Errors Caused by Non-Work-Conjugate Stress and Strain Measures and Necessary Corrections in Finite Element Programs

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Many finite element programs including standard commercial software such as ABAQUS *use an incremental finite strain formulation that is not fully work-conjugate, i.e., the work of stress increments on the strain increments does not give a second-order accurate expression for work. In particular, the stress increments based on the Jaumann rate of Kirchhoff stress are work-conjugate with the increments of the Hencky (logarithmic) strain tensor but are paired in many finite element programs with the increments of Green's Lagrangian strain tensor. Although this problem was pointed out as early 1971, a demonstration of its significance in realistic situations has been lacking. Here it is shown that, in buckling of compressed highly orthotropic columns or sandwich columns that are very "soft" in shear, the use of such nonconjugate stress and strain increments can cause large errors, as high as 100% of the critical load, even if the strains are small. A similar situation may arise when severe damage such as distributed cracking leads to a highly anisotropic tangential stiffness matrix, or when axial cracks between fibers severely weaken a uniaxial fiber composite or wood. A revision of these finite element programs is advisable, and will in fact be easy—it will suffice to replace the Jaumann rate with the Truesdell rate. Alternatively, the Green's Lagrangian strain could be replaced with the Hencky strain.* [DOI: 10.1115/1.4000916]

1 Introduction

The finite strain tensors used in practice all belong to the class of Doyle–Erickson tensors $\varepsilon=(\mathbf{F}^{\mathbf{T}}\mathbf{F}^{m/2}-\mathbf{I})/m$, where $\mathbf{I}$ and $\mathbf{F}$ are the unit tensor and the deformation gradient tensor, respectively, and m is a parameter, which is equal to 2 for the Green's Lagrangian strain tensor, equal to 1 for Biot strain tensor, equal to −2 for Almansi Lagrangian strain tensor, and tends to 0 for the logarithmic (Hencky) strain tensor [1,2]. The work-conjugate objective stress rates, giving the correct second-order incremental work, are the Truesdell rate for $m=2$, Biot rate for $m=1$, Jaumann rate of Kirchhoff stress for $m\rightarrow 0$, and Lie derivative of Kirchhoff stress for $m=-1$ (Table 1 in [1], Table 11.4.1 in [2]). The transition between the formulations for one or another m value was shown to require that the tangential moduli tensor C_{ijkl} be transformed [1,2] as follows:

$$C_{ijkl}^{(m)}=C_{ijkl}^{(2)}+\frac{1}{2}(2-m)(S_{ik}\delta_{jl}+S_{jk}\delta_{il}+S_{il}\delta_{jk}+S_{jl}\delta_{ik}) \quad (1)$$

where the subscripts refer to Cartesian coordinates $(i=1,2,3)$, S_{ij} is the Cauchy stress tensor, δ_{ij} is the unit tensor (Kronecker delta), and the superscripts (m) and (2) refer to the m-value. In particular, if the tangent moduli are constant in one formulation, they vary linearly with S_{ij} in all other formulations.

For several decades before 1971, there were arguments regarding the proper formulation of incremental deformations and stability criteria for three-dimensional bodies under initial stress. The arguments were resolved in 1971 by showing that various formulations are equivalent provided that work-conjugate increments of stress and finite strain are used [2] and the transformation in Eq. (1) is observed. The different mathematical formulations were thus unified in a general treatment of the infinitesimal elastic stability problem [1]. It was shown that when a certain finite strain measure is selected to describe the incremental deformation, then the associated conjugate incremental stress and the corresponding constitutive model must be used in order to recover the same end result that is independent of the choice of the objective stress increment and the finite strain measures. The tangential elastic moduli transformations that are necessary for the equivalence of the different formulations were presented [1]. Through an example, the apparent difference between the well-known Engesser and Harringx formulas for shear buckling of beam-columns was reconciled. Recently, the problem of buckling of soft-in-shear structures such as sandwich plates with very soft cores or general highly orthotropic bodies under triaxial initial stress has been clarified [3,4] and the conditions under which the incremental moduli can be assumed to be constant have been delineated.

