Improved Estimation of Long-Term Relaxation Function from Compliance Function of Aging Concrete

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Abstract: Based on asymptotic considerations, this paper develops an improved approximate formula for estimating the relaxation function from the given compliance function of concrete, which is considered as aging, linearly viscoelastic material. Compared with the formula developed in 1979 by Bažant and Kim, the new formula prevents any violation of the thermodynamic requirement of nonnegativeness of the relaxation function. It is significantly more accurate for long-time relaxation of concrete loaded at a young age, and, for this reason, it is particularly useful for compliance functions that correctly describe multidecade creep, which is the case for model B3 compliance function (a 1995 international RILEM recommendation) and not, for example, for the compliance functions of the American, European, Japanese, and Canadian standard recommendations, for which the benefit is smaller. The main application of the new formula is to evaluate the aging coefficient of the age-adjusted effective modulus method (AAEM) from the compliance function specified by the standard recommendation. The AAEM, developed in 1972 at Northwestern University and embodied in most standard design recommendations including those of the American Concrete Institute (ACI) and the Fédération internationale du béton (*fib*), provides an approximate estimate of the creep effects in structures according to the principle of superposition, which itself is a simplification neglecting nonlinear and diffusion effects. **DOI: 10.1061/(ASCE)EM.1943-7889.0000339.**

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Introduction

According to the standard recommendation of the American Concrete Institute (ACI) Committee 209 (1982) and many other societies, the creep response of concrete structures in the presence of long-time stress redistributions can be approximately estimated by the age-adjusted effective modulus method (AAEM) [Bažant 1972a, 1972b; ACI Committee 209 1982, 2008b; Comité Euro-international du béton-Fédération Internationale de la précontrainte (CEB-FIP) 1990; Fédération internationale du béton (*fib*) 1999]. The AAEM requires that the relaxation function $R(t, t')$ be calculated from the compliance function $J(t, t')$. Function $R(t, t')$ or $J(t, t')$ is defined as the uniaxial stress or strain at time t caused by a unit strain or unit stress applied at age t'. Although there exists a computer program to compute $R(t, t')$ from $J(t, t')$ with high accuracy (Bažant 1972a) in a fraction of a second, an analytical formula is more handy for quick calculations of structured effects.

An approximate formula for $R(t, t')$ was developed by Bažant and Kim (1979). It was calibrated on the basis of the relaxation function accurately computed from the compliance function specified by the ACI Committee 209 (1971) recommendations. However, recently it appeared that this compliance function, which still survives in the ACI-209 recommendation (ACI Committee 209 2008a), gives a poor representation of multidecade concrete creep. Part of the problem is that this formula—as well *fib*, Gardner and Lockman, and Japan Society of Civil Engineers (JSCE) formulas—has an upper bound, while in reality the long-time creep curves of concrete approach logarithmic curves, which are unbounded (Brooks 2005; Bažant et al. 2010, 2011c, 2011d; Gardner and Lockman 2001; JSCE 1996).

Thus, it is not surprising that the old Bažant-Kim formula has recently been found to have a considerable error for long times (>5 years) when compared with accurate linearly viscoelastic calculations for a more realistic compliance function such as that given by model B3 (Bažant and Baweja 1995, 2000). Moreover, the old formula may, for long times and for strain applied at a young age, give stress relaxation curves that cross into negative values. Such crossings are thermodynamically prohibited. This is proven by the fact that every realistic $J(t, t')$ can be approximated with any desired accuracy by the Kelvin chain model (Bažant and Huet 1999). Because the spring stiffnesses and dashpot viscosities are positive, nondecreasing functions of concrete age, a change of stress sign under constant strain can never occur. Therefore, an improved formula for $R(t, t')$ is needed and will be developed here.

Approximating the Relaxation Function

The old Bažant-Kim formula (Bažant and Kim 1979) can be written in the form:

$$R_{BK}(t, t') = A - B \tag{1}$$

$$A = \frac{1 - \Delta_0}{J(t, t')} \quad B = \frac{c_1\, \alpha(t, t')}{J(t, t - \eta)} \tag{2}$$

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$$\alpha(t, t') = \frac{J(t' + \xi, t')}{J(t, t - \xi)} - 1 \tag{3}$$

$$\xi = \frac{t - t'}{2}, \quad \eta = 1\,\text{day} \tag{4}$$

where $\Delta_0 = 0.008$ and $c_1 = 0.115$.

