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Sandwich buckling formulas and applicability of standard computational algorithm for finite strain

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Abstract

Although, for homogeneous columns, the differences between Engesser's and Haringx's formulas for shear buckling have been explained in 1971 by the dependence of shear modulus on the axial stress, for soft-core sandwich columns the choice of the correct formula has baffled engineers for half a century. Recently, Bažant explained this difference by a variational analysis which showed that an agreement is achieved if the shear modulus of the light core is considered to depend on the compressive stress in the skins even when small-strain elasticity applies. To clarify this paradoxical dependence, first the variational framework is briefly reviewed. Subsequently, the mathematical results from Bažant's recent study are physically reinterpreted, with the conclusion that only the Engesser-type theory (rather than Haringx-type theory) corresponds to constant shear moduli as obtained, for example, by the torsional test of a tube made from the foam. This is a rather fundamental point for applications because the discrepancy between these two theories can be very large in the case of short columns with thin skins. The implications for standard finite element programs are then explored by computing the critical loads of several sandwich columns with different material and geometric properties. The finite element computations show agreement with the Engesser-type formula predictions, while the Haringx-type prediction can be obtained with the finite element program somewhat artificially—by updating the core modulus as a function of the axial stress in the skins.

Keywords: C. Finite element analysis (FEA); B. Buckling; B. Elasticity; C. Computational modelling; Stability

1. Introduction

The load capacity of sandwich structures has been studied for over half a century and major advances have been achieved. However, the existing theories in the literature do not give an unambiguous picture. With the recent introduction of composites into construction of large structures, such as the hulls, decks, bulkheads, masts and antenna covers for very large ships (a problem of considerable interest to the Navy), the problem recently gained in importance and the remaining problems must now be resolved.

Sandwich shell failures caused by fracture of the skins, cores and interfaces are often combined with the loss of stability, and therefore the problems of buckling, face wrinkling, delamination, fracture, damage and scaling cannot be separated. In the field of elastic stability analysis, there still exists one fundamental unresolved problem that impinges on all the failure problems of sandwich structures—namely the role of shear of highly deformable sandwich cores. This problem is particularly acute for sandwich plates and shells with stiff fiber-composite laminate skins and very light polymeric foam cores, for which the skin-to-core elastic moduli ratio can be as high as 2000 (which is the case for Divinycell 100 foam).

Some fundamental questions arise in the calculation of critical loads of sandwich structures with a very high skin-to-core elastic moduli ratios:

1. How to explain the differences between the predictions of the Engesser-type and Haringx-type buckling formulae?
2. Which formula corresponds to a constant shear modulus as identified, for example, from the torsional test of a circular tube in small strain?
3. Can finite element programs predict the correct critical load?

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The first problem has recently been successfully resolved in Refs. [7,8], based on variational analysis associated with different finite strain measures. As part of the solution, a seemingly paradoxical behavior, consisting in an apparent influence of the stress in skins on the stiffness of core, has been explained. However, the physical interpretation of the analytical results in Ref. [8] has been oversimplified and needs a reinterpretation. So, the last two aforementioned problems remain, and will be dealt with in this paper. Therefore, the purpose of the paper is: (1) to decide which theory (i.e. a theory of Engesser type, Haringx type, or another type) corresponds to constant shear moduli; and (2) to match the theory to critical loads computed by finite-strain finite element analysis of sandwich columns.

It will be shown by means of an energetic argument that, in the case of sandwich structures, the Engesser-type theory is the only one that allows using for the core a constant tangential shear modulus when the strains are small enough for the core to remain in the elastic range. Then it will be demonstrated that the standard finite element programs, based on the updated Lagrangian algorithm, capture the critical load correctly. Some details of the finite element algorithm for finite strain will also be discussed to give a deeper insight into the problem.

2. Variational analysis of critical loads of columns with shear

2.1. Apparent paradox in shear-beam theories for sandwich buckling

There used to be lively polemics among the proponents of different three-dimensional stability formulations associated variationally with different finite strain measures (see, e.g. the preface of Biot's book [10]), with different objective stress rates, and with different incremental differential equations of equilibrium (particularly those proposed by Hadamard, Biot, Trefftz, Truesdell, Pearson, Hill, Biezeno, Hencky, Neuber, Jaumann, Southwell, Cotter, Rivlin, Engesser, Haringx, etc.—see Ref. [9] (p. 732 and chapter 11) and Ref. [3]). These polemics were settled in 1971 by the demonstration [4] that all these formulations become equivalent if it is realized that the tangential elastic moduli of the material cannot be taken the same but must have different values in each formulation. It was also concluded that these differences matter if initial stresses at the critical state of buckling are not negligible compared to the elastic moduli ([9] Sec. 11.4).

The differences between various stability criteria are, for most buckling problems, insignificant because the initial stresses are negligible compared to the tangential moduli. However, there are some important exceptions. A very important one is the buckling of sandwich plates with a very soft core. Another is the buckling of fiber composites with a highly orthotropic fiber reinforcement and a very soft matrix.

In sandwich plates, which are highly sensitive to buckling [1,11,15,16,25–27], the initial (i.e. pre-buckling) axial stress in the skins of a sandwich column is negligible compared to the elastic modulus of the skins, and the initial axial stress in the foam core is zero. Consequently, it may at first seem that the shear stiffness of the core should be constant, independent of the initial axial force in the skins, which would imply that there should be no differences among the critical load formulae associated with different finite strain measures.