The lack of work-conjugate stress and strain increments nevertheless remains to be an aspect that plagues various finite element (FE) programs, including commercial software such as ABAQUS. The present objective is to show that the consequences can be serious. This is done by the example of a two-dimensional (2D) analysis of buckling of a uniaxially compressed orthotropic column in plane strain that is very soft-in-shear, i.e., has a shear modulus that is very small compared with the axial modulus, which is a realistic situations for sandwich structures typified by laminate sandwich plates made with Divynicell 100 foam. To this end, the proper FE equations for the buckling problem are derived, and different FE formulations for the equilibrium equation governing the bifurcation instability of the orthotropic column are discussed. The predictions from the different formulations are mutually compared. Subsequently, the results obtained when the correct conjugate relationships are not preserved are examined and it is shown that large errors (as high as 100%) can be incurred.

2 Numerical Example Demonstrating Error Magnitude

Let us compare two finite element models (FEMs) for the eigenvalue problems of bifurcation buckling. The first corresponds to the Green's Lagrangian strain measure $(m=2)$, Eq. (A20), while the second uses the Jaumann stress rate with a constant modulus. The latter case is deliberately chosen to show the errors that can be incurred by incorrect choices of stress and strain increments that are not work-conjugate.

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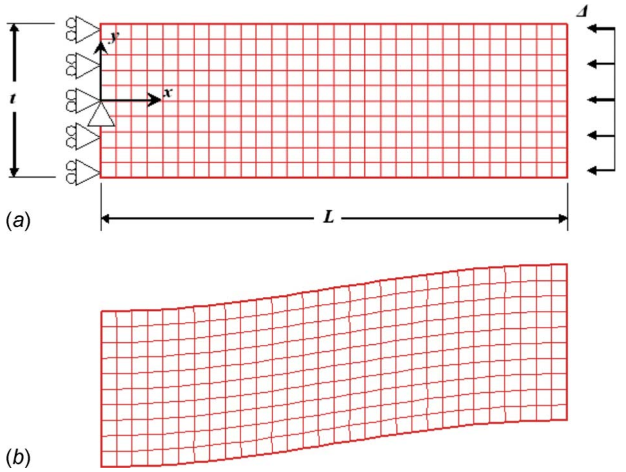

Fig. 1 (*a*) Finite element model of a orthotropic column and (*b*) typical deformed shape of the column in the eigenbuckling problem

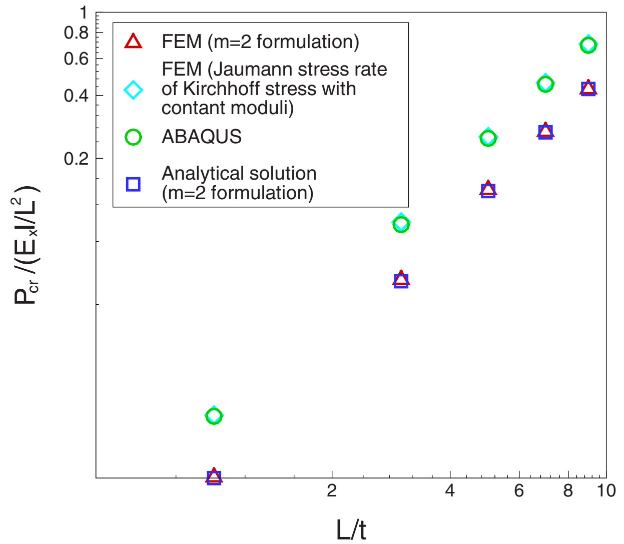

Fig. 2 Comparison of buckling loads from different formulations

The example chosen for study (Fig. 1) is an orthotropic column consisting of an elastic 2D orthotropic continuum. Four-noded bilinear elements are used to generate a sufficiently fine mesh. The material properties of the column are listed in Table 1. It is noted that the axial modulus is 2000 times the shear modulus, which is in the range of values encountered in typical sandwich panels. The commercial finite element analysis (FEA) package ABAQUS is also used in the present study for the purpose of comparison. In the numerical analysis of the eigenvalue problem with ABAQUS, the column is meshed with CPE8, eight-node plane strain elements.