For nonaging materials such as polymers, function J depends only on the time lag $t - t'$, and thus, $\alpha(t, t') = B = 0$. In that case, $A \approx 1/J(t, t')$, which is the well known estimate of the relaxation function for nonaging linearly viscoelastic polymers. This estimate also coincides with the classical effective modulus approximation for concrete (McMillan 1916), in which the aging after the time of loading is ignored, and the structural creep analysis is replaced by elastic analysis with the effective (or sustained) modulus, traditionally expressed as

$$E_{ef} = \frac{1}{J(t, t')} = \frac{E(t')}{1 + \varphi(t, t')} \tag{5}$$

Here, $\varphi(t, t') = [J(t, t') - 1/E(t')]/[1/E(t')]$ = creep coefficient = ratio of the creep strain to the elastic strain; $E(t') = 1/J(t', t')$ = elastic (instantaneous) modulus at age t'. For the B3 model, $E(t')$ is calculated as $1/J(t' + 0.01, t')$, which is in good agreement with the conventional elastic modulus E_{28}.

Concrete creep exhibits a marked decrease of creep and relaxation with the age at loading, which is caused by both cement hydration and the relaxation of the initial microprestress within the nanostructure. This causes the compliance function to depend not only on the time lag $t - t'$, but also on t and t' individually. The stronger the aging, the larger the value of $\alpha(t, t')$; so, parameter $\alpha(t, t')$ in B accounts for the effect of aging, which tends to reduce the relaxation. But the effect of aging must be subject to the aforementioned thermodynamic restriction. To satisfy it, it is necessary to modify the old Bažant-Kim formula, making sure that the resulting value of R will remain positive even if $A < B$. Note that B/A is small for short durations $t - t'$ and large for long durations. Therefore, it is useful to rewrite Eq. (1) as

$$R_{BK}(t, t') = A\left(1 - \frac{B}{A}\right) \tag{6}$$

Because the short-time asymptotic properties of this old formula are realistic, the expression in parentheses should be modified such that it is not altered for small B/A and remains positive for large B/A. This is achieved by the following modified function with an additional positive empirical constant q:

$$\tilde{R}(t, t') = A\left(1 + \frac{B}{qA}\right)^{-q} \tag{7}$$

This function has the following desirable asymptotic properties: (1) Regardless of the value of q, the first term of the binomial expansion of this function, which dominates for small B/A, asymptotically approaches Eq. (6) for short times, which is what we want. (2) For large B/A, this function asymptotically approaches 0 and can never be negative, thus satisfying the thermodynamic requirement of nonnegativeness of the relaxation function. Exponent q controls how early or late the relaxation curve will flatten out to approach 0, or how early or late the improved formula will deviate from the old Bažant-Kim formula. The larger the value of q, the later the approach to the zero asymptote will occur and the sharper the slope will decrease.

Various functions of $J(t - t')$ were accurately integrated from the compliance rate functions $\dot{J}(t, t')$ of model B3 for various ages t' at loading and for various typical compositions of concrete. The corresponding relaxation functions $R(t, t')$ were then accurately solved from the Volterra integral equation relating $R(t, t')$ and $J(t - t')$ (Bažant 1982; RILEM Committee TC 69 1988; Jirásek and Bažant 2002). By optimizing the fits of functions $R(t, t')$ with a generalized reduced gradient algorithm for individual compositions, followed by adjustments to fit the full range of compositions, the value of q has been determined. It has also been found that parameter c_1 should also depend on age t' at loading and Δ_0 should be 0. This simplifies the formula and gives the following result for c_1:

$$c_1 = 0.0119 \ln t' + 0.08, \quad q = 10 \tag{8}$$

where t' must be given in days. Thus, the formula for the improved approximation of the relaxation function, proposed here, reads as follows:

$$\tilde{R}(t, t') = \frac{1}{J(t, t')}\left[1 + \frac{c_1\,\alpha(t, t')J(t, t')}{qJ(t, t - \eta)}\right]^{-q} \tag{9}$$

The relaxation functions for loading ages $t' = 10$, 100, and 1000 days, and for four typical concretes given in Table 1, accurately calculated according to model B3, are plotted as solid lines in Fig. 1. The old Bažant-Kim approximations $R_{BK}(t, t')$ are plotted in the same figure as the dotted lines, and the new approximations $\tilde{R}(t, t')$ as the dashed lines. It is clear that a marked improvement is achieved. The improvement is particularly significant for long durations $t - t'$ for sustained stress applied at young age ($t' = 10$ days).

