Universal Size Effect Law and Effect of Crack Depth on Quasi-Brittle Structure Strength

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Abstract: In cohesive fracture of quasi-brittle materials such as concrete, rock, fiber composites, tough ceramics, rigid foams, sea ice, and wood, one can distinguish six simple and easily modeled asymptotic cases: the asymptotic behaviors of very small and very large structures, structures failing at crack initiation from a smooth surface and those with a deep notch or preexisting deep crack, the purely statistical Weibull-type size effect, and the purely energetic (deterministic) size effect. Size effect laws governing the transition between some of these asymptotic cases have already been formulated. However, a general and smooth description of the complex transition between all of them has been lacking. Here, a smooth universal law bridging all of these asymptotic cases is derived and discussed. A special case of this law is a formula for the effect of notch or crack depth at fixed specimen size, which overcomes the limitations of a recently proposed empirical formula by Duan et al., 2003, 2004, 2006.

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Introduction

In plastic limit analysis or elasticity with strength limit, the nominal strength of structure is size independent. However, for quasi-brittle materials such as concrete, rock, fiber composites, tough ceramics, rigid foams, sea ice, stiff soils, and wood (i.e., heterogeneous brittle materials with a fracture process zone (FPZ) that is not negligible compared to structural dimensions), the nominal strength depends on structure size. This phenomenon is called the size effect. There are two kinds of size effect: (a) statistical, described by the Weibull (1939a,b, 1951) theory of random local material strength, and (b) energetic (deterministic). In the latter, one discerns principally the Type I size effect, occurring in structures that fail at crack initiation from a smooth surface, and the Type II size effect, occuring in structures with a deep notch or deep stress-free (fatigued) crack formed stably before reaching the maximum load.

The large size and small size asymptotic behaviors have already been bridged by closed-form size effect laws, both for the purely energetic size effect (Type II) and energetic-statistical size effect (Type I) (Bažant 1984, 1997, 2001, 2002, 2004; Bažant and Chen 1997; Bažant and Planas 1998). However, bridging the size effects for the cases of no notch and a deep notch (or crack) is a more difficult problem and is important for predicting the behavior of structures with shallow but nonzero cracks. This problem was tackled by Bažant (1997) and a kind of universal size effect law was derived; however, with two serious limitations: (1) it was purely energetic and deterministic (i.e., the Weibull statistical asymptote for Type I size effect was not captured), and (2) the dependence of nominal strength on the notch or crack depth was not smooth.

The objective of the present study is to derive an improved universal size effect law that is free of these two limitations and smoothly describes the behavior transitional among all the six simple asymptotic cases. As a byproduct, the dependence of the nominal strength on the notch depth over the entire range of depth will also be obtained, and compared to previous work of Duan et al. (2006).

Review of Energetic Types I and II Size Effect Laws

When a quasi-brittle structure has a deep notch or a large traction-free (i.e., fatigued) crack that has formed before reaching the maximum load, the size effect on the mean nominal strength of structure is essentially energetic, with the negligible statistical component (Bažant and Xi 1991). This size effect, termed Type II size effect, may approximately be described by the size effect law proposed by Bažant(1984)

$$\sigma_N = Bf_t'(1 + D/D_0)^{-1/2} \tag{1}$$

where $\sigma_N = P/bD$ or P/D^2 = nominal strength for scaling in two or three dimensions; P = maximum applied load or load parameter; D = characteristic size of structure; b = thickness in the third dimension of a structure scaled in two dimensions; and B and D_0 = parameters depending on structural geometry.

Eq. (1) was later rederived more generally by Bažant and Kazemi (1990) and Bažant (1997) using asymptotic approximations of the energy release function of a propagating crack based on equivalent linear elastic fracture mechanics (LEFM); see also Bažant and Planas (1998). Truncating the expansion after the second term, one can express the size effect law in terms of fracture characteristics

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$$\sigma_N = \sqrt{\frac{E'G_f}{g'(\alpha_0)c_f + g(\alpha_0)D}} \tag{2}$$

where the parameters are expressed as

$$D_0 = c_f \frac{g'(\alpha_0)}{g(\alpha_0)} \quad Bf_t' = \sqrt{\frac{E'G_f}{c_f g'(\alpha_0)}} \tag{3}$$

Here $\alpha_0 = a_0/D$=relative initial crack length; c_f=effective length of fracture process zone (considered as a material parameter); $g(\alpha) = D(bK_I/P)^2$=dimensionless energy release function of equivalent LEFM characterizing the specimen geometry (K_I =stress intensity factor); $g'(\alpha_0) = \mathrm{d}g(\alpha)/\mathrm{d}\alpha$ at $\alpha = \alpha_0$, with the prime denoting derivatives; $E' = E$ or $E/(1-\nu^2)$ for plane stress or plane strain (ν=Poisson's ratio); E=Young's modulus; and G_f=fracture energy.

