

Finite weakest-link model of lifetime distribution of quasibrittle structures under fatigue loading

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Abstract

The design of various engineering structures, such as buildings, infrastructure, aircraft, ships, as well as microelectronic components and medical implants, must ensure an extremely low probability of failure during their service lifetime. Since such a low probability is beyond the means of histogram testing, we must rely on some physically based probabilistic model for the statistics of structural lifetime. Attention is focused on structures consisting of quasibrittle materials. These are brittle materials with inhomogeneities that are not negligible compared with the structure size, as exemplified by concrete, fiber composites, tough ceramics, rocks, sea ice, bone, wood, and many more at the micro- or nano-scale. This paper presents a finite weakest-link model of the fatigue lifetime of quasibrittle structures that fail at the fracture of one representative volume element (RVE). In this model, the probability distribution of critical stress amplitude is first derived by assuming a prescribed number of loading cycles and a fixed stress ratio. The probability distribution of fatigue lifetime is then deduced from the probability distribution of critical stress amplitude through the Paris law for fatigue crack growth. It is shown that the present theory matches well with the experimentally measured lifetime histograms of various engineering and dental ceramics, which systematically deviate from the two-parameter Weibull distribution. The theory indicates that the mean fatigue lifetime of quasibrittle structures must strongly depend on the structure size and geometry. Finally, the present model indicates that the probability distribution of fatigue lifetime can be determined from the mean size effect analysis.

Keywords

Fracture mechanics, size effect, lifetime statistics, scaling, Weibull distribution, transition rate theory

1. Introduction and review of previous advances

Many modern engineering structures, such as aircraft, ships, bridges, biomedical implants, etc., are often designed for cyclic loading. In order to ensure a tolerable failure probability for these structures during their service lifetime, understanding the probability distribution of fatigue lifetime is of paramount importance. When a long lifetime is required, which is usually the case for most engineering structures, it is often difficult to obtain the lifetime histogram by waiting until the structure fails. Therefore, it is imperative to develop some

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physically based theory, by which the cumulative distribution function (cdf) of fatigue lifetime can be indirectly determined without relying on histogram testing.

This study focuses on the structures made of quasibrittle materials, which are brittle heterogeneous materials such as concrete, rocks, coarse-grained and toughened ceramics, dental ceramics, fiber composites, fiber-reinforced concretes, rocks, masonry, mortar, stiff cohesive soils, grouted soils, rigid foams, sea ice, consolidated snow, wood, paper, carton, bone, many high-tech, biological and bio-inspired materials, and most materials on the micro- and nano-scales. The salient feature of quasibrittle structures is that the size of fracture process zone (FPZ) is not negligible compared with the structure size. This leads to a size-dependent failure behavior: small structures fail in a quasi-plastic manner and large structures fail in a brittle manner, which has been clearly demonstrated by size effect analysis, numerical simulations and experiments [1–5].

In particular, we are interested in a broad class of structures of the so-called positive geometry, which are characterized by an initially positive derivative of the stress intensity factor at constant load with respect to the crack length. They fail (under a controlled load) as soon as a macro-crack is initiated from a damaged material representative volume element (RVE). For the purpose of calculating the failure statistics, such a class of structures can be modeled as a chain of RVEs. Note that here the RVE is defined as the smallest material volume whose failure triggers the failure of entire structure, which is different from the definition in the homogenization theory. It has been shown that the size of the RVE is a material property, which is about 2–3 times the inhomogeneity size [6].

For large-size structures, the size of the RVE is negligible compared with the structure size. In such a case, the structure can be statistically represented by a chain of an infinite number of RVEs, and therefore the probabilistic distribution of fatigue lifetime must follow the two-parameter Weibull distribution based on the stability postulate of extreme value statistics [7]. However, the infinite weakest-link model is not applicable to quasibrittle structures because the RVE size is not negligible compared with the structure size. Instead, we must consider a finite weakest-link model as suggested in the previous studies on the statistics of monotonic strength and creep lifetime [6, 8–10].

The mean lifetime of structures under cyclic loading has been researched for a long time. A common approach is to express the mean fatigue lifetime as a function of the applied stress amplitude, i.e. stress–life curve (S – N curve). Wöhler [11] was the first to experimentally investigate the S – N curve by performing the rotating bend tests on various alloys. Basquin [12] proposed an inverse power law for the S – N curve for materials under fully reversed cyclic loading. It has been also showed that the S – N curve varies with the applied minimum-to-maximum stress ratio (i.e. R -ratio) [13–16].

Within the framework of fracture mechanics, the S – N curve can be derived from the Paris law for the fatigue crack growth [17], where the fatigue crack growth rate is expressed as a power-law function of the amplitude of the stress intensity factor. The Paris law was initially proposed as an empirical law [17] for metals. It has been experimentally shown that the Paris law exponent for metals is about four.