Fig. 1 is plotted for the case of basic creep, i.e., creep at no moisture exchange, which also closely approximates the creep of thick cross sections. Fig. 2 shows analogous comparisons for the drying creep of a slab of thickness $D = 0.15$ m (6 in.), at environmental relative humidity 60%, with $J(t - t')$ calculated according to model B3. The parameters $t_0 = 7$ days, $k_s = 1$ (infinite plate), $\alpha_2 = 1.2$ (sealed or normal curing in air with initial protection) were used in the drying creep calculations. Again, the new approximation is seen to be much closer to the exact solution than the old one.

Application to Age-Adjusted Effective Modulus Method

In this method, derived by Bažant (1972b), the changes of stresses and deformations from time t_1 of the first loading of the structure until time t of interest are approximately calculated using the incremental (age-adjusted) modulus $E''(t, t_1)$, and the creep increments are calculated from the creep coefficient as if the stress remained constant. The structure is assumed to be under a constant load and constant imposed displacement (if any), although stepped loading can be handled by superposing several separate AAEM calculations. The

Table 1. Typical Concrete Compositions Based on Water-Cement Ratio, Aggregate-Cement Ratio, Cement Content, and 28-Day Compressive Strength

Composition	Water-cement ratio (w/c)	Aggregate-cement ratio (a/c)	Cement content (lb/ft^3)	28-day compressive strength (psi)
Low-strength	0.69	6.7	16.7	2,900
Medium-strength	0.54	5.1	21.4	4,600
High-strength	0.38	3.7	27.7	7,700
Nassar 1986	0.60	7.0	13.7	4,000