The Type II size effect law in Eq. (1) (Bažant 1984, 2001, 2002) applies to most notched fracture specimens and also to most failure types of reinforced concrete structures. Structures exhibiting Type II size effect are also the objective of good design because large stable crack growth prior to maximum load endows the structure with large energy dissipation capability and significant ductility.

Many quasi-brittle structures, however, fail at crack initiation from a smooth surface, as soon as the fracture process zone or boundary layer of cracking fully develops. In that case, the size effect is of Type I (Bažant 2001, 2002). It was analyzed by Bažant and Li (1995) based on stress redistribution caused by a boundary layer of densely distributed microcracking that has, at the peak load, a size-independent critical thickness D_b. In a more general way, related to fracture mechanics, the same Type I size effect was deduced by Bažant (1997) from the limiting case of energy release and dissipation for crack length approaching zero. Because function $g(\alpha)$ vanishes for $\alpha \to 0$ while its first and second derivatives do not, the third term of the large-size asymptotic series expansion of function $g(\alpha)$ about the point $\alpha = 0$ (or $D \to \infty$) must be retained if the size effect should be captured; this gives

$$\sigma_N = \left(\frac{E'G_f}{g'(0)\bar{c}_f + g''(0)\bar{c}_f^2/2D}\right)^{1/2} = f_r^\infty \left(1 - \frac{2D_b}{D}\right)^{-1/2} \approx f_r^\infty \left(1 + \frac{rD_b}{D}\right)^{1/r} \tag{4}$$

in which the asymptotic approximations $(1-2x)^{-1/2} \approx 1 + x \approx (1+rx)^{1/r}$ for $x = D_b/D \ll 1$ (having an error of the order of x^2) have been used and the following notations have been made:

$$f_r^\infty = \sqrt{\frac{E'G_f}{g'(0)\bar{c}_f}} \quad D_b = \frac{\langle -g''(0)\rangle}{4g'(0)}\bar{c}_f \quad \bar{c}_f = \kappa c_f \tag{5}$$

Here f_r^∞, D_b, and r=positive constants for geometrically similar specimens; f_r^∞ has the meaning of nominal strength for a very large structure; D_b has the meaning of effective thickness of the boundary layer; $\bar{c}_f$=effective length (depth) of fracture process zone for fracture initiation from a smooth surface; the operator $\langle . \rangle$ (Macauley bracket) means the positive part, i.e., $\langle x \rangle = \max(x, 0)$; and κ=constant$\geqslant 1$ but close to 1, which characterizes the ratio of the effective sizes of the cracking zones (or FPZ) at a smooth surface and at the tip of a deep notch or crack. From observations, it appears that the FPZ for crack initiation is generally larger than the FPZ for a crack starting from a deep notch. The empirical coefficient r had to be introduced because the case $r=1$ gives only one among infinitely many cases that give the same first two terms of the asymptotic expansion and are equally plausible from the mathematical viewpoint. Considering only $r=1$ would, thus, be an arbitrary, unreasonable, restriction. According to experimental data, the optimum r-value generally lies between 1/2 and 1 [depending on the coefficient of variation of random material strength (Bažant and Pang 2006, 2007).

Note that the second expression in Eq. (4) would give complex σ_N-values when D is not large enough. This feature is, of course, unrealistic though not surprising because this expression was obtained as a large-size asymptotic approximation. However, this feature is eliminated by the last expression in Eq. (4), which is equally justified as Eq. (4) because it has the same first two terms of the large-size asymptotic expansion in powers of a/D, yet is realistic for a broad range of D (practically all the cross sections larger than the representative volume of material)—except for $D \to 0$. According to the cohesive crack model (Barenblatt 1959), which is a continuum model, the limit of σ_N for $D \to 0$ should be finite. This condition may be satisfied by the following modification having no effect on the large size asymptotic expansion (Bažant 1997):

$$\sigma_N = f_r^\infty \left(1 + \frac{rD_b}{D + l_p}\right)^{1/r} \tag{6}$$

where l_p=material characteristic length that represents the size of the representative volume element of quasi-brittle material. This length equals about two to three aggregate sizes in concrete, and is about the same as the minimum possible spacing of parallel cohesive cracks, or as the effective width of the fracture process zone across the direction of propagation (Bažant and Pang 2006, 2007). Although the value of l_p is empirical, its introduction is necessary for mathematical reasons, as a means to satisfy the asymptotic requirement for $D \to 0$ while ensuring the effect of l_p to be negligible for $D \gg l_p$. Note that l_p differs from the Irwin (1958) characteristic length $l_0 = EG_f/f_t^2$, which characterizes the length of the fracture process zone in the direction of propagation.