Various researchers [18–20] physically justified the Paris law with an exponent equal to four for metals by considering the tensile yielding or slip in the reversed plastic zone at the crack tip. Barenblatt and Botvina [21] explained the Paris law with its associated scale effect based on the principle of incomplete self-similarity. Recent experimental studies showed that the Paris law is also applicable to quasibrittle materials such as concrete [22] and ceramics [23, 24] except that the exponent is much higher, compared with metals. In a recent study [10], the Paris law with a high exponent for quasibrittle materials was physically explained based on certain plausible assumptions about the damage accumulation in the cyclic FPZ within a multiscale framework. The model shows that the Paris law exponent is equal to two at the nano-scale and increases by an order of magnitude at the material macro-scale.

In addition to the deterministic analysis of fatigue failure of structures, various analytical models have been developed for the probability distribution of fatigue lifetime. The early approach is to directly incorporate the cyclic loading history into the statistical model of creep lifetime [25–27]. However, such an approach ignores the difference between the failure mechanisms of materials under monotonic and cyclic loading. Consequently, the model cannot capture two important experimental observations: (1) the power-law exponents of S – N curves for monotonic and cyclic loadings have been experimentally shown to be very different [28] while the model implies these exponents to be the same; and (2) the model predicts that the structure subjected to a cyclic load varying from P_{min} and P_{max} would fail slower than it would under a constant load P_{max} . However, experiments showed that the structure under cyclic loading could fail faster [29].

A simple fracture mechanics-based approach is to randomize the parameters of Paris law [30, 31]. However, the resulting cdf of fatigue lifetime is determined by the cdfs of the parameters of the Paris law, which are

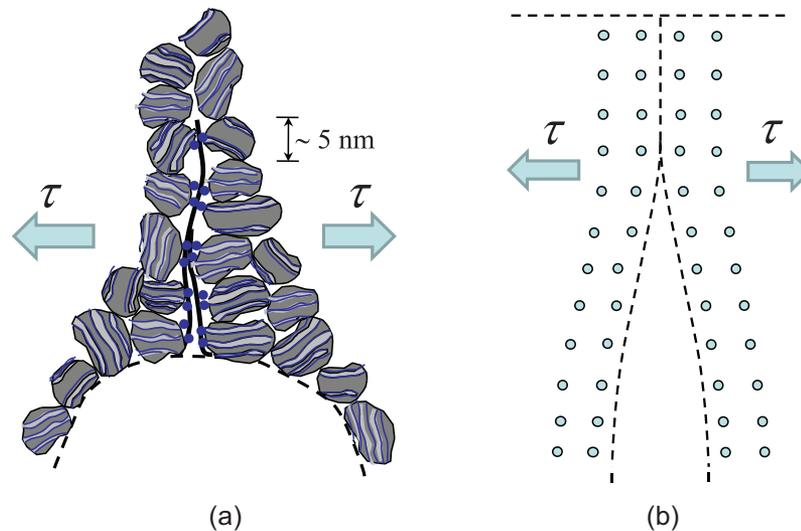


Figure 1. Propagation of a nano-crack in (a) a disordered nano-particle network and (b) an atomic lattice block.

usually assumed empirically. A more complicated approach is to model the fatigue crack growth as a Markov-chain process [32, 33], from which one could derive the probability distribution of fatigue lifetime. The main difficulty associated with this approach is that one has to assume some functions for the transition rates of the process, which are difficult to justify on a physical basis.

Parallel to analytical modeling, extensive efforts have been devoted to the experimental investigation of the probability distribution of fatigue lifetime of structures made of quasibrittle materials such as engineering and dental ceramics [23, 24, 34, 35] and bones [36]. The lifetime histograms are then fitted by some simple probability distribution functions, such as the Weibull distribution. However, it has been consistently observed that the measured histograms of fatigue lifetime deviate from a straight line on the Weibull scale, which indicates that the two-parameter Weibull distribution (having a zero threshold) does not give a good fit.

To improve the fit, the three-parameter Weibull distribution (with a finite threshold) has been widely used [35]. This distribution implies that, for a prescribed load amplitude, there exists a threshold value of the number of cycles under which the material has a zero failure probability. However, this is inconsistent with the well-established transition rate theory with random walk analysis [9, 37], which predicts that, in theory, the material always has a finite lifetime even under zero load albeit at small stress this lifetime could be longer than the age of the universe. Evidently, the observed systematic deviation of lifetime histogram from the two-parameter Weibull distribution has not been physically understood. This calls for a new probabilistic model.

In recent work [8, 9, 37, 38], a unified model has been developed for the statistics of static strength and creep lifetime of quasibrittle structures. The model was developed by statistically modeling the structure as a finite chain of RVEs. The probability distributions of strength and creep lifetime of one RVE were derived from atomistic fracture mechanics and a hierarchical statistical model for series and parallel couplings. It has been shown that the model could successfully describe the deviation of the histograms of strength and creep lifetime from the two-parameter Weibull distribution. This paper presents an extension of this recently developed probabilistic model to the statistics of fatigue lifetime [10]. This extension shows that the cdf of fatigue lifetime can be determined directly from the mean size effect analysis.