The results from various FE models are compared in Fig. 2. The analytical results for $m=2$ are also shown in Fig. 2 in order to validate the results from the finite element models. The analytical result that corresponds to the buckling of an axially loaded orthotropic column according to the formulation with $m=2$ has been obtained by using the methods outlined in Ref. [5]. In that paper, the global periodic buckling and local buckling (as well as the nonperiodic edge buckling) of a sandwich beam in plane strain is treated analytically using 2D elasticity for modeling the face sheet and the core. The buckling loads in Fig. 2 are normalized by $E_{xx}I/L^2$, where I is the centroidal moment of inertia.

The buckling loads obtained from "FEM ($m=2$)" and "analytical solution ($m=2$)" are virtually identical, which is expected since they are based on the same formulation. Note that the buckling loads from "FEM (Jaumann stress rate with the constant moduli)" and ABAQUS are identical and different from the first two buckling loads for different column lengths. Furthermore, the latter buckling loads are consistently higher than the present analytical buckling loads and the corresponding FE results. The differences in buckling loads are seen to increase as the slenderness ratio (L/t) of the column decreases.

When the relative errors between the buckling loads from FEM ($m=2$) and FEM (Jaumann stress rate with the constant moduli) are compared in Fig. 3, it is clearly seen that the error becomes large when the slenderness ratio of column is small, i.e., the errors become large in the regime where the shear deformations cause a large reduction in the buckling load.

As described in its theoretical manual, ABAQUS uses the tangential moduli $C^{(J)}_{ijkl}=C_{ijkl}$ for $C^{(J)}_{ijkl}$ appearing in the first term of Eq. (A27), instead of $C^{(J)}_{ijkl}=C^{(0)}_{ijkl}-\delta_{kl}\sigma_{ij}$. Consequently, in the ABAQUS FE formulation, the terms associated with the volume integral $\int_V \sigma^0_{ij}\dot{e}_{ik}\delta\dot{e}_{kj}dV$ and $\int_V \sigma'_{ij}\dot{e}_{ik}\delta\dot{e}_{kj}dV$ in $\mathbf{K}^0$ and $\mathbf{K}'$ of Eq. (A20) are nonvanishing. When the body is in compression, these extra terms produce, through the matrices $\mathbf{K}^0$ and $\mathbf{K}'$, additional stiffness compared to Eq. (A20). This leads in the eigenvalue problem to buckling loads that are much too high, as shown in Fig. 2. A switch from the Jaumann rate to the Truesdell rate is required.

Alternatively, the Jaumann rate of Krichhoff stress could be retained if Green's Lagrangian strain were replaced by the Hencky (logarithmic strain), which has the advantage of directly giving the principal stretch logarithms called the "true" strains, which are finite strain measures favored in materials science. Computation

Table 1 Material properties of the orthotropic column

Material property	Value
E_{xx}	$2000\times G_{xy}$ GPa
E_{yy}	$2\times G_{xy}$ GPa
G_{xy}	7.17 GPa
ν_{xy}	0.29
t	10 mm

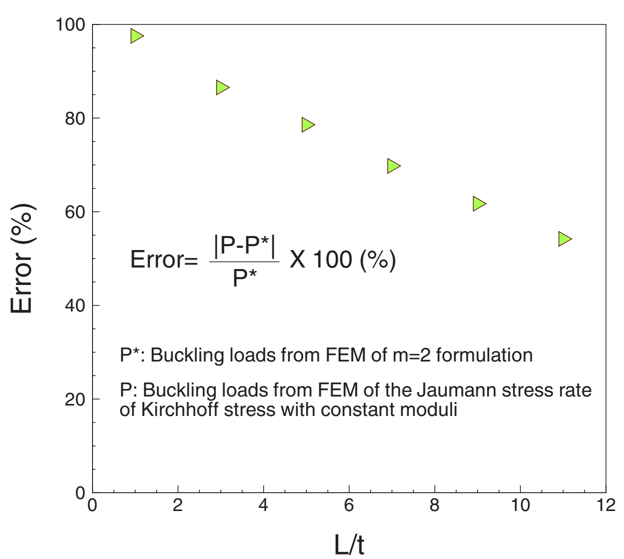

Fig. 3 Relative error of the buckling loads

of the Hencky strain tensor used to be an obstacle but a highly accurate easily computable approximation has been presented in Ref. [6].

