $\ll 1$ is not well defined for practical purposes; it may mean less than 0.001, 0.01, 0.1 or even 0.7, but always less than unity. Therefore D_{lb} defines only a lower bound for the size for which Karihaloo's analysis *might* be mathematically acceptable, and is devoid of any physical meaning with regard to experiments. The actual lower limit of applicability of Karihaloo's solution within a given tolerance is in fact much larger (see Section 3).

Apart from the foregoing mathematical reasoning, there are at least two further "external" reasons for rejecting the existence of a D_{lb} in the physical sense: (1) Tests involving specimen sizes less than Karihaloo's theoretical value D_{lb} can be (and have actually been) carried out, with reasonable results for the nominal strength; and (2) numerical computations of the nominal strength of three-point bend specimens using Karihaloo's softening law and other laws close to it can be (and have actually been) carried out with great accuracy for beam depths much smaller than D_{lb}.

Therefore, it appears that D_{lb} has no physical meaning. Rather, it represents a bound on the mathematical acceptability of Karihaloo's analysis. This leaves unanswered the question of how far down the large-size asymptotic approximation can be pushed, and what to do to obtain a closed form expression for the intermediate range, in which, for concrete at least, most if not all the experiments lie.

2.5. Asymptotic matching versus asymptotic expansion

Using simplified energy-based reasoning, Bažant (1984) derived a size effect law which, with some change of notation, coincides with the asymptotic expression (8). It can be written as

$$\sigma_{Nu} = \frac{\sigma_0}{\sqrt{1 + D/D_0}}, \tag{11}$$

where σ_0 and D_0 are constants that are related to the material fracture parameters and the geometrical shape as follows (for details, see, e.g., Bažant and Kazemi, 1990; Bažant and Planas, 1998):

$$\sigma_0 = \sqrt{\frac{E'G_f}{g'(\alpha_0)g(\alpha_0)c_f}}, \qquad D_0 = \frac{g'(\alpha_0)}{2g(\alpha_0)}c_f\,. \tag{12}$$

Here G_f is the fracture energy, and c_f the critical effective crack extension for infinite size (entirely analogous to G_F and $\Delta a_{c\infty}$ for a cohesive crack, but, in principle, of a broader scope since no particular nonlinear fracture model is implied). Now, as seen before, the large-size asymptotic expansions of the foregoing simple size effect formula (11) and of Karihaloo's formula (6) have the same form. However, for small sizes (relative to the material characteristic length $l_{p\infty}$), (11) and (6) are completely different. The salient aspect of (11) is that this formula matches the asymptotic trends for *both* the large and small sizes, since it is generally agreed (and has been numerically verified for cohesive cracks with high accuracy) that for very small sizes the size effect must asymptotically vanish.

This is not to say that the simple size effect formula (11) covers all the possibilities. In fact, more general laws have been developed (for a review, see Bažant and Planas, 1998, Chapter 9), and others are under development (Bažant, 1999). The important point, however, is that an equation for the size effect that would be valid over the full size range $(0, \infty)$ is difficult, it not impossible, to derive merely by refining the asymptotic expansion (i.e., adding more terms) at only one end. The reason, of course is that the radius of convergence may be limited and, even if it were infinite (which is hard to prove), the number of terms required for a good approximation far from the asymptote would become too large to be practical (for example, for using the expansion $\exp(x^{-1}) = 1 + x^{-1} + x^{-2}/2! + x^{-3}/3! \cdots$ for small x, the

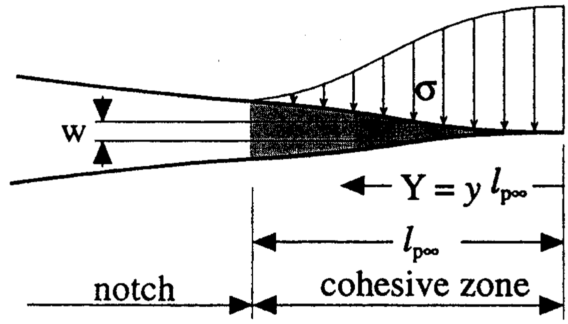

Figure 1. Cohesive zone at the tip of a semi-infinite crack.

number of terms that need to be taken to achieve good accuracy is $n \gg x^{-1}$; for $x = 0.1$, this means $n \gg 10$).