Consequently, it came as a surprise that the Engesser-type [12–14] buckling formula for sandwich columns, which is associated with the Doyle-Ericksen finite strain tensor of order $m = 2$, gave, for short sandwich columns, much smaller critical loads than the Haringx-type [18,19] formula, which is associated with the Doyle-Ericksen tensor of order $m = -2$.

The differences between the Engesser-type and Haringx-type buckling formulae for sandwich columns, shown in Fig. 1, have been analyzed in detail in a recent paper [7,8] and will now be reviewed. The discussions of these differences began about 60 years ago [2,4–6,17,28,29,31,32,34–36]. However, no consensus on the theory has yet emerged [5,6].

First let us recall the class of Doyle-Ericksen finite strain tensors $\boldsymbol{\epsilon} = (\mathbf{U}^m - \mathbf{I})/m$ (where m = real parameter, $\mathbf{I}$ = unit tensor, and $\mathbf{U}$ = right-stretch tensor). These tensors, which include virtually all the strain measures ever used, have the second-order approximation

$$\epsilon_{ij}^{(m)} = e_{ij} + \frac{1}{2}u_{k,i}u_{k,j} - \alpha e_{ki}e_{kj}, \quad e_{ki} = \frac{1}{2}(u_{k,i} + u_{i,k}), \quad \alpha = 1 - \frac{1}{2}m \tag{1}$$

[4]; e_{ij} = small (linearized) strain tensor and the subscripts refer to Cartesian coordinates $x_i, i = 1, 2, 3$. The stability criteria expressed in terms of any of these strain

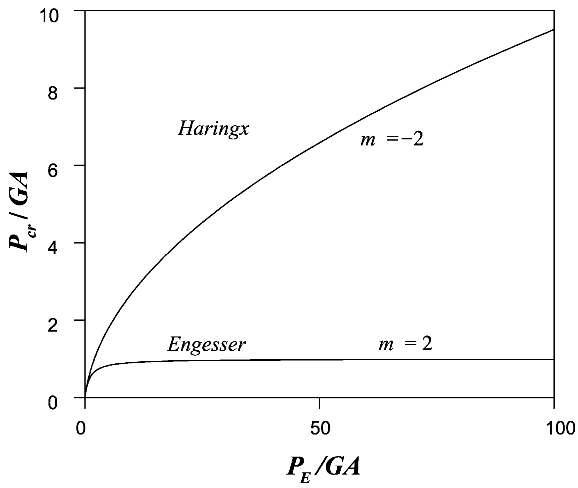

Fig. 1. Difference between the Engesser-type and Haringx-type formulae for a sandwich with $E_{\text{skin}}/G_{\text{core}} \approx 1400$.

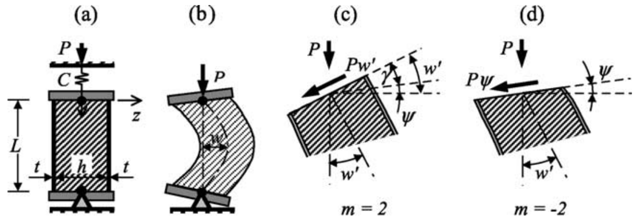

Fig. 2. Sandwich column in (a) initial state and (b) deflected state; (c) shear force on cross-section normal to deflection curve; (d) shear force on rotated cross-section that was normal in the initial state.

measures are mutually equivalent if the tangential moduli associated with different m-values satisfy Bažant's [4] relation:

$$C_{ijkl}^{(m)} = C_{ijkl} + \frac{1}{4}(2-m)(S_{ik}\delta_{jl} + S_{jk}\delta_{il} + S_{il}\delta_{jk} + S_{jl}\delta_{ik}) \quad (2)$$

(see also Ref. [9], p. 727); C_{ijkl} = tangential moduli associated with Green's Lagrangian strain ($m = 2$), and S_{ij} = current stress (Cauchy stress).

Engesser [12–14] and Haringx [18] presented different formulae for the first critical load in buckling of columns with significant shear deformations (Fig. 2). They read:

$$P_{\mathrm{cr}} = \frac{P_{\mathrm{E}}}{1 + (P_{\mathrm{E}}/\mathrm{GA})} \qquad \text{(Engesser)} \quad (3)$$

$$P_{\mathrm{cr}} = \frac{\mathrm{GA}}{2}\left(\sqrt{1 + \frac{4P_{\mathrm{E}}}{\mathrm{GA}}} - 1\right) \qquad \text{(Haringx)} \quad (4)$$

where

$$P_{\mathrm{E}} = (\pi^2/l^2)\mathrm{EI} \quad (5)$$

Here E, G = elastic Young's and shear moduli, P_{E} = Euler's critical load, l = effective buckling length, and EI, GA = bending stiffness and shear stiffness of cross-section. The discrepancy between these two formulae, regarded before 1971 as a paradox, was shown [4,9] to be caused by a dependence of the tangential shear modulus $C_{1212} = G$ on the axial stress $S_{11} = -P/A$, which is different for different choices of the finite strain measure, i.e. for different m. Engesser's formula corresponds to Green's Lagrangian strain tensor ($m = 2$), and Haringx's formula to Lagrangian Almansi strain tensor ($m = -2$), with the shear moduli related according to Eq. (2) as

$$G^{(2)} = G^{(-2)} + P/A \quad (6)$$

(a negligible difference in the E-values is ignored). The difference in shear moduli in Eq. (6), of course, becomes significant only if the axial stress $S_{11} = -P/A$ is not negligible compared to G. Such a situation arises for the continuum approximation of built-up (lattice) columns or for highly orthotropic fiber composite columns.