2. Fatigue strength of a nano-scale element

In this study, we first derive the probability distribution of fatigue strength, which is defined as the critical stress amplitude leading to the failure of a structure after a prescribed number of cycles and at a fixed R -ratio. Following the previous studies [8, 9], the only scale at which a clear physical basis for failure statistics exists is the atomic scale. This is because the failure of interatomic bonds is typically (even for dynamic loads) a quasi-steady process, in which the interatomic bond breaks are rare compared with the frequency of atomic thermal vibrations. This implies that the probability of failure of interatomic bonds is proportional to the frequency of their breakage [8, 9].

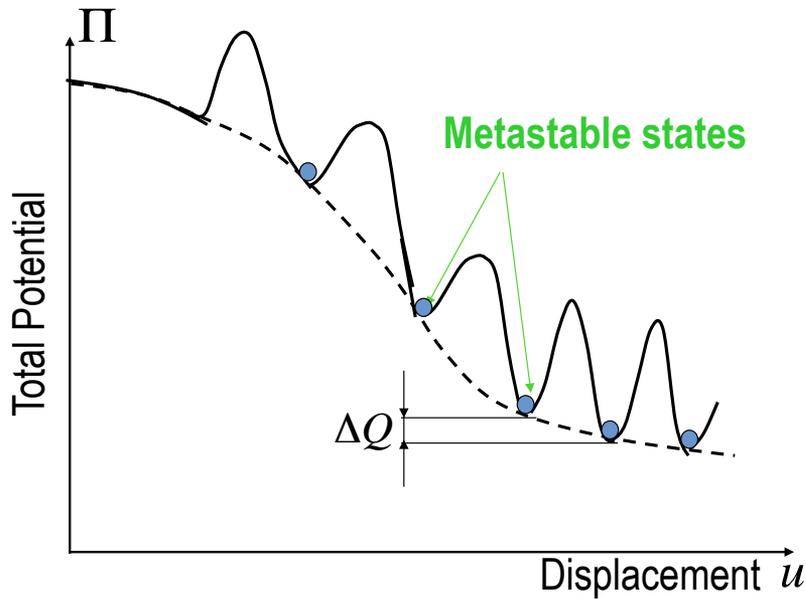


Figure 2. Potential–displacement curve of a nano-scale element.

Consider a nano-scale element such as a disordered nano-particle network or a regular atomic lattice block (as shown in Figure 1(a) and (b)). In a nano-scale material element, a nano-crack propagates by breaking the nano-particle connections (Figure 1(a)) or the interatomic bonds (Figure 1(b)) along the crack surface. The bond force–separation relation of the nano-particle connections or the interatomic bonds, which acts as a cohesive law for the nano-crack, can be obtained by differentiating their potential functions with respect to the separation. This approach may be attributed to the seminal contribution by Barenblatt on the cohesive crack model [39]. The nano-scale element fails if the nano-crack propagates to a certain critical length, which involves breakage of numerous nano-particle connections or interatomic bond pairs.

At the nano-scale, the rupture of nano-particle connections or the breakage of interatomic bonds can be considered as a thermal activated process. Since many bonds must fail, there are many undulation waves on the potential curve of the nano-scale element (Figure 2). Since many bonds must fail, it must be expected that the energy difference ΔQ between two adjacent potential wells, which represent two adjacent metastable states, must be very small compared with the activation energy barrier Q_0 .

Since we are dealing with the rare transitions between two states (before the bond and after the bond breakage) over a large energy barrier (i.e. $\Delta Q \ll Q_0$), the frequency of rupture of a nano-particle connection in a disordered nano-particle network or breakage of an atomic bond in a nano-scale lattice block can be obtained from Kramer’s formula [40, 41]:

$$f_1 = \nu_T(e^{(-Q_0+\Delta Q/2)/kT} - e^{(-Q_0-\Delta Q/2)/kT}) \approx 2\nu_T e^{-Q_0/kT} \sinh[\Delta Q/2kT] \tag{1}$$

where Q_0 is the free activation energy barrier, k is the Boltzmann constant, T is the absolute temperature, $\nu_T = kT/h$, h is Planck’s constant (energy of a photon)/(frequency of its electromagnetic wave) = $6.626 \cdot 10^{-34}$ Js, and ΔQ is the energy difference between two adjacent metastable states, which can be related to the energy release due to fracture [8]:

$$\Delta Q = \delta_a \left[\frac{\partial \Pi^*(P, a)}{\partial a} \right]_P = V_a(\alpha) \frac{\tau^2}{E} \tag{2}$$

Here $V_a(\alpha) = \delta_a(\gamma_1 \alpha l_a^2) k_a^2(\alpha)$ is the activation volume, δ_a is the spacing of nano-particle connections or atomic spacing, $k_a(\alpha)$ is the dimensionless stress intensity factor of the nano-element, l_a is the characteristic dimension of the nano-scale element, α is the relative crack length a/l_a (where a is the equivalent crack length based on the equivalent linear elastic fracture mechanics), γ_1 is the geometry constant such that $\gamma_1 a$ is the perimeter of the radially growing crack front, and E is Young’s modulus of the nano-structure.