That is why the size effect formula (11) must be understood as an *asymptotic matching* formula (and so must some other more complex formulae). The crucial point to realize is that, in the plot of log σ_{Nu} versus log D, all the physically admissible size effect formulae must approach for $D \to 0$ a horizontal asymptote (which evidently implies that no mathematical cut-off at a certain D_{lb} may be imposed). This is not to mean, of course, that one could test specimens as small as desired and still use continuum (homogenized) approaches such as the cohesive crack model. But this is a completely different subject, which is in fact addressed by all testing standards – the minimum admissible size of a specimen relative to its microstructural features.

3. Other aspects of Karihaloo's asymptotic method

3.1. Critical effective crack extension

The preceding section demonstrates that Karihaloo's size effect equation (6) would be correct up to and including first order terms in D^{-1} if $\Delta a_{c\infty}$ happened to be equal to $\ell_{p\infty}/2$. Since closed form expressions for $\Delta a_{c\infty}$ exist, and Karihaloo's results include the stress distribution, it is an easy matter to check whether or not this condition is met. The expression for $\Delta a_{c\infty}$ originally deduced by Planas and Elices (1991) can be cast in the form (Smith, 1995):

$$\frac{\Delta a_{c\infty}}{\ell_{p\infty}} = 1 - 2\frac{\displaystyle\int_0^1 \sigma y^{1/2}\mathrm{d}y}{\displaystyle\int_0^1 \sigma y^{-1/2}\mathrm{d}y} \tag{13}$$

in which the integrals are performed over the cohesive zone, y is the distance from the cohesive crack tip divided by $\ell_{p\infty}$ (Figure 1), and σ is the cohesive stress distribution. According to Karihaloo's results, the cohesive stress distribution is given by

$$\sigma = f_t(1-y)^{3/2}(1+\tfrac{3}{2}y) \tag{14}$$

Inserting this expression into (13), both integrals are seen to reduce further to two Euler's beta functions, and the final result is found to be

$$\Delta a_{c\infty} = \frac{31}{56}\ell_{p\infty} = 0.5536\ell_{p\infty}\,. \tag{15}$$

This result shows that Karihaloo's size effect is not even first-order accurate, although the error represents only about 10% of the second (first-order) term. The possible sources of the error are discussed in Section 3.5. However, further analysis of Karihaloo's results shows that, even if the asymptotic size effect were a relatively good approximation to the exact asymptotic equation, other results are anyway abnormal. Two striking anomalous results will be discussed next, concerning: (1) the dependence of the cohesive zone length at peak load on the specimen size, and (2) the dependence of the softening curve on the size.

3.2. COHESIVE ZONE AND THE QUESTION OF 'MINIMUM' SIZE

Before proceeding any further, it should be pointed out that Karihaloo's results for infinite size (those labeled with subscript ∞, such as $l_{p\infty}$) are exact. But the introduction of the first-order term in the expression (5) for the stress ahead of an elastic crack leads to anomalous results.

In particular, the length of the cohesive zone at peak load l_p is given by

$$l_p = l_{p\infty}\left(1 - \frac{D_{lb}}{D}\right)^{-1} . \tag{16}$$

Now, this result indicates that l_p *increases* as size D decreases. This is contrary to all numerical analyses, in which the size of the cohesive zone *decreases* with decreasing D (Planas and Elices, 1991). This is also contrary to experimental knowledge, since in the experiments the cohesive zone is always confined within the ligament of the specimen, more so in bending of notched beams, in which the cohesive zone is limited by the compression zone that exists on the side of the specimen opposite to the notch mouth. Thus, the proper conclusion should be that this solution cannot be right. Instead, Karihaloo puts an *ad hoc* limitation on the size, requiring D to be larger than a certain value $D_{\min}$ determined so that $l_p \leq D$ for $D \geq D_{\min}$. The result follows trivially from Equation (16);

$$D_{\min} = l_{p\infty} + D_{lb} = l_{p\infty}\left(1 + \frac{g'(\alpha_0)}{2g(\alpha_0)}\right) . \tag{17}$$

Now, obviously, this $D_{\min}$ cannot be given any physical meaning. It is simply an artificial lower bound imposed so that the mathematical results would not be illogical. An even larger lower bound can be obtained if one chooses to require that, at least for bending, l_p must be less than the ligament length $D - a_0 = (1 - \alpha)D$. This, however, does not remove the incorrect dependence of l_p on D in Equation (16).