3 Comment on Jaumann Rate of Cauchy Stress

In addition to the extra terms that have been discussed, one more aspect deserves attention. Bazant [1] showed that the Jaumann rate of Cauchy stress is not energetically conjugate to any admissible finite strain measure. This deficiency causes the equilibrium equations expressed in terms of the Jaumann rate of Cauchy stress to be not suitable for evaluating stability, unless the material is incompressible [1,2]. The Jaumann rate of Cauchy stress can be regarded as a special case of the objective stress rate corresponding to the Biezeno–Hencky formulation $m=0$,

$$\hat{\sigma}_{ij}^{(0)} = \dot{\sigma}_{ij} - \sigma_{kj}\dot{\omega}_{ik} + \sigma_{ik}\dot{\omega}_{kj} + \sigma_{ij}v_{k,k} \tag{2}$$

if the last term $\sigma_{ij}v_{k,k}$ is omitted. The Jaumann stress rate considers only the rotations of the coordinate frame and neglects coordinate scaling. This is analogous to the rotation-based expression for finite strain sometimes used for thin-walled structures [7]. For such structures, however, no error arises compared with the Green's Lagrangian formulation.

4 Orthotropically Damaged Material and Practical Situations With Conjugacy Problem

Material damage such as a system of dense parallel microcracks can also lead to a highly orthotopic or anisotropic trangential stiffness matrix of the macroscopic homogenizing continuum. With a high degree of damage, the elastic modulus in the direction normal to the microcrack planes and the shear moduli on these planes can be one or several orders of magnitude smaller than the elastic moduli in the directions parallel to the microcrack planes. This situation commonly arises with realistic constitutive models for quasibrittle materials, e.g., concretes, rocks and ceramics. In unidirectional fiber composites or wood under axial compression, microcracks form parallel to the fibers, and the action of compressed fibers further increases the degree of orthotropy of the macroscopic homogenizing continuum.

These situations, which are not uncommon in numerical simulations of failure of these materials, are similar to the previous example of a soft-in-shear sandwich. The use of non-work-conjugate stresses and strains may then lead to errors of a similar magnitude.

5 Closing Comment

Significant errors due to the use of nonconjugate stress and strain increments in finite element programs are, of course rare, and get manifested only in special situations. One such situation, which is of practical significance, is addressed in this short paper. Other cases, such as the critical load for shear buckling of a sandwich plate with a very soft core (Divinycell foam cores) or of an elastomeric bearing for bridges or seismic isolation, consisting of alternating lamina of steel and soft elastomer, have also shown large errors. Further similar situations arise in isotropic or orthotropic materials when they are highly damaged by a system of parallel microcracks, rendering their tangential stiffness matrix highly orthotropic. Such situations often occur on approach to the peak load of structures made of concrete, rock, ceramics, fiber composites or wood.

Consequently, caution must be exercised in the use of commercial programs such as ABAQUS in such special situations. On the other hand, the switch to a fully work-conjugate formulation would be easy and is recommended.

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Appendix: Analytical Formulation of the Problem

To provide a more detailed understanding, it is helpful to present the general 3D formulation of the buckling problem of the 2D column with the details of the associated FE formulation. The general form of the equilibrium equations in the buckled state may be obtained from the principle of virtual work as

$$\int_V S_{ij}\frac{\partial \delta v_i}{\partial X_j}dV = \int_\Omega t_i \delta v_i d\Omega + \int_V b_i \delta v_i dV \tag{A1}$$

where S_{ij} is the first Piola–Kirchhoff stress, δv_i is the virtual velocity field, t_i is the nominal traction on the boundary Ω of the initial state, b_i is the body force per unit volume of the base state, and V is the volume of the body in its reference configuration. The corresponding rate form of Eq. (A1) is

$$\int_V \dot{S}_{ij}\frac{\partial \delta v_i}{\partial X_j}dV = \int_\Omega \dot{t}_i \delta v_i d\Omega + \int_V \dot{b}_i \delta v_i dV \tag{A2}$$