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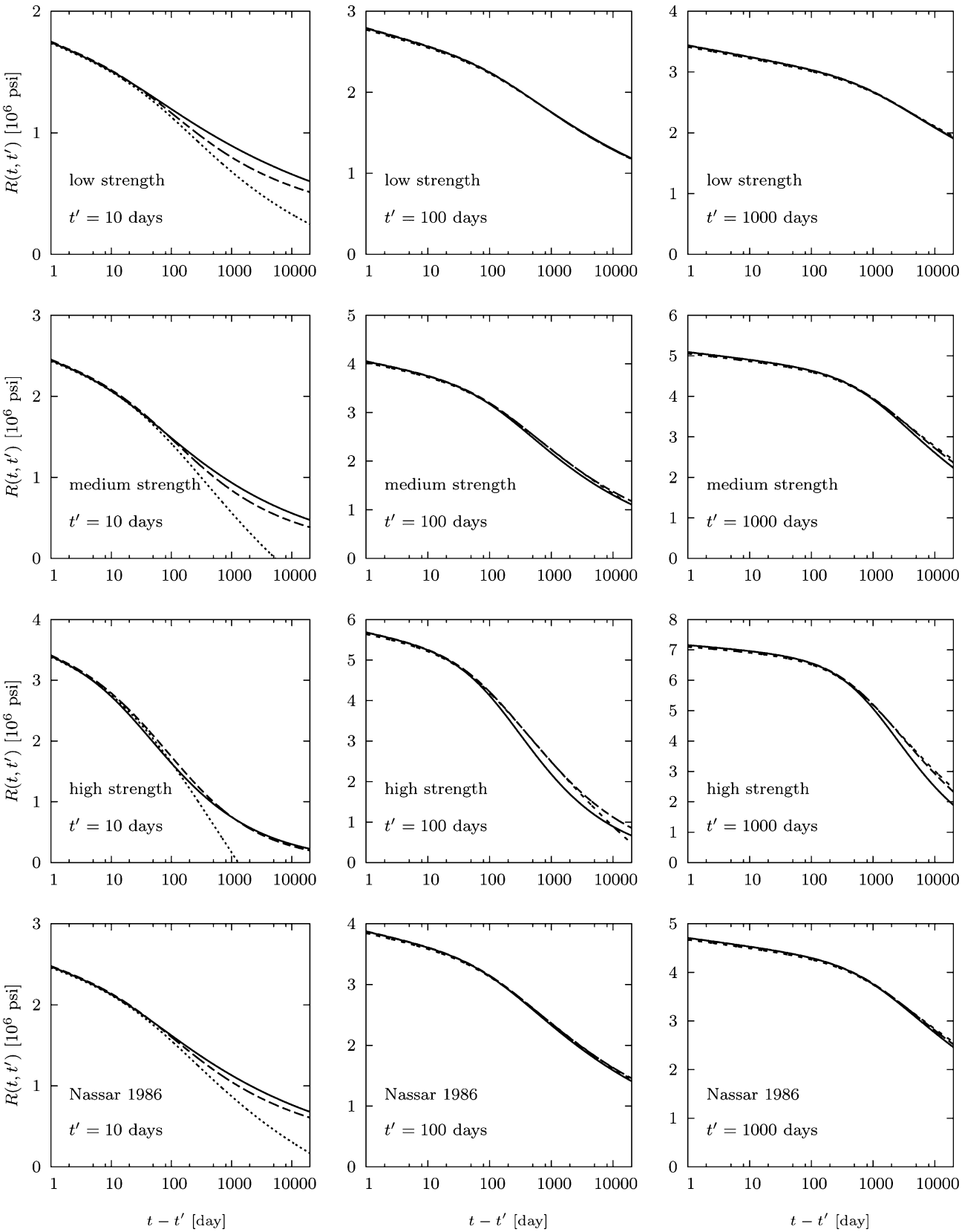

Fig. 1. Basic creep: comparisons of the updated approximate formula for the relaxation function with the exact solution and the 1979 Bažant-Kim approximate formula; Nassar 1986 data are from Nassar and Al-Manaseer (1986)

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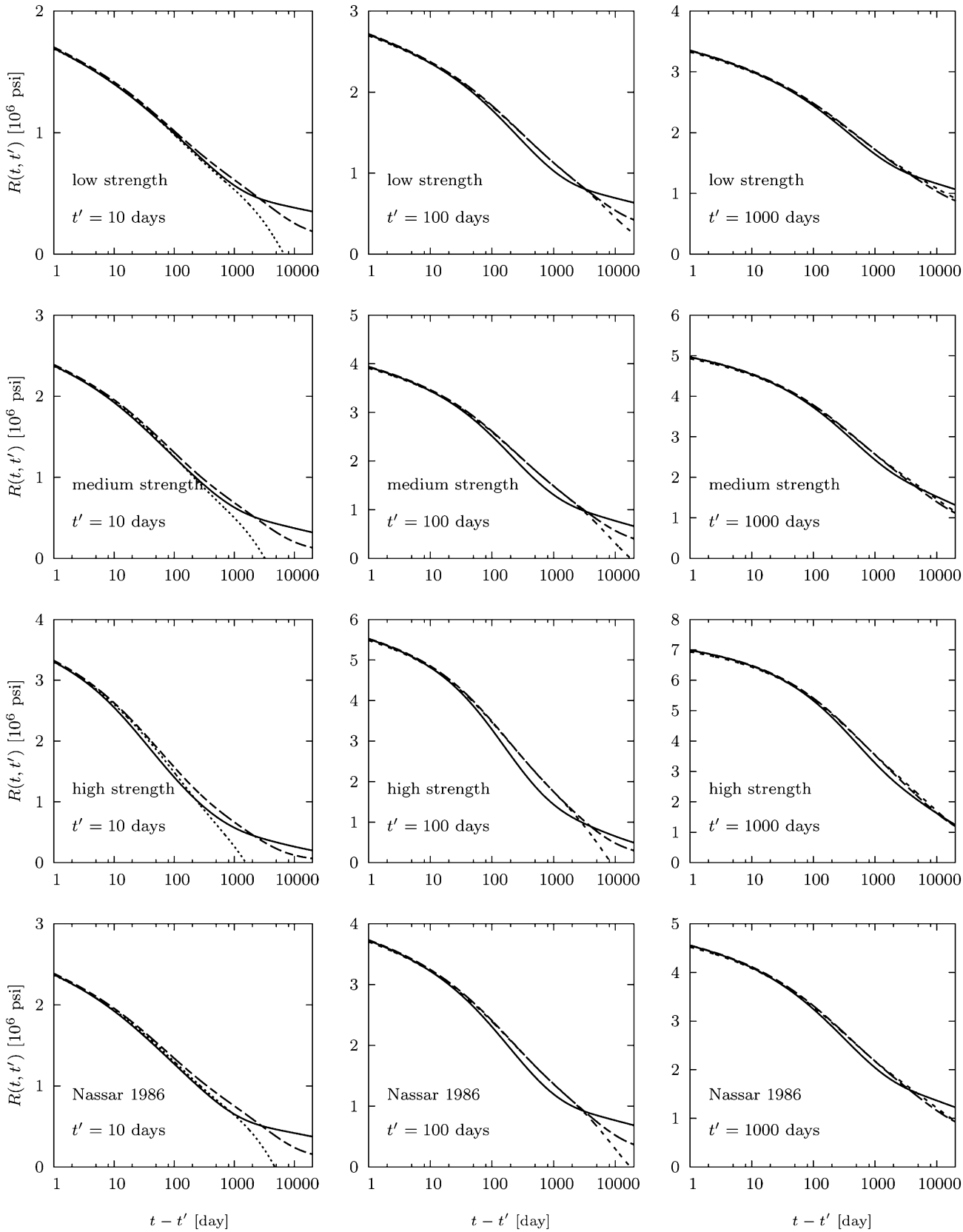