It is also instructive to present Karihaloo's size effect formula (6) in the usual plot of nominal strength against specimen size in the logarithmic scales. For this purpose, with the help of (9), we rewrite (6) in the equivalent form

$$\sigma_{Nu} = \sigma_1\sqrt{\frac{D_{lb}}{D} - \frac{D_{lb}^2}{D^2}} , \tag{18}$$

where $\sigma_1 = \sqrt{E'G_F/D_{lb}}$. Figure 2 shows that, according to Karihaloo's formula, the nominal strength is not a monotonically decreasing function of specimen size. For sizes between 0 and D_{lb}, the nominal strength is undefined (imaginary), for sizes between D_{lb} and $2D_{lb}$ it increases from zero to the maximum value $\sigma_1/2$, and only for sizes larger than $2D_{lb}$ does it start decreasing and approaches the LEFM asymptote. However, an increase of nominal strength with increasing size of the three-point bend specimen has never been observed in

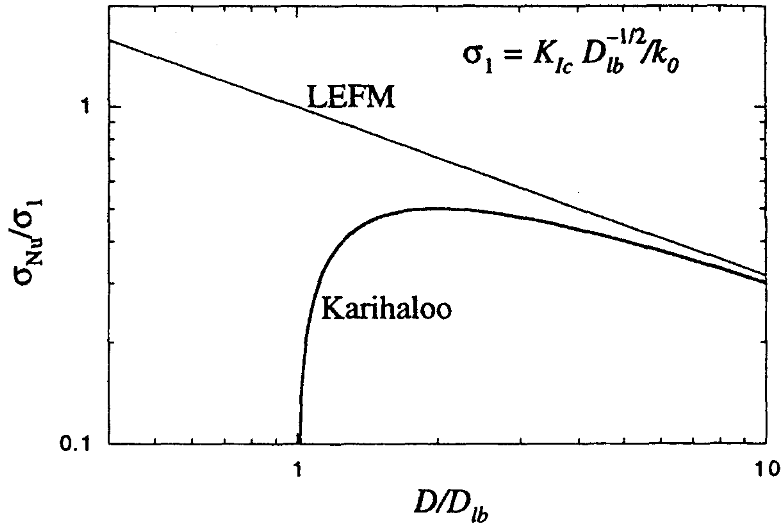

Figure 2. Karihaloo's size effect formula in the logarithmic plot of relative nominal strength σ_{Nu}/σ_1 against relative size D/D_{lb}.

experiments, and so a reasonable agreement between formula (6) and experimental data can be expected only for sizes above $2D_{lb}$, which is still larger than Karihaloo's $D_{\min}$. For example, for the most favorable relative notch depth $\alpha_0 = 0.3$, and for normal concrete with $L_{p\infty} =$ 70 mm, the depth below which Karihaloo's formula gives positive values of nominal strength but indicates the incorrect trend is $2D_{lb} = 360$ mm.

3.3. THE SOFTENING CURVE

In Karihaloo's analysis, the softening curve is not defined as a premise but is obtained parametrically from the analysis, because what is assumed *a priori* is the shape (depending on three parameters) of the crack opening profile on the cohesive zone. The stress is expressed in terms of parameter y by Equation (14), and the corresponding expression for the crack opening is

$$w = w_c y^{3/2} \frac{35 + y(7 - 6y)(4 + A)}{39 + A}, \tag{19}$$

where w_c is the critical crack opening (the crack opening for which the cohesive stress becomes zero) and A is a size-dependent parameter which, according to Karihaloo's results, can be recast to read

$$A = \frac{4D_{lb}}{D - D_{lb}} \tag{20}$$

Thus A varies between 0, for $D \to \infty$, and ∞ for $D = D_{lb}$. The resulting softening curves are shown in Figure 3. As can be seen, the softening curve is size dependent, which is contrary to one of the basic assumptions of the cohesive crack model and means that the material definition is not objective. Furthermore, the softening curve displays an inadmissible 'snap-back' – a decrease of crack opening with decreasing stress. This happens for sizes $D > \frac{15}{11} D_{lb}$ (at that size, the tangent to the softening curve at $w = w_c$ becomes vertical).

Figure 3 further reveals that the softening curve is reasonably size-independent only when, roughly, $D > 5D_{lb}$. for three-point bend specimens with notches between $0.15D$ and $0.5D$, for which $D_{lb} \approx 3l_{p\infty}$, this means that the softening curves are reasonably constant only