The left hand side of Eq. (A2) can be expressed in terms of the rate of Kirchhoff stress, $\dot{\tau}$ so that it is written as

$$\int_V \dot{S}_{ij}\frac{\partial \delta v_i}{\partial X_j}dV = \int_V \dot{\tau}_{ij}\frac{\partial \delta v_i}{\partial x_j} - \left(\tau_{ik}\frac{\partial v_j}{\partial x_k}\right)\frac{\partial \delta v_i}{\partial x_j}dV \tag{A3}$$

Equation (A3) is obtained from Eq. (A2) using the relations

$$S_{ij} = \tau_{ik}\frac{\partial X_j}{\partial x_k}$$
$$\left(\frac{\partial \delta x_i}{\partial X_j}\right)^{\cdot} = \frac{\partial \delta v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j} \tag{A4}$$

Since the deformation during the transition from the unbuckled to the buckled state is infinitesimal, the Kirchhoff stress and its rate can be approximated as

$$\tau_{ij} = J\sigma_{ij} \simeq \sigma_{ij}$$
$$\dot{\tau}_{ij} = J\dot{\sigma}_{ij} + \dot{J}\sigma_{ij} \simeq \dot{\sigma}_{ij} + v_{k,k}\sigma_{ij} \tag{A5}$$

where $J=1+u_{k,k}$ (Jacobian of the transformation).

It was shown [3,8] that if the elastic moduli in a column are kept constant (i.e., stress independent), the Green's Lagrangian strain and its associated formulation must be used. In this case, $\dot{\sigma}_{ij}$ may be rewritten using Truesdell's stress rate

$$\dot{\sigma}_{ij} = \hat{\sigma}_{ij} + \sigma_{kj}v_{i,k} + \sigma_{ki}v_{j,k} - \sigma_{ij}v_{k,k} \tag{A6}$$

where the superscript "∧" denotes the stress rate. The reason for choosing Truesdell's stress rate is that it is work-conjugate to the Green's Lagrangian strain tensor. The substitution of Eq. (A6) into Eq. (A3) yields

$$\int_V \dot{S}_{ij}\frac{\partial \delta v_i}{\partial X_j}dV = \int_V \hat{\sigma}_{ij}\delta v_{i,j}dV + \int_V (\sigma_{kj}v_{i,k} + \sigma_{ki}v_{j,k} - \sigma_{ik}v_{j,k})\delta v_{i,j}dV \tag{A7}$$

Using the symmetry properties of stress $\sigma_{ij}=\sigma_{ji}$ we can simplify the foregoing equation to

$$\int_V \dot{S}_{ij}\frac{\partial \delta v_i}{\partial X_j}dV = \int_V \hat{\sigma}_{ij}\delta\dot{e}_{ij}dV + \int_V \sigma_{kj}v_{i,k}\delta v_{i,j}dV \tag{A8}$$

where $\dot{e}_{ij}=(1/2)(v_{i,j}+v_{j,i})$ is the linearized small strain in terms of the velocity fields.

The rates of the surface tractions t_i and the body forces b_i in Eq. (A2) can be expressed as

$$\dot{t}_i = \frac{\partial t_i}{\partial F_{jk}}\dot{F}_{jk}$$
$$\dot{b}_i = \frac{\partial b_i}{\partial F_{jk}}\dot{F}_{jk} \tag{A9}$$

since t_i and b_i are depend on the change of geometry through the deformation gradient F_{ij}. When the initial and the current configurations are almost identical, we have

$$\frac{\partial t_i}{\partial F_{jk}}\dot{F}_{jk} \simeq \frac{\partial t_i}{\partial F_{jk}}v_{j,k}$$
$$\frac{\partial b_i}{\partial F_{jk}}\dot{F}_{jk} \simeq \frac{\partial b_i}{\partial F_{jk}}v_{j,k} \tag{A10}$$