Review of Energetic-Statistical Type I Size Effect Law

Since the material strength is random, a macrocrack can initiate at many different points in the structure. Therefore, the size effect of Type I must, for $D/l_p \to \infty$, approach the Weibull statistical size effect. Based on the nonlocal Weibull theory, which combines the energetic and statistical size effects [as conceived by Bažant and Xi (1991) and extended by Bažant and Novák (2000a,b)], the following generalization of Eq. (4) was derived by Bažant and Novák (2000a,c):

$$\sigma_N = f_r^\infty \left[\left(\frac{l_s}{D}\right)^{rn/m} + \frac{rD_b}{D}\right]^{1/r} \tag{7}$$

For small D, this formula converges to Eq. (4), and for large D it converges to Weibull size effect $\sigma_N \propto D^{-n/m}$. A similar statistical generalization of the extended energetic formula in Eq. (6) reads (Bažant 2004; Bazănt et al. 2007)

$$\sigma_N = f_r^\infty \left[\left(\frac{l_s}{l_s + D}\right)^{rn/m} + \frac{rD_b}{l_p + D}\right]^{1/r} \tag{8}$$

where l_s=second (statistical) characteristic length. Although its value is empirical, l_s must be introduced for the same mathematical reasons as already explained for l_p below Eq. (6).

In the size effect of Type II, the randomness of material strength has no significant effect on the mean nominal strength σ_N of the structure, which is the objective of formulating the present universal size effect law. The randomness controls only the statistical distribution of σ_N.

There also exists a Type III size effect (Bažant 2001, 2004), which, however, is so close to Type II that it is hardly distinguishable experimentally, and will not be considered here.

Asymptotic Conditions Required in Different Types of Size Effect

The cohesive crack model has emerged as the most realistic among simple models for quasi-brittle fracture. According to this deterministic model, cohesive fracture must have the following asymptotic properties for vanishing and infinite structure sizes D (Bažant 2001, 2002):

$$\text{For } D \to 0: \quad \sigma_N \propto 1 - k_0 D + O(D^2) \quad \text{(all types)} \tag{9}$$

$$\text{For } D \to \infty \text{ and } \alpha_0 \to 0: \quad \sigma_N \propto 1 + k_1 D^{-1} + O(D^{-2}) \quad \text{(Type I)} \tag{10}$$

For $D \to \infty$ and large α_0:

$$\sigma_N \propto D^{-1/2}[1 - k_2 D^{-1} + O(D^{-2})] \quad \text{(Type II)} \tag{11}$$

where k_0, k_1, k_2=positive constants (Type III is omitted, since it is similar to Type II). The deterministic size effect laws in Eqs. (1) and (6) or Eqs. (2) and (6) satisfy these asymptotic properties. This may be checked by the following second-order approximations (derived by binomial power series expansions):

$$\sigma_0(1 + D/D_0)^{-1/2} \underset{D/D_0 \to 0}{\Rightarrow} \sigma_0(1 - D/2D_0) \tag{12}$$

$$\sigma_0(1 + D/D_0)^{-1/2} \underset{D/D_0 \to \infty}{\Rightarrow} \sigma_0\sqrt{D_0/D}(1 - D_0/2D) \tag{13}$$

$$f_r^\infty\left(1 + \frac{rD_b}{l_p + D}\right)^{1/r} \underset{D/l_p \to 0}{\Rightarrow} f_r^\infty\left(1 + \frac{rD_b}{l_p}\right)^{1/r}\left(1 - \frac{D}{(1 + l_p D_b/r)L_p}\right) \tag{14}$$

$$f_r^\infty\left(1 + \frac{rD_b}{l_p + D}\right)^{1/r} \underset{D/D_b \to \infty}{\Rightarrow} f_r^\infty\left(1 + \frac{D_b}{D}\right) \tag{15}$$

where L_p=constant.

These equations show that the formulas for Types I and II size effects have very different asymptotic properties for $D \to 0$ and $D \to \infty$; see Fig. 1. In Type II, the asymptote for $D \to \infty$ is $\sigma \propto D^{-1/2}$ (a straight line of slope $-1/2$ in double logarithmic scale), which is the size effect of similar cracks for perfectly brittle behavior, governed by LEFM. For $D \to 0$, the size effect curve of σ_N versus D approaches a horizontal asymptote, i.e., the size effect disappears, which is typical of plasticity.