At the nano-scale, the breakage of individual interatomic bonds or nano-particle connections can be considered as an independent process, which implies that the frequency of bond breakage does not depend on the

history of the processes that lead to the current state [42]. Therefore, the frequency of failure of a nano-scale element can be calculated from the sum of the frequencies of breakage of interatomic bonds or nano-particle connections that causes the propagation of nano-crack to reach a critical length. Furthermore, previous studies [8, 9] have demonstrated that the argument in the sine hyperbolic function in (1) is usually very small, i.e. $\Delta Q/2kT < 0.1$. Based on (1) and (2), the frequency of failure of a nano-element under a constant stress τ can be written as

$$f_a = \nu_T e^{-Q_0/kT} \left[\int_{\alpha_0}^{\alpha_c} V_a(\alpha) d\alpha \right] \frac{\tau^2}{EkT} \quad (3)$$

where α_0 and α_c denote the original and critical relative nano-crack lengths.

Now consider that the macro-scale structure is subjected to a cyclic load of constant remote stress amplitude $\Delta\sigma = \sigma_{max} - \sigma_{min}$, with the stress ratio $R = \sigma_{min}/\sigma_{max}$. For the nano-scale element, it is expected that the corresponding stress profile will stabilize after a few cycles. Since we are interested in high cycle fatigue, the first few cycles, during which the residual stress builds up, can be neglected. The final stabilized stress history of the nano-element can be described by the nanoscale stress amplitude $\Delta\tau = \tau_{max} - \tau_{min}$ and the nanoscale stress ratio $R_\tau = \tau_{min}/\tau_{max}$. This can further be related to the macro-scale stress amplitude and stress ratio as $\Delta\tau = c_1 \Delta\sigma$ and $R_\tau = c_2 R$; here c_1 and c_2 are constants which could be determined only through a detailed micro-mechanical analysis.

Similar to the assumption of independent breakage of nano-particle connections or interatomic bonds, it may be further assumed that the energy bias between two adjacent energy wells ΔQ depends only on the current stress, but not on the stress history (bond restoration, which would lead to a random walk of crack tip, are considered to be unlikely). Therefore, for a given number of cycles N_0 , the failure probability of the nano-scale element, which is proportional to the frequency of the failure, can be written as

$$P_f \propto \frac{\nu_T e^{-Q_0/kT}}{N_0 t_c} \left[\int_{\alpha_0}^{\alpha_c} V_a(\alpha) d\alpha \right] \int_0^{N_0 t_c} \frac{\tau^2(t)}{E_1 kT} dt \quad (4)$$

$$\propto \frac{e^{-Q_0/kT} (1 + \langle R_\tau \rangle + \langle R_\tau \rangle^2)}{3E_1 h (1 - R_\tau)^2} \left(\int_{\alpha_0}^{\alpha_c} V_a(\alpha) d\alpha \right) (c_1 \Delta\sigma)^2 \quad (5)$$

where t_c is the duration of one loading cycle, and $\langle x \rangle = \max(x, 0)$. The essential point here is that, as shown by Equation (5), the cdf of the fatigue strength of a nano-scale element must follow a power law with zero threshold.

3. Multiscale transition of statistics of fatigue strength

The statistics of fatigue strength at the nano-scale needs to be related to the statistics of fatigue strength of the macro-scale RVE. To establish such a link, some sort of multi-scale framework is required. Recent studies [6, 8, 9, 43] showed that the strength cdfs at the nano-scale and the macro-scale can be related through a hierarchical model consisting of a bundle of only two long subchains, each of which consists of sub-bundles of two sub-subchains, each of which consists of two sub-sub-bundles, etc., until the nano-scale element is reached (Figure 3). The chain model statistically represents the localization of damage, and the bundle model represents the compatibility condition between one scale and its subscale as well as the load re-distribution mechanism as the material is partially damaged. In this study, we use the same hierarchical model to derive the functional form of the cdf of fatigue strength of one RVE.

3.1. Chain model

Consider a chain of elements subjected to fatigue loading with a load amplitude ΔP . The fatigue strength of the chain is simply equal to the minimum fatigue strength of all the elements. Based on the joint probability theorem, the cdf of fatigue strength of the chain can be calculated as

$$P_{f,chain}(\Delta\sigma) = 1 - \prod_{i=1}^{n_c} [1 - P_1(s_i \Delta\sigma)] \quad (6)$$