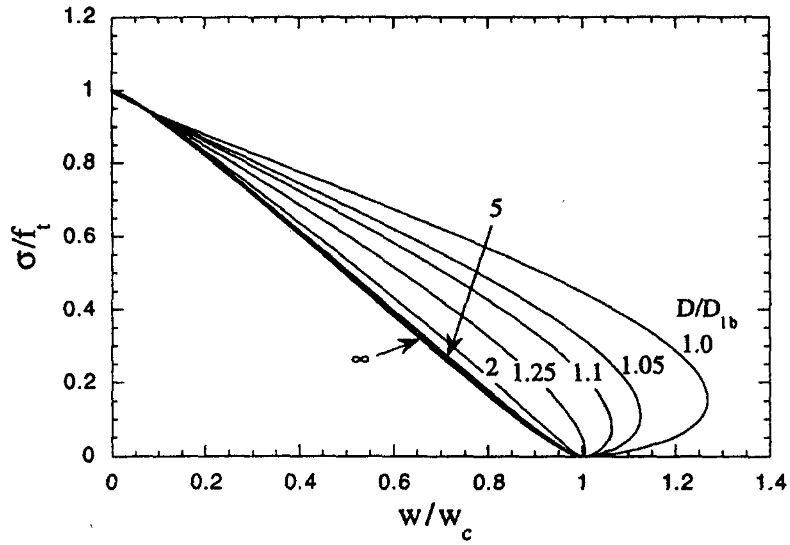

Figure 3. Softening curves for various sizes according to Karihaloo's results.

or $D > 15l_{p\infty}$, which are extreme, totally unrealistic, specimen sizes, exceeding four times Karihaloo's $D_{\min}$. For example, for normal concrete with $l_{p\infty} = 70$ mm, the depth of the specimen would have to be at least 1 meter.

3.4. Comparison with numerical results

To finish the main analysis of Karihaloo's results, Figure 4 compares Karihaloo's size effect equation (6) with the numerical results obtained using the discrete smeared-tip superposition method (for details on this method see Bažant and Planas, 1998, and Planas et al., 1999b). The softening curve used is that defined in Karihaloo's results for $D \to \infty$, given by Equations (14) and (19) with $A = 0$. The underlying finite element mesh had 100 equal elements on the central cross section, and a notch-to-depth ratio $\alpha_0 = 0.5$ was used. The data circles in Figure 4 represent numerical results, the full line Karihaloo's results, and the dotted line a smooth interpolation of the numerical results and the exact asymptotic expansion (8), with $\Delta a_{c\infty}$ given by Equation (15). Figure 4a displays the size effect for nominal strength relative to LEFM predictions in linear scale, and Figure 4b plots σ_{Nu}/σ_1 versus D/D_{lb} using logarithmic scales on both axes. As can be seen it is indeed possible to find numerical results for sizes D well below D_{lb} (in fact, most laboratory tests correspond to beam depths much less than D_{lb}). Moreover, Karihaloo's results approach the numerical-asymptotic interpolation curve sufficiently closely (within 1%) only for $D > 8D_{lb}$, which is much larger than $D_{\min}$ (for this geometry, $D_{\min} = 4.1734l_{p\infty} = 1.315D_{lb}$).

The bilogarithmic plot more clearly shows the contrast between the anomalous behavior of Karihaloo's solution for small sizes and the transitional behavior of the numerical solution.

3.5. Final remarks

In the preceding analysis, the interpretation of Karihaloo and thought-provoking analysis has been put into question because of several concepts shown to be physically unrealistic. A completely different matter would be to ascertain the precise influence of the various simplifications implied in Karihaloo's approach. Although this lies beyond the scope of this brief critique, a few more words are nevertheless in order, regarding two fundamental aspects.