In Type I size effect, the fracture process zone, represented by the boundary layer of distributed cracking, becomes negligible compared to the specimen size when $D \to \infty$. There is then negligible stress redistribution and failure occurs when the maximum elastically calculated stress attains the material strength value f_r^∞. So, for $D \to \infty$, the size effect asymptotically vanishes; see the horizontal asymptote in Fig. 1.

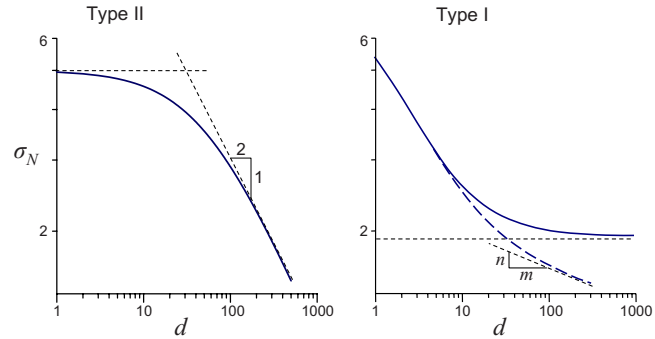

Fig. 1. Two types of size effect behavior; left: Type II; right: Type I

For vanishing structure size $D \to 0$, the cohesive crack model implies that the material strength is mobilized at all the points of the failure surface (or crack). This is equivalent to a crack filled by a perfectly plastic glue, in which case the size effect also vanishes.

After analyzing the asymptotic expansion and determining the asymptotic properties for different size effects, Bažant (1995, 1997) derived a kind of universal size effect formula, which has both Type I (crack initiation) and Type II (large notch) as its limit cases; it reads

$$\sigma_{Nu} = Bf_t'\left[1 + \left(\frac{D}{D_0}\right)^r\right]^{-1/2r}\left[1 + s\frac{2l_f D_0}{(2\gamma l_f + D) + (D_0 + D)}\right]^{1/s} \tag{16}$$

in which r, s, γ=empirical parameters, the values of which can be set approximately as $r=s=1$ [see their discussion in Bažant (1995, 1997)]. The constant boundary layer thickness D_b from Eq. (5) is here replaced by parameter l_f depending on the initial crack (or notch) length

$$l_f = \frac{\langle -g''(\alpha_0)\rangle}{4g'(0)}\kappa c_f \tag{17}$$

In this previous attempt for a universal size effect formula, the first term contains the size effect law for notched specimen, while the second term contains the law for crack initiation. The three-dimensional plot of this formula is given in Fig. 9.1.3 in Bažant and Planas (1998), and also in Fig. 6 of Bažant (1997). There are, however, two shortcomings of Eq. (16), which need to be remedied.

First, Eq. (16) at $\alpha=0$ (crack initiation) converges for $D \to \infty$ to a horizontal asymptote while correctly it should converge to the power law $D^{-n/m}$ of the Weibull statistical size effect, i.e., to a straight line of slope $-n/m$ in the logarithmic scale (m=Weibull modulus—for concrete $m \approx 24$, and for most materials $m=10$ to 50). For fracture specimens and many structures, the fracture geometry scales in two dimensions, i.e., $n=2$, whether or not the structure is scaled in two or three dimensions (i.e., the beam width has no effect on the nominal strength).

Second, Eq. (16) is not smooth, which is evident in its three-dimensional picture shown in Bažant (1995, 1997) and in Bažant and Planas (1998). The surface has a sudden change of slope at $\alpha \approx 0.1$, which is caused by the Macauley bracket in Eq. (17).

Fig. 2 shows the plot of the energy release function and its first and second derivatives for a three-point bend beam with the span-depth ratio of $S/D=4$. This is a standard ASTM test specimen geometry, for which the approximate energy release function is given in handbooks and textbooks (Tada et al. 1985; Murakami

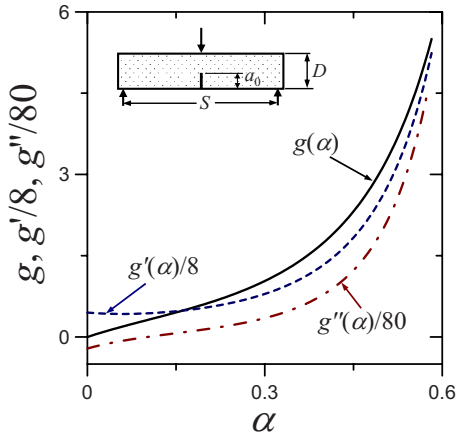

Fig. 2. Energy release function and its first and second order derivatives for a three-point bend beam