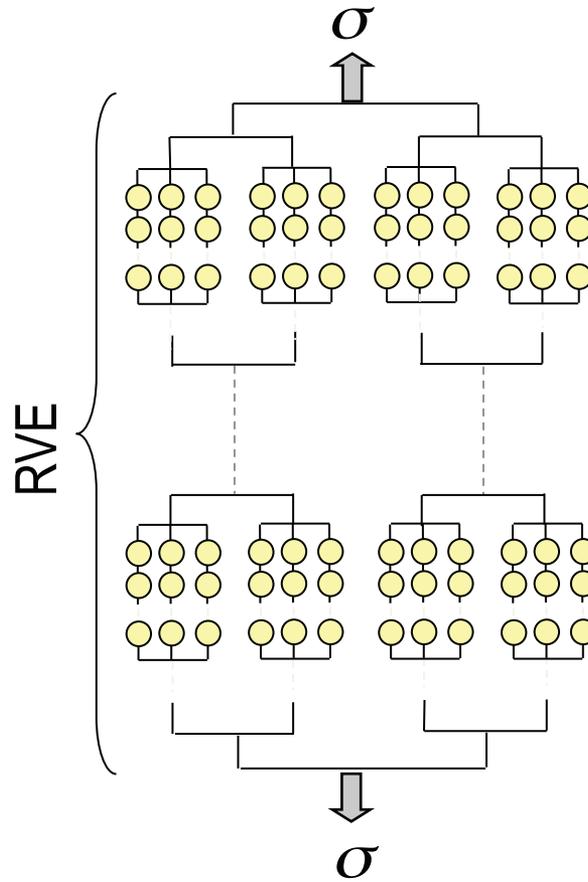


Figure 3. Statistical hierarchical model.

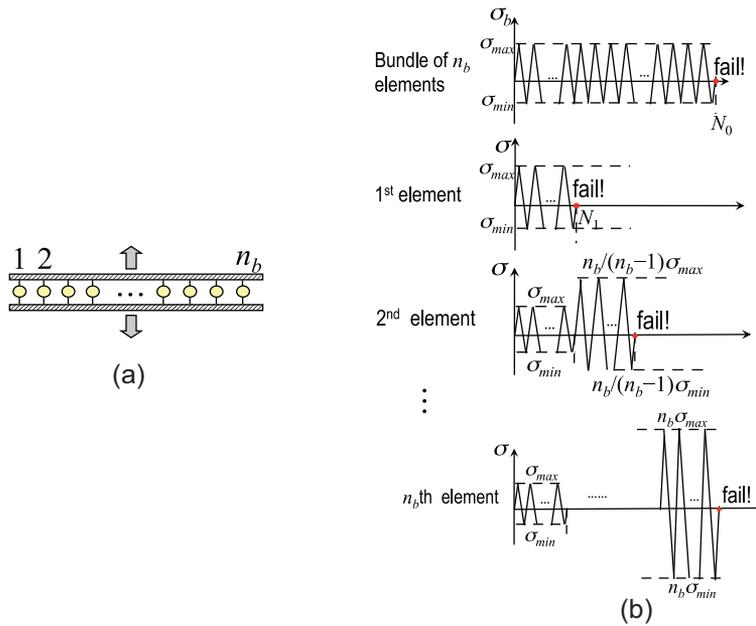


Figure 4. (a) Bundle model and (b) loading history of elements in the bundle.

where $\Delta\sigma = \Delta P/A_0$ is the nominal stress amplitude of the chain (where A_0 is the nominal cross-sectional area of the chain), s_i is the dimensionless constants such that $s_i\Delta\sigma$ is equal to the nominal stress amplitude of the

i th element, P_1 is the cdf of fatigue strength of one element, and n_c is the number of elements in the chain. It is clear that, if the cdf of fatigue strength of each element has a power-law tail, then the cdf of the fatigue strength of the entire chain also has a power-law tail. When the number of elements approaches infinity, the cdf of the fatigue strength of the chain converges to the Weibull distribution.

3.2. Bundle model

Compared with the chain model, the calculation of the cdf of the fatigue strength of a bundle is more complicated. Consider a bundle of n_b elements connected between two rigid plates (Figure 4(a)). Let $\Delta\sigma_b$ and $\Delta\sigma_i$ ($i = 1, \dots, n_b$) denote the fatigue strengths of the bundle and its elements, respectively, where the fatigue strength is defined as the critical stress amplitude for N_b loading cycles. Let us further number the elements in the order of increasing random fatigue strengths $\Delta\sigma_i$. Since it is not unduly restrictive to assume that all of the fibers have equal elastic stiffness [8, 9], the corresponding load history of the bundle and its each element can be depicted by Figure 4(b).

To relate the fatigue strength of the bundle to the fatigue strengths of its elements, one needs some damage accumulation rule. A simple approach to formulate this rule is through the kinetics of fatigue crack growth [44], which can be described by the Paris law [5, 17, 29]:

$$\frac{da}{dN} = A\Delta K^n \quad (7)$$

where da/dN is the crack growth rate, ΔK is the amplitude of the stress intensity factor, and A, n are empirical constants. Experiments showed that A can be considered as constant, unless ΔK or K_{max} is large in which case A depends on the remote applied stress ratio R . It is clear that all of the elements are subjected to a constant stress ratio (Figure 4(b)). Therefore, we could apply the Paris law with the same A to each element for the entire duration of cyclic load. Consider that every element contains a subcritical crack, which could represent a distributed damage zone based on equivalent linear elastic fracture mechanics. If we assume that the i th element fails once its subcritical crack propagates from its initial length a_{0i} to a critical length a_{ci} , then by integrating the Paris law, we have