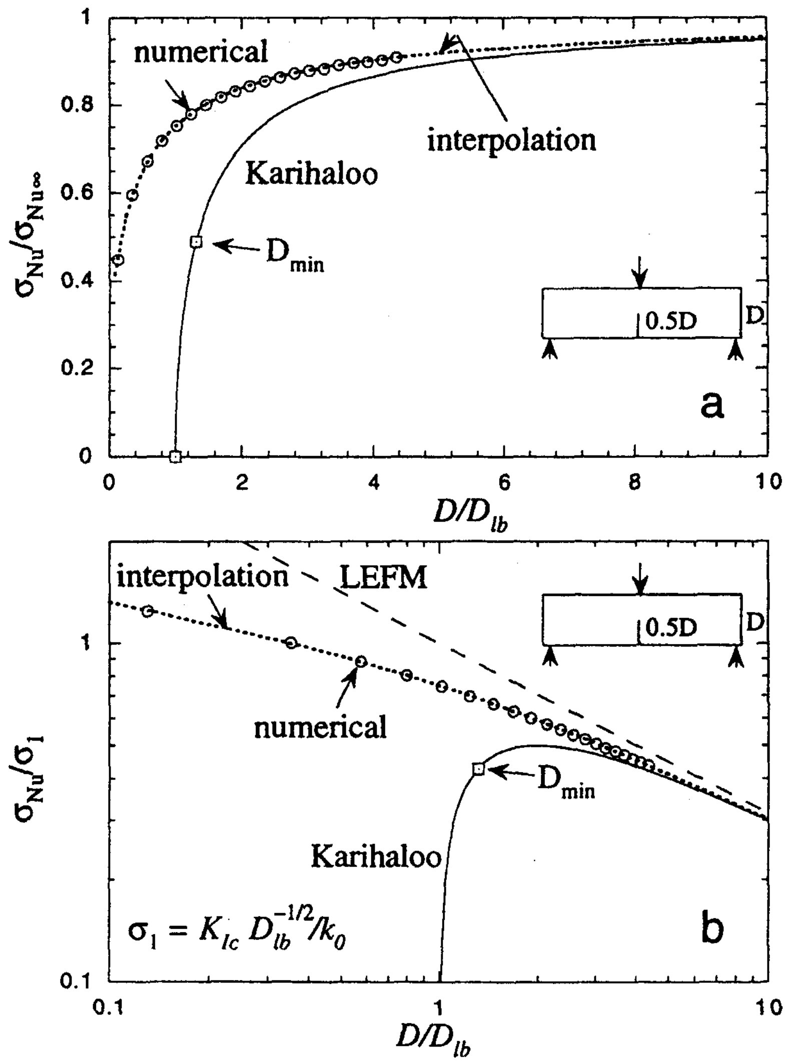

Figure 4. Comparison of Karihaloo's size effect curve (full line) with accurate numerical results (open circles) and an interpolation curve satisfying the first-order asymptotic approximation (dotted line): (a) curves of nominal strength relative to LEFM prediction, and (b) curves of nominal strength relative to fixed reference strength σ_1, with logarithmic scales on both axes. The results are for three-point-bend notched beams with a notch-to-depth-ratio of 0.5.

First, Karihaloo uses in his integral equation the kernels corresponding to infinite size. He in fact follows, analytically, the approach of Horii, Hasegawa and Nishino (1989). However, as pointed out in a previous discussion of this approach (Planas, 1989), two terms would have to be used also in these kernels in order to achieve a consistent first-order (two-term) asymptotic approximation.

Second, Karihaloo assumes the peak load to occur when the stress at the initial crack tip first becomes zero (or $w = w_c$). However, this is known to be true only for infinite size specimens ($l_{p\infty} \ll D$), but not for finite ones. For finite size specimens, all the available numerical solutions with the cohesive crack show that, at the peak load, the stress at the initial crack tip has not yet completely softened to zero (this is verified by the results of Petersson, 1981, and Planas and Elices, 1991, 1993, as well as unpublished results obtained with two very different numerical approaches by Bažant and Beissel, 1994, and by Bažant and Li, 1995).

Aside from these two aspects, a comment is deserved by Karihaloo's conclusion that changing the softening curve (which is nearly linear for large size, as seen in Figure 3) would alter the numerical factors in the results only slightly. This is not so. The asymptotic analysis of Planas and Elices (1991, 1992, 1993) clearly showed that $l_{p\infty}$ and $\Delta a_{c\infty}$ as well as their ratio depend strongly on the shape of the softening curve, for a material with a given fixed f_t and G_F (for a summary, see Bažant and Planas, Section 7.2.4). In fact, the ratio $\Delta a_{c\infty}/l_{p\infty}$ can be as low as $\frac{1}{3}$ for nearly rectangular softening and as large as 0.85 for a quasi-exponential softening; for linear softening this ratio is 0.57 (Planas and Elices, 1991). It thus seems that the ratio $\Delta a_{c\infty}/l_{p\infty}$ is close to 0.5 in Karihaloo's results because the softening curve for infinite size is very close to linear, as can be seen in Figure 3. This puts another limit to Karihaloo's approach, since it provides no simple way to implement softening curves with different shapes, which may be required, for example, for accurate modeling of concrete fracture.

4. Conclusions

From the foregoing analysis, it turns out that Karihaloo's theory is a simplified asymptotic analysis that gives approximate values of the strength of precracked structures provided the softening of the material is nearly linear and the size of the structures is rather large.

For notched (precracked) three-point-bend beams with a notch-to-depth ratio of 0.5 Karihaloo's solution deviates from more accurate solutions by less than 1% for beam depths larger than about $8D_{lb}$. The deviation increases to 5% when the beam depth is decreased to about $3.7D_{lb}$. Using Karihaloo's solution for smaller depths does not give realistic results.

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