For elastic sandwich columns, the motivation of this study, a new paradox has recently been noticed, as a consequence of the numerical and experimental studies of Huang and Kardomateas [21], Kardomateas [22–24], Simitses and Shen [30] and Gjelsvik [17]. Let L = length of sandwich column, l = effective length, and P = axial force. The core has thickness h and shear modulus G. The skins have axial elastic modulus E (Fig. 2a) and thickness t, $t \ll h$ and $E \gg E$-modulus of the core, and so the entire axial force and bending moment are carried by the skins, while the entire shear force is carried by the core. Therefore, $\mathrm{EI} = Ebt(h+t)^2/2 + Ebt^3/6 \approx Ebth^2/2$ = bending stiffness of the sandwich ($t \ll h$), and $\mathrm{GA} = Gbh$ = shear stiffness of the sandwich, b being the cross-section width. With these notations

$$P_{\mathrm{cr}} = \frac{P_{\mathrm{E}}}{1 + (P_{\mathrm{E}}/Gbh)} \qquad \text{(Engesser type)} \quad (7)$$

$$P_{\mathrm{cr}} = \frac{Gbh}{2}\left[\sqrt{1 + \frac{4P_{\mathrm{E}}}{Gbh}} - 1\right] \qquad \text{(Haringx type)} \quad (8)$$

where $G = G_{\mathrm{core}}$ = shear modulus of the core, and $P_{\mathrm{E}} = (\pi^2/l^2)Ebth^2/2$ = Euler load.

In similarity to Eq. (6), it may be checked that, if the replacement

$$G_{\mathrm{core}} \leftarrow G_{\mathrm{core}} - \frac{2t}{h}\sigma_{\mathrm{skins}} \quad (9)$$

with $\sigma_{\mathrm{skins}} = -P_{\mathrm{cr}}/2bt$ is made in the Engesser-type formula (7), the Haringx-type formula (8) results [7]. This replacement, however, appears paradoxical; the shear modulus in the *core* cannot depend on the axial stress in the *skins*. Furthermore, since the axial stress in the core is negligible compared to the shear modulus of core, it appears paradoxical, in view of Eq. (2), that G-moduli associated with different strain measures need to be distinguished. We thus have a new kind of paradox. The resolution of this paradox, presented in Refs. [7,8], will now be reviewed and a new analysis of the relation of shear stiffness to experiments presented.

2.2. Finite strain variational analysis

In the sandwich beam theory, the skins and the core are constrained by the hypothesis of planar (though non-normal) cross-sections. Keeping it in mind, one may adapt the general variational analysis of column buckling, expressing the incremental potential energy of the column accurately up to the second order in displacement gradients [4,9].

We introduce Cartesian coordinates x_i ($i = 1,2,3$); Fig. 2a. The incremental displacements from the initial undeflected configuration of the column carrying axial load P are u_i; $u_3 = w(x)$ = small lateral deflection, and $u_1 = u(x,y,z)$ = small axial displacement; ψ a small rotation of the cross-section (Fig. 2c,d). The shear angle $\gamma = \theta - \psi$ (Fig. 2c,d) where $\theta = w'$ = slope of the deflection curve. The second-order incremental potential energy $\delta^2\mathscr{W}$ for small deflections $w(x)$ and small axial displacements $u(x)$ is

$$\delta^2\mathscr{W} = \int_0^L \int_A \left[S^0(y,z)(\epsilon_{11}^{(m)} - e_{11}) + \frac{1}{2}E^{(m)}(y,z)e_{11}^2 + \frac{1}{2}G^{(m)}(y,z)\gamma^2 \right] \mathrm{d}A\, \mathrm{d}x + \int_A \int_0^L \frac{1}{2}E^{(m)}(y,0)(u_0/L)^2 \mathrm{d}A\, \mathrm{d}x \tag{10}$$

([9], chapter 11); $y = x_2$ and $z = x_3$ = coordinates of the cross-section whose area is A; $S^0(y,z)$ = initial axial normal stress; $E^{(m)}(y,z), G^{(m)}(y,z)$ = tangential elastic moduli.

Imposing the condition that the cross-sections of core remain plane, and setting

$$\alpha = 1 - \tfrac{1}{2}m,$$

one can obtain from Eq. (10) the following expression:

$$\delta^2\mathscr{W} = \frac{1}{2}\int_0^L \left\{ R^{(m)}\psi'^2 + \left[H^{(m)} + \frac{1}{4}(2-m)P \right] \times (w' - \psi)^2 - Pw'^2 \right\} \mathrm{d}x \tag{11}$$