$$\int_0^{N_i} [\Delta\sigma(N')]^{n_e} dN' = \int_{\alpha_{0i}}^{\alpha_{ci}} \frac{d\alpha}{A_i l_{ie}^{n_e-1} k_i^{n_e/2}(\alpha)} \quad (8)$$

where A_i, n_e are the Paris law constant and exponent for the element i , l_{ie} is the characteristic size of the element, $k_i(\alpha)$ is the dimensionless stress intensity factor of the element, and $\alpha = a/l_{ie}$ is the dimensionless crack length. Meanwhile, we can also apply the Paris law to the element for the loading case that is used to define its fatigue strength:

$$\Delta\sigma_i^{n_e} N_b = \int_{\alpha_{0i}}^{\alpha_{ci}} \frac{d\alpha}{A_i l_{ie}^{n_e-1} k_i^{n_e/2}(\alpha)} \quad (9)$$

By calculating the integral on the left-hand side of Equation (8) for the cyclic stress history shown in Figure 4(b) and comparing it with Equation (9), we have

$$\Delta\sigma_i^{n_e} N_b = \int_0^{N_i} [\Delta\sigma(N')]^{n_e} dN' \quad (10)$$

$$= \Delta\sigma_b^{n_e} N_1 + \left(\frac{n_b \Delta\sigma_b}{n_b - 1}\right)^{n_e} (N_2 - N_1) + \dots + \left(\frac{n_b \Delta\sigma_b}{n_b - i + 1}\right)^{n_e} (N_i - N_{i-1}) \quad (11)$$

where N_k is the number of cycles at which the k th element fails ($k \leq i$); further we have $N_0 = 0$ and $N_{n_b} = N_b$. With Equation (11), the fatigue strength of the bundle can be expressed as a function of the fatigue strengths of its elements:

$$\Delta\sigma_b = \left\{ \sum_{i=1}^{n_b} \left[\frac{(n_b - i + 1)^{n_e}}{n_b^{n_e}} - \frac{(n_b - i)^{n_e}}{n_b^{n_e}} \right] \Delta\sigma_i^{n_e} \right\}^{1/n_e} \quad (12)$$

If the fatigue strength of the bundle does not exceed a certain value S , then the fatigue strengths of all of the elements must be bounded by the region $\Omega(S)$, which can be described by the following system of inequalities:

$$\left\{ \sum_{i=1}^{n_b} \left[\frac{(n_b - i + 1)^{n_e}}{n_b^{n_e}} - \frac{(n_b - i)^{n_e}}{n_b^{n_e}} \right] \Delta\sigma_i^{n_e} \right\}^{1/n_e} \leq S \tag{13}$$

$$\Delta\sigma_1 \leq \Delta\sigma_2 \leq \dots \leq \Delta\sigma_{n_b} \tag{14}$$

Assuming that the fatigue strengths of all of the elements are independent random variables, then the cdf of the fatigue strength of the bundle can be calculated as

$$G_{n_b}(S) = n_b! \int_{\Omega(S)} \prod_{i=1}^{n_b} f_i(\Delta\sigma_i) d\Delta\sigma_1 d\Delta\sigma_2 \dots d\Delta\sigma_{n_b} \tag{15}$$

where $f_i(\Delta\sigma_i)$ is the probability density function (pdf) of the fatigue strength of the i th element. If we further assume that the function $f_i(\Delta\sigma_i)$ has a power-law tail, i.e. $f_i = (\Delta\sigma_i/s_i)^{p_i}$ and set $y_i = \Delta\sigma_i/S$, then Equation (15) yields

$$G_{n_b}(S) = n_b S^{\sum_i p_i} \int_{\Omega(1)} \prod_{i=1}^{n_b} (y_i/s_i)^{p_i} dy_1 \dots dy_{n_b} \tag{16}$$

where $\Omega(1)$ represents the feasible region $\Omega(S)$ normalized by S . It is clear that the integral in Equation (16) leads to a constant. Therefore, we conclude that if the cdf of the fatigue strength of each element has a power-law tail, then the cdf of the fatigue strength of the bundle will also have a power-law tail, and the power-law exponent will be equal to the sum of the exponents of the power-law tails of the cdfs of the fatigue strength of all of the elements in the bundle.

Another interesting property is the functional form of the cdf of fatigue strength of a large bundle. To analyze it, let us consider two extreme values of n_e :