Fig. 2. Drying creep: comparisons of the updated approximate formula for the relaxation function (dashed lines) with the exact solution (solid lines) and the 1979 Bažant-Kim approximate formula (dotted lines)

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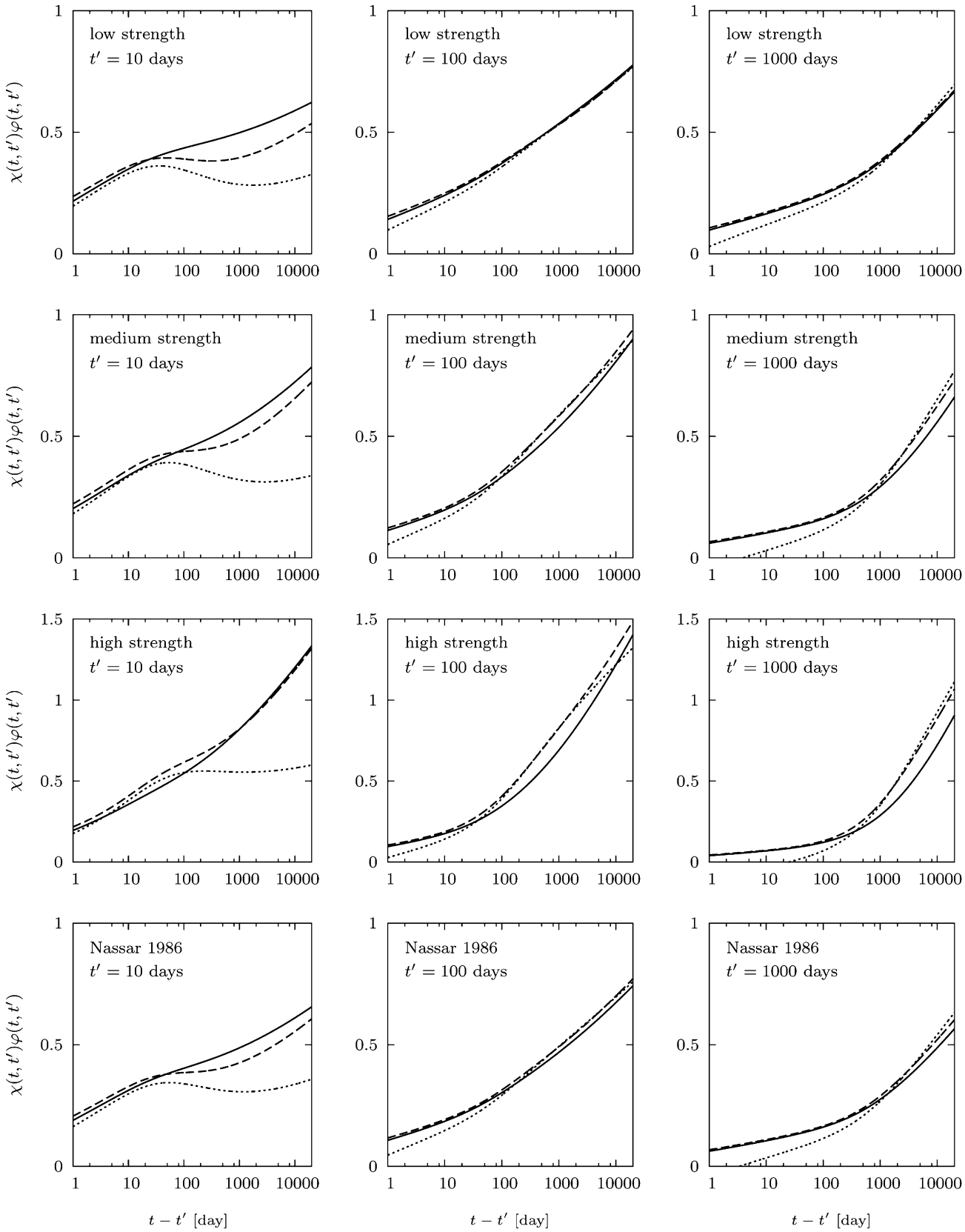