Here $R^{(m)} = E^{(m)}(1/2)bth^2$ = bending stiffness, $H^{(m)} = G^{(m)}bh$ = shear stiffness of the cross-section. The necessary condition of stability loss and bifurcation is that the first variation of the second-order work $\delta^2\mathscr{W}$ during any kinematically admissible deflection variations $\delta w(x)$ and $\delta u(x)$ must vanish (Trefftz condition). This condition leads to a system of two ordinary linear homogeneous differential equations for $w(x)$ and $\psi(x)$, with coefficients depending on P. It is found [7] that a non-zero solution exists if and only if

$$\frac{1}{4}(2-m)P^2 + \left[H^{(m)} + \frac{1}{4}(2+m)P_\mathrm{E}^{(m)} \right] P - H^{(m)}P_\mathrm{E}^{(m)} = 0 \tag{12}$$

where P_E = Euler load $= P_\mathrm{E}^{(m)} = \pi^2 R^{(m)}/L^2$. This quadratic equation has, for $m = 2$ and $m = -2$, the following solutions, which are analogous to Engesser's and Haringx's formulae, respectively [7,8]

$$\text{for } m = 2: \quad P_\mathrm{cr} = \frac{P_\mathrm{E}^{(2)}}{1 + (P_\mathrm{E}^{(2)}/H^{(2)})} \tag{13}$$

$$\text{for } m = -2: \quad P_\mathrm{cr} = \frac{H^{(-2)}}{2}\left[\sqrt{1 + \frac{4P_\mathrm{E}^{(-2)}}{H^{(-2)}}} - 1 \right] \tag{14}$$

It has been shown [4,9] that the case $m = 2$ is associated by work with Truesdell's objective stress rate, and the case $m = -2$ with Cotter and Rivlin's (convected) objective stress rate (or Lie derivative of Kirchhoff stress).

Further it is possible to obtain from Eq. (12) an infinite number of sandwich buckling formulae, each associated with any chosen value of m. Curiously, however, no investigators has proposed critical load formulae associated with other m values, although many investigators (e.g. Ref. [10]; or Biezeno, Hencky, Neuber, Jaumann, Southwell, Oldroyd, Truesdell, Cotter, Rivlin—see Ref. [9], chapter 11) introduced formulations for objective stress rates, three-dimensional stability criteria, surface buckling, internal buckling, and incremental differential equations of equilibrium associated with $m = 1, 0$ and -1.

2.3. Paradox resolution: definition of shear stiffness for stressed sandwich

In similarity to Eq. (6), one may expect the shear stiffnesses for the Engesser-type and Haringx-type formulae to be related as $H^{(2)} = H^{(-2)} + Ph/2t$. When this relation is substituted into Eq. (13) and the resulting equation is solved for $P = P_\mathrm{cr}$, (14) indeed ensues. However, unlike homogeneous columns weak in shear, the foregoing transformation cannot be physically justified on the basis of the general transformation of tangential moduli in Eq. (2), nor its special case in Eq. (6), because the axial stress S^0 in the core is negligible.

Why should the shear modulus of the core be adjusted according to the axial stress in the skins? This seems to be a paradox. To resolve it, we must examine the definition of the shear stiffness H of a sandwich.

Imagine a homogeneous pure shear deformation of an element Δx of the sandwich column; $u_1 = u_{1,1} = u_{1,3} = e_{11} = 0$, $u_{3,1} = \gamma$, $e_{13} = e_{31} = \gamma/2$. Based on Eq. (10), the second-order incremental potential energy of the element is found to be

$$\delta^2\mathscr{W} = \Delta x \int_A \left[-\frac{P}{2bt}\left(\tfrac{1}{2}u_{k,1}u_{k,1} - \alpha e_{k1}e_{k1} \right) + \tfrac{1}{2}G\gamma^2 \right] \mathrm{d}A \tag{15}$$

where the superscript m is omitted for G to emphasize that, because the core is in small strain, the shear modulus is independent of the particular choice of finite strain measure. Upon rearrangements, the incremental potential energy density per unit height of the column ($\Delta x = 1$) can be brought to the form [7]:

$$\delta^2\mathscr{W} = \left(Gbh + \frac{2-m}{4}P \right)\frac{\gamma^2}{2} - \frac{Pw'^2}{2} \tag{16}$$

Expression (16) should be compared with the second-order work obtained by the elementary reasoning described in Fig. 3a–c. The deformation process of the beam element is decomposed into two parts: bending (Fig. 3b) and shear (Fig. 3c). Bending of the skin, considered inextensible, causes axial shortening of the element (in the direction of P) and thus contributes to the overall work with the second order term $-Pw'^2/2$. However, because each of the skins (like normal beams) does not deform by shear (see the detail in Fig. 3c), no work contribution is associated with Fig. 3c. It must be noted that the skins exhibit shear only if considered together. Therefore, the second-order work consists only of the strain energy contribution from the shear of the core and the second-order work contribution of

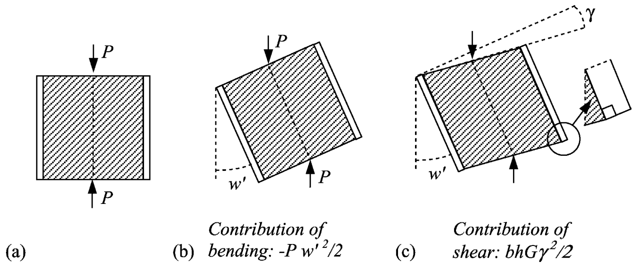

Fig. 3. Contributions to the second order work of a column element of unit height due to work of constant axial force P on rotation w', and to shear strain γ in the core.

axial force P due to axial skin shortening, i.e.:

$$\delta^2 \mathscr{W} = Gbh\frac{\gamma^2}{2} - \frac{Pw'^2}{2} \tag{17}$$

Equivalence of Eqs. (16) and (17) is achieved if and only if $m = 2$. Therefore, the Engesser-type theory must be applied. In this, and only in this, theory, the shear modulus is equal to that obtained in a small-strain pure shear test, for example, in the simple torsion test of a thin-wall tube made of the foam. The use of Haringx-type formula is off course equivalent but the shear modulus of the core cannot be kept constant; rather, it must be corrected for the effect of the axial forces $F = P/2$ carried by the skins.