1987; Broeck 1988; Kanninen and Popelar 1985). Here we use the following more accurate approximation, derived by Pastor et al. (1995)

$$g(\alpha) = k(\alpha)^2 \quad k(\alpha) = \frac{p_4(\alpha)\sqrt{\alpha}}{(1+2\alpha)(1-\alpha)^{3/2}} \tag{18}$$

in which

$$p_4(\alpha) = 1.900 - \alpha[-0.089 + 0.603(1-\alpha) - 0.441(1-\alpha)^2 + 1.223(1-\alpha)^3] \tag{19}$$

It is typical for this and other geometries that $g''(\alpha_0)$ changes its sign from negative to positive. This occurs at $\alpha_0 \approx 0.1$, which is where the slope of the surface is discontinuous, because of discontinuity of $\langle -g''(\alpha_0)\rangle$.

Universal Size Effect Law

An improved universal size effect law, which has already been reported without derivation at a recent conference (Bažant and Yu 2004), will now be derived in detail, based on asymptotic arguments. For α close to α_0, the dimensionless energy release function $g(\alpha)$ may be approximated by its first three terms of the Taylor series expansion at α_0

$$\sigma_N = \sqrt{\frac{E'G_f}{g(\alpha)D}} \approx \left(\frac{E'G_f}{g_0D + g_0'c_f + (g_0''c_f^2/2D)}\right)^{1/2} \quad (\text{for } D \gg a_0) \tag{20}$$

where, for brevity, $g_0 = g(\alpha_0)$, $g_0' = g'(\alpha_0)$, $g_0'' = g''(\alpha_0)$. For failure at crack initiation (Type I), $g_0 = g(\alpha_0) = g(0) = 0$. To separate this case from Type II, for which $g_0 > 0$ [and the third term with g_0'' must be separated, or else the opposite asymptotic properties of cohesive crack model for $D \to 0$ could not be matched; Bažant (2001)], the last expression, applicable only for large enough D, may be rearranged as follows:

$$\sigma_N = \left(\frac{E'G_f}{g_0D + g_0'c_f}\right)^{1/2}\left(1 + \frac{g_0''c_f^2}{2D(g_0D + g_0'c_f)}\right)^{-1/2} \approx \sqrt{\frac{E'G_f}{g_0D + g_0'c_f}}\left(1 - \frac{rg_0''c_f^2}{4D(g_0D + g_0'c_f)}\right)^{1/r} \tag{21}$$

Here we inserted an arbitrary coefficient r. This insertion is permitted, and in fact required for generality, because the asymptotic expansions of the last two expressions in terms of powers of (a_0/D) are independent of r up to the quadratic terms, and because there is no reason for r to be 1. However, note that although both previous expressions coincide exactly for $r=-2$, a negative r could not be used because typically $g_0'' < 0$, which would make σ_N imaginary.

When the specimen has a deep notch (i.e., when a_0 is not negligible compared to the cross section dimension), the preceding formula must be made identical to Eq. (2). So, g_0'' must be replaced by a function smoothly approaching 0 when α_0 becomes large enough (i.e., when a_0 becomes non-negligible compared to the cross section dimension). To this end, we can replace g_0'' by $g_0''e^{-k\alpha_0^q}$, where k and q=positive empirical constants controlling the transition. Eq. (21) may, thus, be rewritten as

$$\sigma_N = \sqrt{\frac{E'G_f}{g_0D + g_0'c_f}}\left(1 - \frac{rc_f^2g_0''e^{-k\alpha_0^q}}{4D(g_0D + g_0'c_f)}\right)^{1/r} \tag{22}$$

To check the general applicability of Eq. (22), note the following two opposite asymptotic cases of crack initiation and of deep notch:

For large α_0:

$$e^{-k\alpha_0^q} \to 0 \text{ and } \sigma_N = \sqrt{\frac{E'G_f}{g_0D + g_0'c_f}} \tag{23}$$

For $\alpha_0 \to 0$:

$$\sigma_N = \sqrt{\frac{E'G_f}{g_0'c_f}}\left(1 + \frac{\langle -g_0''\rangle}{4g_0'}\frac{rc_f}{D}\right)^{1/r} \tag{24}$$

For $D \to 0$, Eq. (22) gives an infinite nominal strength σ_N, this would violate the small-size asymptotic limit of the cohesive crack model, which is always finite. We can circumvent this problem by replacing $4D$ by $4(D+l_p)$ in Eq. (22), where l_p gives the center of transition to a horizontal asymptote and represents a material characteristic length, which should be approximately equal to the maximum aggregate size (Bažant and Pang 2006, 2007). For $D/l_p \to \infty$, Eq. (22) is approached asymptotically. With this modification, the foregoing asymptotic cases satisfy the asymptotic conditions of the cohesive crack model [Eqs. (23) and (24)]; hence, Eq. (22) satisfies them too.