- (1) When $n_e = 1$, Equation (12) yields $\Delta\sigma_b = n_b^{-1} \sum_{i=1}^{n_b} \Delta\sigma_i$, which is the same as the formulation of the strength distribution of bundles with plastic elements [6]. Based on the central limit theorem, as n_b increases, the cdf of $\Delta\sigma_b$ must follow the Gaussian distribution except for its far left tail. The rate of convergence to the Gaussian distribution is about $O(n_b^{-1/2})$ [6, 43]. Meanwhile, as the cdf of $\Delta\sigma_b$ approaches the Gaussian distribution, its power-law tail is drastically shortened at the rate of $P_{m_b} \sim (P_{t1}/n_b)^{n_b}$ (where P_{t1} and P_{m_b} are the probabilities to which the power-law tails extend for one element and for the bundle with n_b elements, respectively) [6].
- (2) When $n_e = \infty$, Equation (12) yields $\Delta\sigma_b = \max[\Delta\sigma_1, (n_b - 1)\Delta\sigma_2/n_b, \dots, \Delta\sigma_{n_b}/n_b]$, which is the same as the formulation of the strength distribution of bundles with brittle elements. In such a case, the probability distribution of $\Delta\sigma_b$ can be calculated by Daniels' recursive equation [45]. Daniels also showed that the cdf of $\Delta\sigma_b$ must converge to the Gaussian distribution as n_b increases. The convergence rate is about $O(n_b^{-1/3}(\log n_b)^2)$ (see [46, 47]). In the meantime, the power-law tail of the cdf of the fatigue strength shrinks at a rate of $P_{m_b} \sim (P_{t1}/3n_b)^{n_b} - (P_{t1}/n_b)^{n_b}$ (see [6, 43]).

We can expect that, for any value of $n_e \geq 1$, the formulation of the cdf of the fatigue strength of the bundle is analogous to the formulation of the strength cdf of the bundle with nonlinear softening elements. Recent studies [6, 8, 9, 48] showed that, regardless of the mechanical behavior of the elements, the strength cdf of the bundles always approaches the Gaussian distribution with an increasing number of elements n_b at the rate of $O(n_b^{-1/2})$ to $O(n_b^{-1/3}(\log n_b)^2)$ except for the power-law tail, where the power-law tail terminates at about $P_{m_b} \sim (P_{t1}/3n_b)^{n_b} - (P_{t1}/n_b)^{n_b}$. Consequently, we expect that this result is also applicable to the cdf of the fatigue strength of the bundle.

3.3. Fatigue strength of one RVE

With the foregoing analysis of chain and bundle models, we could now calculate the cdf of the fatigue strength of one RVE based on the hierarchical model shown in Figure 3. It is evident that the mathematical formulations

of the chain model for the fatigue strength and the static strength are exactly the same. The mathematical formulation of the bundle model for the fatigue strength is analogous to that for the static strength. It has been shown that the exponent of the Paris law increases from 2 at the nano-scale to some large number (e.g., 10 for concrete and 30 for alumina ceramics) at the macro-scale [10]. Therefore, it is expected that the cdf of the fatigue strength of bundles at the nano-scale is similar to the strength cdf of plastic bundles, and the cdf of the fatigue strength of bundles at the macro-scale is similar to the strength cdf of a brittle bundle. Considering the same hierarchical model for the static strength and fatigue strength of one RVE, it is clear that the cdf of the fatigue strength of one RVE must have the same functional form as the cdf of static strength of one RVE. This fact has been numerically verified in [10].

Previous studies [6, 8, 9, 43] showed that the cdf of the static strength of one RVE can be approximated by a Gaussian distribution onto which a Weibull distribution (power-law tail) is grafted at a probability of about 10^{-3} to 10^{-4} . Based on the similarity of the chain and bundle models between the cases of static and fatigue strengths, the cdf of the fatigue strength of one RVE can be approximately described by the same function, i.e.

$$P_1(\Delta\sigma_f) = 1 - e^{-(\Delta\sigma_f/s_0)^m} \quad (\Delta\sigma_f \leq \Delta\sigma_{gr}) \quad (17)$$

$$P_1(\Delta\sigma_f) = P_{gr} + \frac{r_f}{\delta_G \sqrt{2\pi}} \int_{\Delta\sigma_{gr}}^{\Delta\sigma_f} e^{-(\sigma' - \mu_G)^2 / 2\delta_G^2} d\sigma' \quad (\Delta\sigma_f > \Delta\sigma_{gr}) \quad (18)$$

Here $\Delta\sigma_f$ is the amplitude of the maximum principal stress of the RVE, m is the Weibull modulus (equal to the tail exponent), s_0 is the scale parameter of the Weibull tail, μ_G and δ_G are the mean and standard deviations of the Gaussian core if $\Delta\sigma_f$ is considered extended to $-\infty$, r_f is the scaling parameter required to normalize the grafted cdf such that $P_1(\infty) = 1$, and P_{gr} is the grafting probability, which is equal to $1 - \exp[-(\Delta\sigma_{gr}/s_0)^m]$. Finally, continuity of the pdf at the grafting point requires that $(dP_1/d\Delta\sigma_f)|_{\Delta\sigma_{gr}^+} = (dP_1/d\Delta\sigma_f)|_{\Delta\sigma_{gr}^-}$.

4. Probability distribution of the fatigue lifetime

We will now relate the cdf of the fatigue strength to the cdf of the fatigue lifetime of one RVE. To accomplish this, we would still need to rely on the kinetics of the fatigue crack growth for one RVE. The cyclic loading initially causes some micro-cracking. After the formation of a sufficient number of micro-cracks, a macro-subcritical crack starts to propagate, which finally leads to fracture of the RVE. In this study, we model the fatigue fracture of the RVE by considering the propagation of a subcritical crack in the RVE. Before the formation of the macro-subcritical crack, this subcritical crack represents the micro-cracking in the RVE based on the equivalent linear elastic fracture mechanics. We may assume that the RVE fails once the subcritical crack propagates to a critical length.