Consequently, a constant shear stiffness H corresponding to a constant shear modulus G can be used only in the Engesser-type formula ($m = 2$).

2.4. Differential equations of equilibrium associated with Engesser's and Haringx's theories

Alternatively, it is possible to derive Engesser's and Haringx's critical load formulae from the differential equations of equilibrium (p. 738 in Ref. [9]). Fig. 2c,d shows two kinds of cross-sections of a sandwich column in a deflected position: (a) the cross-section that is normal to the deflected column axis, on which the shear force due to axial load is $Q = Pw'$ and (b) the cross-section that was normal to the column axis in the initial undeflected state, on which the shear force due to axial load is $\bar{Q} = P\psi$. For a simply supported (hinged) column, the bending moment is $M = -Pw$ in both cases. The force–deformation relations are $M = Ebth^2\psi'/2$ and Q or $\bar{Q} = Gbh\gamma = Gbh(w' - \psi)$ in case (a) or (b), respectively. Eliminating M, γ, ψ and Q or $\bar{Q}$, one gets the corresponding two forms of a linear homogeneous differential equation for $w(x)$, of which the first is found to lead to Engesser's formula (3) and the latter to Haringx's formula (4). Thus it is concluded that Engesser's formula ($m = 2$) is obtained when the shear deformation γ is assumed to be caused by the shear force acting on the cross-section that is normal to the deflected axis of column, and Haringx's formula ($m = -2$) when γ is assumed to be caused by the shear force acting on the rotated cross-section that was normal to the beam axis in the initial state [7,8].

The foregoing equilibrium reasoning, however, does not show that the values of shear stiffness in both formulae must be different. Especially, it does not show that only the shear stiffness for the cross-section perpendicular to the deflected axis can be kept constant.

For further interesting implications for buckling of highly orthotropic fiber composites, built-up lattice columns, layered elastomeric bearings and spiral springs, see Refs. [7,8].

3. Finite element computation of critical loads of sandwich structures

3.1. Numerical results

The consequences of the preceding variational analysis for standard finite element programs have been explored by computing the critical loads of sandwich columns. In the computations, different values of Young's modulus E of the skin varying in the range between $E = 10$ GPa and $E = 105$ GPa are considered, while Poisson's ratio is kept at a constant value, $\nu = 0.26$. The core is characterized by $E = 75$ MPa, $\nu = 0.25$ and $G = 30$ MPa. Not surprisingly, the computer results, as well as the predictions of the Engesser-type and Haringx-type formulae, exhibit high sensitivity to the skin-to-core moduli ratio E/G. The columns considered are characterized by a ratio $L/(h+2t)$ varying between $L/(h+2t) = 10$ and $L/(h+2t) = 30$, and the core-to-skin thickness ratios considered are $h/t = 5$ and $h/t = 20$. The column is free standing, being fixed at the base, and so the effective length is $l = 2L$.

The values of the bending stiffness EI and shear stiffness GA in Eqs. (3) and (4) are replaced by the following more accurate effective stiffnesses [21] in which the small but non-negligible shear stiffness of the skins and the bending stiffness of the core are taken into account

$$\overline{\mathrm{EI}} = b\left[E_s\frac{t^3}{6} + \frac{1}{2}E_s t(t+h)^2 + E_c\frac{h^3}{12}\right] \tag{18}$$

$$\overline{\mathrm{GA}} = \frac{1}{2b}\left\{\frac{E_s^2}{4\overline{\mathrm{EI}}^2 G_s}\left[a^4 t - \frac{2}{3}a^2(a^3-d^3) + \frac{1}{5}(a^5-d^5)\right] + \frac{E_s^2}{\overline{\mathrm{EI}}^2 G_c}\left[t^2c^2d + \frac{2}{15}\frac{E_c^2}{E_s^2}d^5 + \frac{2}{3}\frac{E_c}{E_s}tcd^3\right]\right\}^{-1} \tag{19}$$

where $a = t + h/2$, $c = (t+h)/2$, $d = h/2$, E_s is the Young modulus of the skin, E_c is Young's modulus of the core, G_s is the shear modulus of the skin, and G_c is the shear modulus of the core. Because the equivalent stiffness defined in Eqs. (18) and (19) takes into account the bending and shear stiffnesses of the skins, the skins are modeled by isoparametric four-node elements, and such elements are also used to model the core. Due to its finite thickness, the skin makes a non-vanishing but small contribution to the overall shear stiffness of the column. The core, similarly, makes

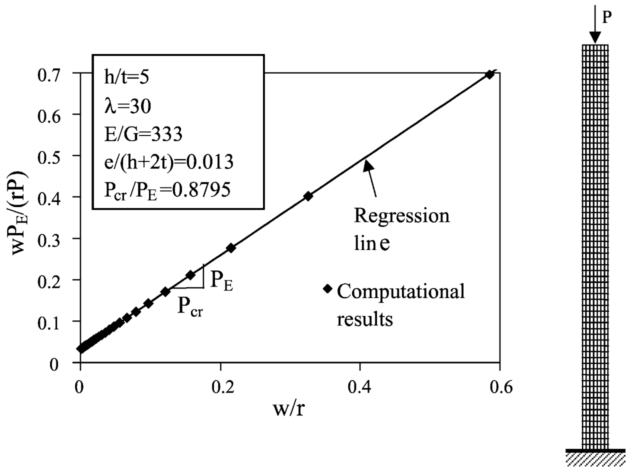

Fig. 4. Southwell plot (on the left) and column considered in the FEM analysis (on the right).