Finally, by comparison with test data, it seems possible to set $q=2$. So, if only the deterministic-energetic size effect is considered, and if $g_0'' < 0$, the new deterministic universal size effect law for mean σ_N may be expressed as follows:

$$\sigma_N = \sqrt{\frac{E'G_f}{g_0'c_f + g_0D}}\left(1 - \frac{rc_f^2g_0''e^{-k\alpha_0^2}}{4(l_p + D)(g_0D + g_0'c_f)}\right)^{1/r} \tag{25}$$

Fig. 3 shows a three-dimensional plot of Eq. (25) for the previously considered three-point bend beam with $S/D=4$, and for the following typical material parameters of concrete: $c_f=200$ mm, $l_p=100$ mm, $E'=28.0$ GPa, $G_f=70$ N/m, and $f_t'=3.0$ MPa (with empirical constants $r=1$ and $k=115$). The cross sections of this surface for constant α_0 represent the size effect curves, Type I for $\alpha_0=0$ and Type II for large α_0, and it can be clearly seen that a smooth transition between these two types is achieved (thanks to the suppression of discontinuous Macauley brackets with the multiplier $e^{-k\alpha^2}$).

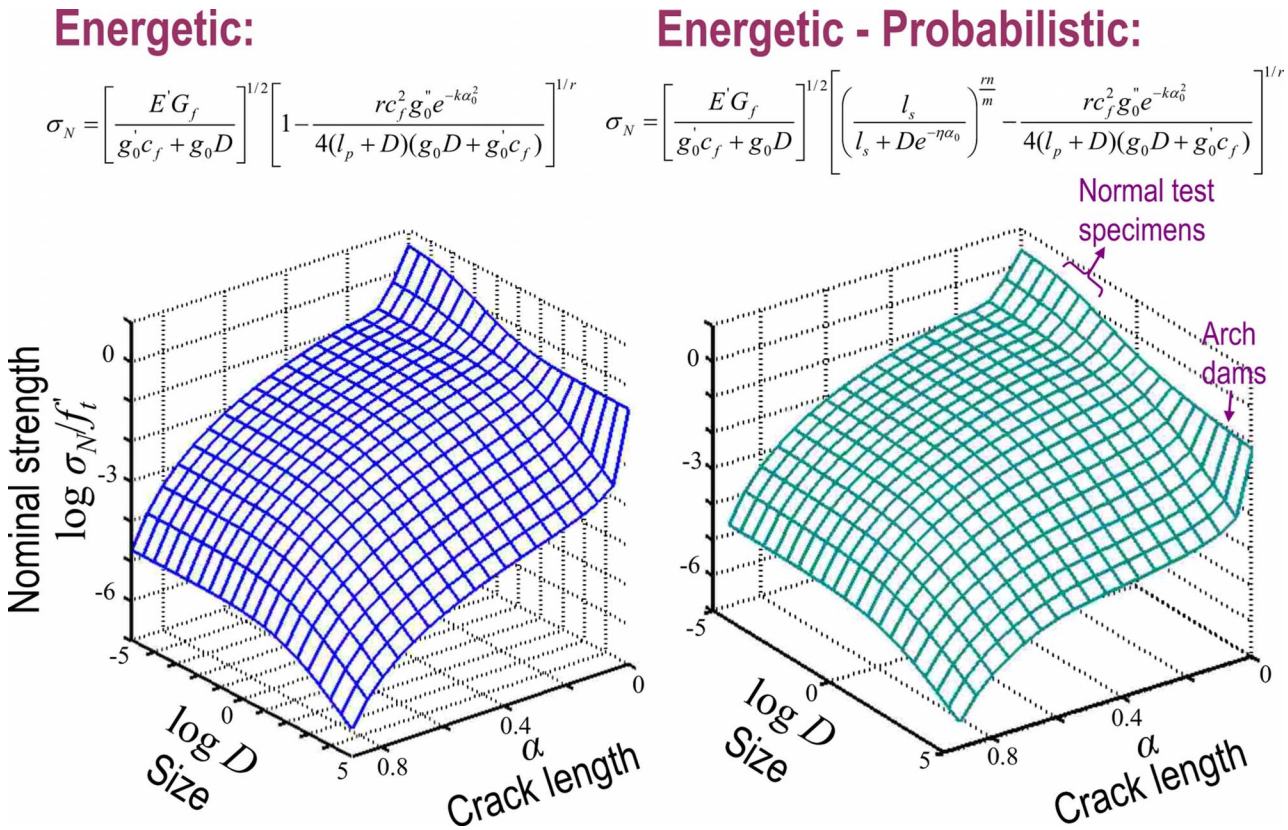

Fig. 3. Improved universal size effect law; left: without Weibull statistics; right: with Weibull statistics

To capture the Weibull statistical size effect, which becomes significant for very large unnotched structures, a statistical part, analogous to Eq. (7), may be superposed on Eq. (25)

$$\sigma_N = \sqrt{\frac{E'G_f}{g_0'c_f + g_0 D}}\left[\left(\frac{l_s}{l_s + De^{-\eta\alpha_0^2}}\right)^{rn/m} - \frac{rc_f^2 g_0'' e^{-k\alpha_0^2}}{4(l_p + D)(g_0 D + g_0' c_f)}\right]^{1/r} \tag{26}$$

(if $g_0'' < 0$). This final formula, representing a general universal size effect law, satisfies three asymptotic conditions for size effect at crack initiation ($\alpha_0 = 0$):

- For no notch, $\alpha_0 \to 0$ ($g_0 \to 0$), and for small enough sizes, $D \ll l_s$, Eq. (26) asymptotically approaches the deterministic-energetic formula

$$\sigma_N = f_r^\infty\left(1 + \frac{rD_b}{l_p + D}\right)^{1/r} \tag{27}$$