According to Eqs. (A8) and (A10) and the constitutive model corresponding to Truesdell's rate of the Cauchy stress

$$\hat{\sigma}_{ij} = C_{ijkl}\dot{e}_{kl} \tag{A11}$$

Equation (A2) now becomes

$$\int_V \delta\dot{e}_{ij}C_{ijkl}\dot{e}_{kl}dV + \int_V \sigma_{kj}v_{i,k}\delta v_{i,j}dV - \int_\Omega \delta v_i\frac{\partial t_i}{\partial F_{jk}}v_{j,k}d\Omega - \int_V \delta v_i\frac{\partial b_i}{\partial F_{jk}}v_{j,k}dV = 0 \tag{A12}$$

To formulate the eigenvalue problem of buckling, the stress, the surface traction, and the body force in Eq. (A12) are decomposed into initial and perturbed quantities, such that

$$\sigma_{ij} = \sigma_{ij}^0 + \lambda\sigma'_{ij}$$
$$t_i = t_i^0 + \lambda t'_i \tag{A13}$$
$$b_i = b_i^0 + \lambda b'_i$$

where λ is a constant multiplier yet to be determined. By substituting Eq. (A13) into Eq. (A12) and rearranging the terms, we obtain the final equation for the buckling problem

$$\int_V \delta\dot{e}_{ij}C_{ijkl}\dot{e}_{kl}dV + \int_V \sigma_{kj}^0 v_{i,k}\delta v_{i,j}dV - \int_\Omega \delta v_i\frac{\partial t_i^0}{\partial F_{jk}}v_{j,k}d\Omega - \int_V \delta v_i\frac{\partial b_i^0}{\partial F_{jk}}v_{j,k}dV - \lambda\left[\int_V \sigma'_{kj}v_{i,k}\delta v_{i,j}dV - \int_\Omega \delta v_i\frac{\partial t'_i}{\partial F_{jk}}v_{j,k}d\Omega - \int_V \delta v_i\frac{\partial b'_i}{\partial F_{jk}}v_{j,k}dV\right] = 0 \tag{A14}$$

When the velocity fields are discretized as

$$\mathbf{v} = \mathbf{N}\dot{\mathbf{q}} \tag{A15}$$

with $\mathbf{N}$ being the assumed shape functions, each term of Eq. (A14) is transformed into,

$$\int_V \delta\dot{e}_{ij}C_{ijkl}\dot{e}_{kl}dV = \delta\dot{\mathbf{q}}^{\mathrm{T}}\left[\int_V \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}dV\right]\dot{\mathbf{q}} \tag{A16}$$

$$\int_V \sigma_{kj}v_{i,k}\delta v_{i,j}dV = \delta\dot{\mathbf{q}}^{\mathrm{T}}\left[\int_V \left(\frac{\partial\mathbf{N}}{\partial\mathbf{x}}\right)^{\mathrm{T}}\sigma\frac{\partial\mathbf{N}}{\partial\mathbf{x}}dV\right]\dot{\mathbf{q}} \tag{A17}$$

$$\int_\Omega \delta v_i\frac{\partial t_i}{\partial F_{jk}}v_{j,k}d\Omega = \delta\dot{\mathbf{q}}^{\mathrm{T}}\left[\int_\Omega \mathbf{N}^{\mathrm{T}}\frac{\partial\mathbf{t}}{\partial\mathbf{q}}d\Omega\right]\dot{\mathbf{q}} \tag{A18}$$

$$\int_V \delta v_i\frac{\partial b_i}{\partial F_{jk}}v_{j,k}d\Omega = \delta\dot{\mathbf{q}}^{\mathrm{T}}\left[\int_V \mathbf{N}^{\mathrm{T}}\frac{\partial\mathbf{b}}{\partial\mathbf{q}}dV\right]\dot{\mathbf{q}} \tag{A19}$$