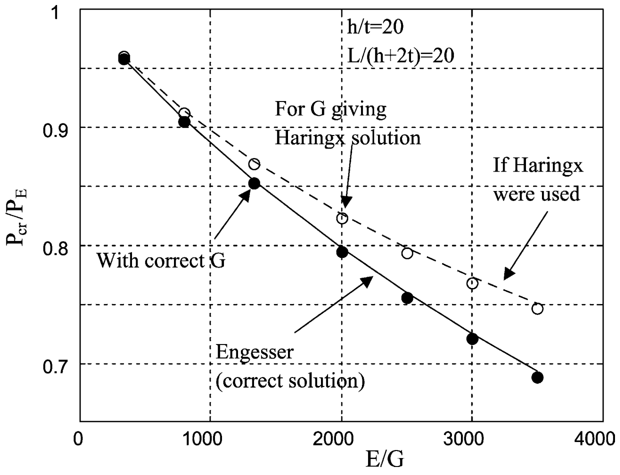

Fig. 6. Critical load for slender column with thin skin.

a non-vanishing but small contribution to the overall bending stiffness of the column. The geometry of the columns analyzed and the mesh used in the computation are shown in Fig. 4 on the right (the real mesh was finer than shown).

The stresses in both the skin and the core are assumed to be negligible compared to their respective elastic moduli, which means that small-strain linear elasticity is followed. Therefore, both the skin and the core are treated as Saint Venant-Kirchhoff materials and the updating algorithm in the analysis is the energy-momentum conserving algorithm as described in Ref. [33]. This material definition represents the natural extension of the small deformation elasticity approach to finite strain. According to dimensional analysis (Buckingham's Π-theorem), the values of P_{cr}/P_E can depend only on the ratios h/t and $L/(h+2t)$, and on the skin-to-core ratio E/G, and therefore the results are plotted in these dimensionless coordinates.

The critical loads of perfect sandwich columns have been calculated in two ways, which give identical results: (1) a small imperfection (very small load eccentricity e) has been assumed and linear regression in Southwell plot has been used to deduce the critical load from the regression slope [9], as shown in Fig. 4. (2) The singularity of the tangential stiffness matrix of a perfect column has been identified by a sign change in the diagonal term of this matrix during triangular decomposition.

The computer results obtained with a standard finite element code (in this particular case, code FEAP, by R.L. Taylor) are shown by the solid circles in Figs. 5–7. Because the program is based on the standard updated Lagrangian formulation corresponding to $m = 2$, the results are expected to agree with the Engesser-type formula, and this is indeed the case (see the lower straight line). It must now be noted that the reason for this agreement is that the updating algorithm assumes the material moduli tensor to be constant with respect to the beginning of each loading step.

Furthermore, in relation to the preceding variational analysis, it has been checked what happens if the modulus of the core is updated in each loading step on the basis of the stress in the skins, as indicated by Eq. (9). As expected, this causes the computational results to agree with

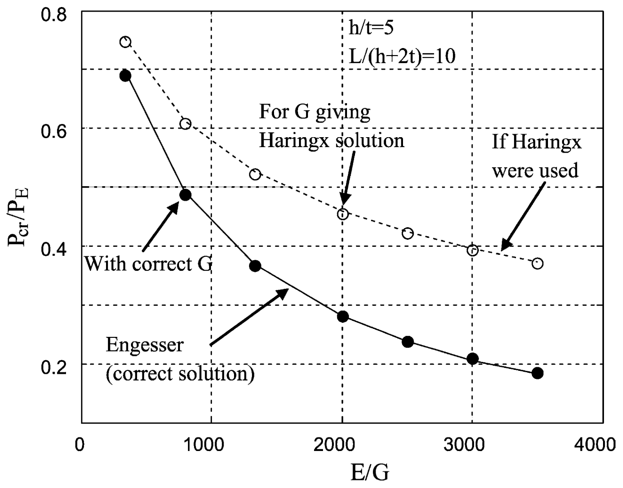

Fig. 5. Critical load for short column with thick skin.