- For no notch, $\alpha_0 \to 0$ ($g_0 \to 0$), and for large enough sizes, $D \gg \mathrm{Max}(l_s, l_p)$, Eq. (26) asymptotically approaches the Weibull type size effect

$$\sigma_N = f_r^\infty (l_s/D)^{n/m} \tag{28}$$

- For $m \to \infty$ and any size D, Eq. (26) coincides with deterministic-energetic formula (25).

The three-dimensional surface of the universal size effect law in Eq. (26) is shown in Fig. 3. The surface is seen to be smooth.

Way of Experimental Identification of Material Parameters

In Eq. (26), there are seven free parameters: m, r, c_f, l_p, l_s, k, and η. First, c_f (as well as $E'G_f$) may be identified from Type II size effect tests on scaled notched specimens.

Knowing c_f, parameters f_r^∞, r, and D_b may then be identified by fitting Eq. (27) to the test results for modulus of rupture (or flexural strength) at different sizes, with a sufficient size range (they may also be obtained by discrete particle simulations), after that $\bar{c}_f$ may be solved from Eq. (4), and then $\kappa = \bar{c}_f / c_f$ according to Eq. (5). Parameter l_p matters only for extrapolation to zero size and can be obtained only by calculating the zero size limit with the cohesive crack model, although estimating it as equal to the maximum aggregate size seems adequate.

Knowing f_r^∞, parameters m (Weibull modulus) and l_s (Weibull scaling parameter) can be identified by fitting Eq. (28) to test data on the statistical size effect, which can be obtained directly only on very large unnotched specimens.

Parameters η and k can be experimentally identified only by testing the Type I-Type II transition, although approximately one can probably assume that, for relative notch depth $\alpha_0 = 0.1$, the values of $k\alpha_0^2$ and $\eta\alpha_0^2$ are 1, which gives $k \approx \eta \approx 100$.

Special Case of Crack Length Effect, Contrasted with Duan-Hu Formula

A semiempirical size effect formula for the maximum load dependence on the crack length at constant size D was proposed by