Fig. 3. Basic creep: comparison of the product of the aging coefficient and the creep coefficient based on the improved approximation for the relaxation function (dashed lines) with the product based on the exact solution (solid lines) and on the previous approximate formula from Bažant and Kim (1979) (dotted lines)

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incremental modulus is expressed as (Bažant 1972b, 1982; RILEM Committee TC 69 1988; Jirásek and Bažant 2002):

$$E''(t, t_1) = \frac{E(t_1)}{1 + \chi(t, t_1)\varphi(t, t_1)} \tag{10}$$

$$= \frac{E(t_1) - R(t, t_1)}{\varphi(t, t_1)} \tag{11}$$

$$= \frac{E(t_1) - R(t, t_1)}{E(t_1)J(t, t_1) - 1} \tag{12}$$

where $\chi(t, t_1)$ = aging coefficient. This coefficient introduces a correction to the effective modulus E_{ef} in Eq. (5) (which is obtained for $\chi \approx 1$ and is applicable for no aging), and is calculated as (Bažant 1972b):

$$\chi(t, t_1) = \frac{E(t_1)}{E(t_1) - R(t, t_1)} - \frac{1}{\varphi(t, t_1)} \tag{13}$$

These expressions would give the exact solution of structural creep response if the strains evolved in time as linear functions of $J(t, t_1)$ or $\varphi(t, t_1)$. The fact that such strain variation is often close to reality explains why the AAEM often gives rather close approximations to the exact solutions according to the principle of superposition.

In the AAEM, the correction to E_{ef} is given by the dimensionless product $\chi(t, t_1)\varphi(t, t_1)$. The values of this product, calculated from the exact $R(t, t')$ using the old approximation $R_{BK}(t, t')$ and the present new approximation $\tilde{R}(t, t')$, are plotted in Fig. 3 for the same cases as in Fig. 1. Again, it is seen that the new approximation is quite close to, and much better than, the old one. The new formula improves the approximation not only for long durations but also for short ones, which is reflected by the improved approximation of $\chi(t, t')\varphi(t, t')$ and prevents nonphysical negative values that were generated by the old formula.