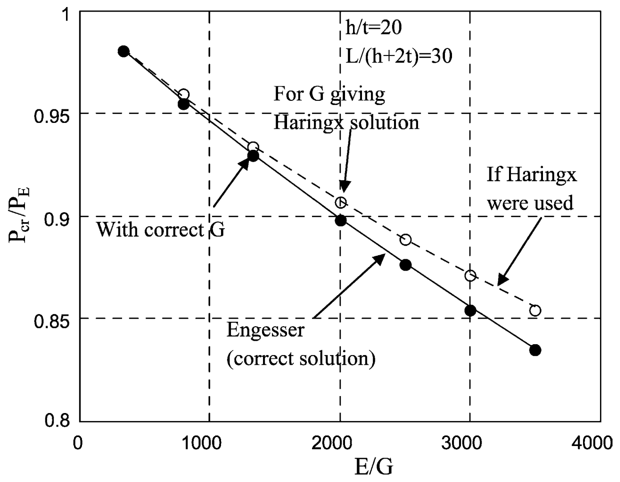

Fig. 7. Critical load for very slender column with thin skin.

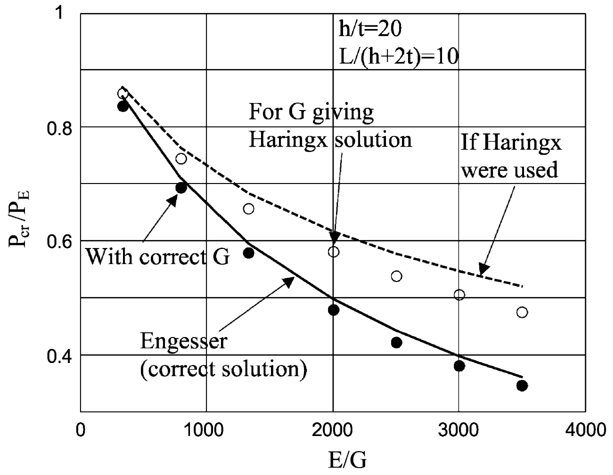

Fig. 8. Critical load for short column with thin skin (misfit is caused by the fact that the beam theory cannot reproduce the correct deformation near the ends).

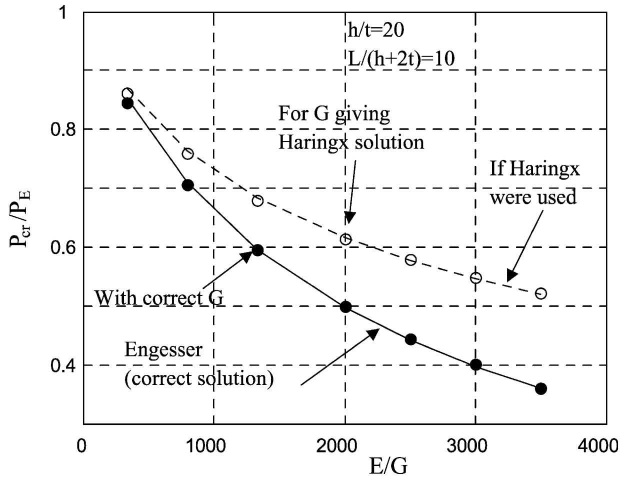

Fig. 10. Critical load for short column with thin skin and stiff plate at the top.

the Haringx-type predictions, as shown in Figs. 5–7 where the empty circles represent the computational results, and the dashed upper line the Haringx-type formula predictions.

The computation shows also the effect of geometry of the column on the computation. The shorter the column, the larger is the discrepancy between the formula prediction and the computation, as shown in Fig. 8 for a column with $L/(h+2t) = 10$ and $h/t = 20$. This discrepancy is also affected by the core-to-skin thickness ratio h/t, such that for lower h/t values there is a better agreement, as shown in Fig. 5. The cause of this discrepancy lies in the end effects for shorter columns and in non-planarity of the deformed cross-sections near the ends (Fig. 9).

Besides, interaction between global buckling and local wrinkling of the skins may reduce significantly the buckling load for short column with thin skins [20]. To force the cross-sections to remain plane and to prevent local wrinkling, a plate with a very high longitudinal stiffness and a very low transversal stiffness is considered to be placed at the top. With this, close agreement with the formula predictions is achieved, as shown in Fig. 10.

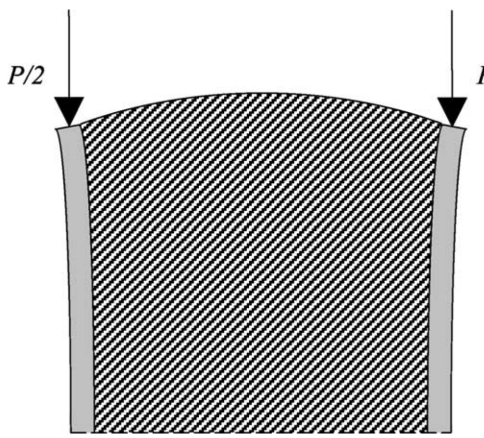

Fig. 9. Deformation of the cross-section at the top of the column (important for short column with very thin skin).

To sum up, according to the foregoing variational analysis and the numerical results it is correct to simulate soft-core sandwich structures with the standard finite element programs using the Lagrangian updating algorithm, which is based on Green's Lagrangian strain tensor corresponding to $m = 2$. The results agree with the Engesser-type critical load formula very well. The Haringx-type formula can give very different critical loads, and it has been confirmed that these critical loads are obtained by the finite element program if the shear modulus of the core is updated in each step according to Eq. (6).