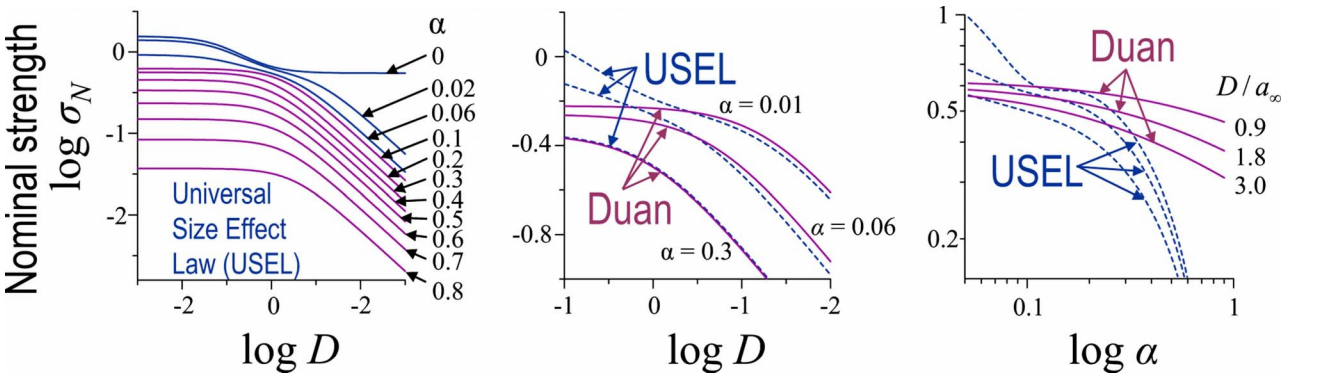

Fig. 4. Profiles obtained from improved universal size effect formulas [Eqs. (25) and (26)] compared to Duan et al. (2006) approximation

Duan et al. (2002, 2003, 2004, 2006) and by Hu and Wittmann (2000)

$$\sigma_n = \sigma_0(1 + a/a_\infty^*)^{-1/2} \tag{29}$$

Here $\sigma_0 = f_t'$ is assumed for small three-point bend specimens, and a_∞^* is a certain constant representing the maximum tensile stress in the ligament based on a linear stress distribution over the ligament, and (in Duan and Hu's notation) $\sigma_n = \sigma_N / A(\alpha_0)$ where $A(\alpha_0) = (1-\alpha_0)^2$ for three-point bend specimens.

Duan and Hu's formula ought to be equivalent to the profile of the universal size effect law (Fig. 4) at constant size D, scaled by the ratio $\sigma_n/\sigma_N = 1/A(\alpha_0)$. The curve of that formula [Eq. (29)] approaches the asymptotic case for $\alpha_0 \to 0$ with a horizontal asymptote (in $\log \alpha_0$ scale). However, the asymptotic limit for $\alpha_0 \to 0$ is independent of the structure size. So we conclude that, according to Duan and Hu's formula, there is no size effect for failure at crack initiation from a smooth surface (Type I), e.g., in the tests for flexural strength (or modulus of rupture). This is an unrealistic feature of their formula, conflicting with extensive experimental evidence for unnotched beams (Bažant and Li 1995; Bažant 1998, 2001, 2002; Bažant and Novák 2000b,c).

Except within a small portion of the ranges of crack length and structure size, the profiles at constant D and α given by the present universal size effect law are not matched by Duan and Hu's formula (after its conversion from σ_n to σ_N). So, even though Eq. (29) seems to work for a narrow range of typical deep-notched fracture specimens of concrete, it does not have broad applicability. This is not surprising because Eq. (29) is not based on the energy release function $g(\alpha)$ in the sense of equivalent LEFM, and because the material strength f_t' in the cohesive crack model, which is a constant, is not applicable to very short or vanishing notches, for which the tensile strength must exhibit the Type I size effect. For very short cracks or notches ($\alpha_0 < 0.1D$), the tensile strength must be treated in the way of either the Guinea et al. (1994a, b) method, or a similar method by Bažant et al. (2002) called the zero-brittleness method.

Conclusions

- There are two simple asymptotic types of size effect in quasibrittle fracture: Type I, which occurs in failures at crack initiation from a smooth surface, and Type II, which occurs in failures starting from a deep notch or crack. To describe the continuous transition between these two types of size effect, an improved universal size effect law is required.
- The improved universal size effect law can be derived by matching asymptotic series expansions for six basic limit cases: (1) the asymptotic behaviors for very small and very large sizes (which can be captured by the first two nonzero terms of the expansion for each case); (2) the large notch and vanishing notch behaviors; and (3) the energetic and statistical parts of size effect.
- In contrast to a previous formulation (Eq. (16)), the present universal size effect law achieves a smooth transition between the Types I and II size effects.
- In contrast also to the previous formulation, the classical Weibull statistical size effect is captured as a limiting case of proposed universal size effect law.
- The dependence of the nominal strength of structure on the notch depth at constant specimen size is a special case of the present universal size effect law. This dependence is more realistic than an empirical formula previously proposed by Duan and Hu. That formula does not have realistic asymptotics and conflicts with the Type I size effect law, which must be the limit case for a vanishing notch depth.

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