Note that it makes no sense to consider comparisons for the aging coefficient $\chi(t, t')$. If they were plotted, one would see big errors (up to 11%) for short times $t - t'$ and early ages t' at loading. But these errors do not matter because $\chi(t, t')$ is always multiplied by $\varphi(t, t')$ in the applications, which is nearly 0 whenever the error in $\chi(t, t')$ is large. What matters is only the product $\chi(t, t')\varphi(t, t')$.

Discussion of Other Aspects

Despite significant nonlinear effects in structures, the linearly viscoelastic theory of aging creep, based on the principle of superposition in time, inevitably represents the basis on which various nonlinearities and diffusion effect are overlaid. In long-term structural creep analysis, one deals with the service stress state in which the stresses are generally low enough to be within the linear range of concrete creep. Significant deviations from the principle of superposition may be caused by cracking. What is of main interest for creep are prestressed structures. In these structures, cracking is mostly suppressed by the prestress, except in the cases of poor design or a poor creep prediction model (Bažant et al. 2010), leading to excessive deflections. More significant deviations from linearity are caused by simultaneous drying, but no known simple method such as the AAEM can take them into account, and a three-dimensional (3D) finite-element creep analysis is needed (Bažant et al. 2010).

It is sometimes doubted whether it is justified to compute the relaxation function using the principle of superposition. For the basic creep of concrete (i.e., creep of sealed specimens with no cracking) during the first few years, these doubts have long been dispelled by the aging linear viscoelastic calculations shown in Figs. 2.2 and 2.6 in the paper by RILEM Committee TC 69 (1988), compared with the tests in the studies by Hanson (1953) and Harboe et al. (1958) in the 1950s, which used the duration of only 2 years.

One might object that the aforementioned relaxation tests were too short and made on concretes too old to be relevant today; however, no better relaxation tests on modern concretes seem to exist. Yet the lack of such tests cannot be too serious because the creep of modern concretes differs only by the numerical values of creep and relaxation data. The source of creep in both the old and modern concretes resides in the calcium silicate hydrate (C-S-H), the key constituent, which is the same in both, and the other noncreeping constituents merely provide various degrees of restraint and modify the rate and degree of hydration. This is why, for example, the form of the B3 compliance function applies to both old and modern concretes, and to low and high strength concretes, while the difference consists only in the numerical values and in the relation of the compliance function parameters to concrete strength and composition.

The foregoing facts are corroborated by the successful match of observed long-term creep deflections (Bažant et al. 2011a, 2011b) with accurate 3D multidecade simulations (Bažant et al. 2010, 2011c, 2011d). These simulations were based on aging linear viscoelasticity, onto which the nonlinear effects of cracking, temperature, drying, and steel relaxation were superposed.

Conclusions

1. Under the assumption of linear aging viscoelastic behavior sanctioned by the current standard design recommendations, the relaxation function of concrete can be accurately estimated from the given compliance function by an explicit formula, with no need for a numerical solution of a Volterra integral equation.
2. A new, more accurate formula derived by asymptotic considerations is proposed for this estimation. Unlike the previous formula of Bažant and Kim, it ensures that the relaxation curves cannot cross into the opposite sign, which is a thermodynamic requirement.
3. The new formula is mainly useful for the compliance function of model B3, which for constant stress gives unbounded creep curves terminating with a logarithmic function of time, in agreement with laboratory test data and structural observations. For this compliance function, the old formula has a large error, especially in the case of long-time relaxation beginning at a young age.
4. For the ACI, *fib*, GL, and JSCE compliance functions, the accuracy gain with the present formula is smaller. However, it does not matter because these compliance functions do not correctly represent multidecade creep anyway.
5. The main purpose of the improved approximate formula for the relaxation function is to compute the age-adjusted effective modulus for approximate analysis of creep effects in structures that follow linear aging viscoelasticity.
6. It must be cautioned, however, that neither the age-adjusted effective modulus method (AAEM), sanctioned by the current standard recommendations of ACI and *fib*, nor accurate solutions of creep effects according to linear aging viscoelasticity, are sufficiently accurate for predicting the response of structures of high creep sensitivity such as large span bridges, large-span roofs, nuclear containments and super-tall buildings (rate-type analysis with a broad retardation spectrum, enhanced for the effects of cracking, variable environment,

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diffusion and nonlinear prestressing steel relaxation, is required for that).

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