3.2. Characteristics of the finite element algorithm for finite strain

The case of sandwich structure gives the opportunity for a deeper insight on how the finite element algorithm deals with finite strain and how the two formulae (of Engesser-type and Haringx-type) are related in the numerical computation. In the case of finite-strain elasticity involving mainly geometric non-linearity dominated by geometrically nonlinear effects of large material rotations, the Saint Venant-Kirchhoff elasticity model is usually applied. In this constitutive model, the second Piola-Kirchhoff stress tensor $\mathbf{S}$ depends linearly on the Green-Lagrange strain $\mathbf{E}$ through the stiffness tensor $\mathbf{C}$ as $\mathbf{S} = \mathbf{C} : \mathbf{E}$. This approach requires a push-forward of the second Piola-Kirchhoff stress and of the tangent stiffness tensor to the current configuration in each loading step. This means that, for each loading step, starting at time t_n and ending at time t_{n+1}, and for each Gaussian integration point of each finite element, the program evaluates the Green-Lagrange strain $\mathbf{E}_{n+\alpha}(m = 2)$ on the basis of the deformation gradient $\mathbf{F}_n$ at t_n and $\mathbf{F}_{n+1}$ at t_{n+1}

$$\mathbf{E}_{n+\alpha} = \frac{1}{2}(w_1\mathbf{E}_{n+1} + w_2\mathbf{E}_n) \tag{20}$$

where

$$\mathbf{E}_{n+1} = \frac{1}{2}(\mathbf{F}_{n+1}^{\mathrm{T}}\mathbf{F}_{n+1} - \mathbf{I}) \tag{21}$$

$$\mathbf{E}_{n} = \frac{1}{2}(\mathbf{F}_{n}^{\mathrm{T}}\mathbf{F}_{n} - \mathbf{I}) \tag{22}$$

where $\mathbf{I}$ is the second-order identity tensor. In Eqs. (21) and (22), the weights w_1 and w_2 are decided according to an energy-momentum conserving algorithm [33]. The parameter α, varying between 0 and 1, identifies the instant within the time step at which the strain is evaluated, which corresponds to a certain chosen integration rule (for example, the mid-point rule, for which $\alpha = 0.5$).

The second Piola-Kirchhoff stress is obtained from the tangent stiffness tensor as

$$\mathbf{S}_{n+\alpha} = \mathbf{C}_{n+\alpha} : \mathbf{E}_{n+\alpha} \tag{23}$$

where the tangent stiffness tensor $\mathbf{C}_{n+\alpha}$ represents the elasticity tensor of small-strain isotropic elasticity evaluated at time $t_{n+\alpha}$ and is kept constant within the time step and with respect to the initial (undeformed) configuration

$$\mathbf{C} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{I} \tag{24}$$

Here λ and μ are Lamé's constants (which depend on E and ν), and $\mathbf{I}$ is the fourth-order identity tensor. The tensors $\mathbf{S}_{n+\alpha}$ and $\mathbf{C}_{n+\alpha}$ in Eq. (23) are then pushed forward to the current configuration through the deformation gradient $\mathbf{F}_n$ and $\mathbf{F}_{n+1}$ to obtain the Cauchy stress $\sigma_{n+\alpha}$ and the spatial tangent stiffness $\mathbf{c}_{n+\alpha}$, which is done according to the relations

$$\sigma_{n+\alpha} = \frac{1}{J_n}\mathbf{F}_n \mathbf{S}_{n+\alpha}\mathbf{F}_n^{\mathrm{T}} \tag{25}$$

$$c_{ijkl}^{n+\alpha} = F_{iI}^{n} F_{jJ}^{n} F_{kK}^{n+1} F_{lL}^{n} C_{IJKL}^{n+\alpha} \tag{26}$$

where J_n is the Jacobian of the transformation (determinant of the deformation gradient $\mathbf{F}_n$) and $\mathbf{c}_{n+\alpha}$ is described in the component form.

Whereas a standard finite element code in which the moduli are considered as constant with respect to the beginning of the step yields the correct result for sandwich structures, agreeing with the Engesser-type formula, the Haringx-type formula, on the other hand, is known to be the correct formula to predict the critical loads of elastomeric bridge bearings or helical springs. If these structures are homogenized, then the standard finite element code cannot be used. Rather, the modulus of the core must be updated as in Eq. (9) because of the general relation (2). An alternative way to obtain the Haringx-type load would be to generalize the finite element code for arbitrary m, as described in Sec. 11.8 of Ref. [9], and then select $m = -2$. This would require also to first push forward the strain tensor and then compute the stress increment in terms of Lie deriva-tive of the Kirchhoff stress, associated with constant moduli with respect to the end of the loading step. However, the implications of the last point go beyond the scope of this paper.

4. Conclusions

1. The variational analysis of the Engesser-type and Haringx-type formulae for buckling of short soft-core sandwich columns clarifies the relationship between these formulae and explains the large discrepancy between their predictions.
2. Bažant's [8] recent explanation of a paradoxical dependence of the shear modulus in the core on the stresses in the skin is confirmed, however, the physical consequences must be interpreted differently. This leads to a fundamental argument in favor of Engesser-type (rather than Haringx type) theory for sandwich columns. Therefore, whenever the core of a sandwich structure is in small strain and a constant shear modulus is considered in calculations, the Engesser-type theory, variationally associated with Green's Lagrangian finite strain tensor, is the only theory to use.
3. It is demonstrated that the critical loads given by the Engesser-type formula can be correctly captured by the standard finite elements programs, which use an updated Lagrangian variational formulation associated with Green's Lagrangian strain tensor.

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