Here, $\mathbf{B}$ is the derivative of $\mathbf{N}$ with respect to $\mathbf{x}$. Thus, the FE formulation for the eigenvalue buckling problem reduces to

$$[\mathbf{K}^0 + \lambda\mathbf{K}']\dot{\mathbf{q}} = 0 \tag{A20}$$

where

$$\mathbf{K}^0 = \int_V \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}dV + \int_V \left(\frac{\partial\mathbf{N}}{\partial\mathbf{x}}\right)^{\mathrm{T}}\sigma^0\frac{\partial\mathbf{N}}{\partial\mathbf{x}}dV - \int_\Omega \mathbf{N}^{\mathrm{T}}\frac{\partial\mathbf{t}^0}{\partial\mathbf{q}}d\Omega - \int_V \mathbf{N}^{\mathrm{T}}\frac{\partial\mathbf{b}^0}{\partial\mathbf{q}}dV \tag{A21}$$

$$\mathbf{K}' = \int_V \left(\frac{\partial\mathbf{N}}{\partial\mathbf{x}}\right)^{\mathrm{T}}\sigma'\frac{\partial\mathbf{N}}{\partial\mathbf{x}}dV - \int_\Omega \mathbf{N}^{\mathrm{T}}\frac{\partial\mathbf{t}'}{\partial\mathbf{q}}d\Omega - \int_V \mathbf{N}^{\mathrm{T}}\frac{\partial\mathbf{b}'}{\partial\mathbf{q}}dV \tag{A22}$$

The Jaumann rate of Kirchhoff stress $\hat{\tau}_{ij}^{(J)}$ has been favored for buckling problems in various commercial FE packages such as ABAQUS. The relation between the Kirchhoff stress rate $\dot{\tau}_{ij}$ and $\hat{\tau}_{ij}^{(J)}$ is

$$\hat{\tau}_{ij}^{(J)} = \dot{\tau}_{ij} - \tau_{kj}\dot{\omega}_{ik} + \tau_{ik}\dot{\omega}_{kj} \tag{A23}$$

where

$$\dot{\omega}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \tag{A24}$$

Therefore, the left hand side of Eq. (A2) becomes

$$\int_V \dot{S}_{ij}\frac{\partial\delta v_i}{\partial X_j}dV = \int_V \hat{\tau}_{ij}^{(J)}\delta\dot{e}_{ij} + \tau_{ij}(\delta v_{i,k}v_{k,j} - 2\dot{e}_{ik}\delta\dot{e}_{kj})dV \tag{A25}$$

Now with the right hand side of Eq. (A2) and with Eq. (A25), Eq. (A2) is rewritten as

$$\int_V \hat{\tau}_{ij}^{(J)}\delta\dot{e}_{ij}dV + \int_V \tau_{ij}(\delta v_{i,k}v_{k,j} - 2\dot{e}_{ik}\delta\dot{e}_{kj})dV - \int_\Omega \frac{\partial t_i}{\partial F_{jk}}\dot{F}_{jk}d\Omega - \int_V \frac{\partial b_i}{\partial F_{jk}}\dot{F}_{jk}dV = 0 \tag{A26}$$

Following the assumptions previously made for the small perturbation, Eq. (A26) becomes

$$\int_V \delta\dot{e}_{ij}C_{ijkl}^{(J)}\dot{e}_{kl}dV + \int_V \sigma_{ij}(\delta v_{i,k}v_{k,j} - 2\dot{e}_{ik}\delta\dot{e}_{kj})dV - \int_\Omega \frac{\partial t_i}{\partial F_{jk}}v_{j,k}d\Omega - \int_V \frac{\partial b_i}{\partial F_{jk}}v_{j,k}dV = 0 \tag{A27}$$

where $C_{ijkl}^{(J)} = C_{ijkl}^{(0)} - \delta_{kl}\sigma_{ij}$ [1]. This last equation represents the constitutive model corresponding to the Jaumann rate of the Cauchy stress. Note that Eq. (A27) is identical to Eq. (A14). This demonstrates that a consistent formulation of the problem must be independent of the choice of the finite strain tensor.

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