Consider both the fatigue strength tests and fatigue lifetime tests on the same RVE. In the fatigue strength test, the RVE is loaded for a prescribed number of cycles N_0 and a fixed stress ratio R , and the critical stress amplitude $\Delta\sigma_c$, at which the RVE fails, is recorded. In the lifetime test, the RVE is loaded for a prescribed stress amplitude $\Delta\sigma_0$ and a fixed stress ratio R , and the critical number of load cycles N_c is recorded. By applying the Paris law to the subcritical crack growth in these two tests, we have

$$N_0 \Delta\sigma_c^n = N_c \Delta\sigma_0^n = \int_{\alpha_0}^{\alpha_c} \frac{d\alpha}{A l_0^{n-1} k^{n/2}(\alpha)} \quad (19)$$

where l_0 is the RVE size, which is about 2–3 times the size of material inhomogeneity [6], $k(\alpha)$ is the dimensionless stress intensity factor of the subcritical crack, $\alpha = a/l_0$ is the lengths of the subcritical crack relative to the RVE size, and α_0 and α_c denote the initial and critical relative lengths of the subcritical crack. Note that, when deriving Equation (19), we apply the Paris law to a single RVE and therefore the Paris law constant A and exponent n are associated with one RVE. Thus their dependence on structure size, which has been studied by a number of researchers [21, 22, 49–51], need not be considered here.

Now denote N_c as the random fatigue lifetime of one RVE subjected to the stress amplitude $\Delta\sigma_0$ and stress ratio R , and $\Delta\sigma_c$ is the random fatigue strength of the same RVE subjected to N_0 loading cycles and stress ratio R . Based on Equation (19), we have

$$\Pr(N_c \leq N_f) = \Pr(\Delta\sigma_c \leq \Delta\sigma_0 N_0^{-1/n} N_f^{1/n}) \quad (20)$$

With the cdf of the fatigue strength (Equations (17) and (18)), we obtain the cdf of the fatigue lifetime of one RVE:

$$\text{for } N_f < N_{gr}: P_1(N_f) = 1 - \exp[-(N_f/s_N)^{\bar{m}}] \tag{21}$$

$$\text{for } N_f \geq N_{gr}: P_1(N_f) = P_{gr} + \frac{r_f}{\delta_G \sqrt{2\pi}} \int_{\gamma N_{gr}^{1/n}}^{\gamma N_f^{1/n}} e^{-(N' - \mu_G)^2 / 2\delta_G^2} dN' \tag{22}$$

where $\gamma = \Delta\sigma_0 N_0^{-1/n}$, $N_{gr} = (\Delta\sigma_{gr} / \Delta\sigma_0)^n N_0$, $s_N = s_0^n N_0 \Delta\sigma_0^{-n}$, and $\bar{m} = m/n$. Clearly, the tail of the cdf of the fatigue lifetime of one RVE follows the Weibull distribution (power-law function). The rest of the cdf follows a Gaussian cdf transformed by a power-law function.

For the purpose of later calculation of the fatigue lifetime cdf of the entire structure, it is convenient to define a new random variable $\mathcal{N} = \Delta\sigma_0^n N$. Then the cdf of \mathcal{N} can be written as

$$\text{for } \mathcal{N}_f < \mathcal{N}_{gr}: P_{1\mathcal{N}}(\mathcal{N}_f) = 1 - \exp[-(\mathcal{N}_f/s_{\mathcal{N}})^{\bar{m}}] \tag{23}$$

$$\text{for } \mathcal{N}_f \geq \mathcal{N}_{gr}: P_{1\mathcal{N}}(\mathcal{N}_f) = P_{gr} + \frac{r_f}{\delta_G \sqrt{2\pi}} \int_{\gamma_{\mathcal{N}} \mathcal{N}_{gr}^{1/n}}^{\gamma_{\mathcal{N}} \mathcal{N}_f^{1/n}} e^{-(\mathcal{N}' - \mu_G)^2 / 2\delta_G^2} d\mathcal{N}' \tag{24}$$

where $\gamma_{\mathcal{N}} = N_0^{-1/n}$, $\mathcal{N}_{gr} = \Delta\sigma_{gr}^n N_0$, and $s_{\mathcal{N}} = N_0 s_0^n$. Note that N_0 is the arbitrary reference number of loading cycles, which is used to define the fatigue strength. Therefore, it is expected that the values of s_0 , μ_G , δ_G , and r_f depend on the choice of N_0 .

Since the structures that fail at the fracture of one RVE can be statistically modeled as a chain of RVEs, the cdf of the structural lifetime under a prescribed cyclic load can be calculated based on the joint probability theorem and the assumption that the fatigue lifetimes of the RVEs are independent random variables:

$$P_f(\mathcal{N}_f) = 1 - \prod_{i=1}^{n_s} [1 - P_{1\mathcal{N}}(\mathcal{N}_f \langle s(\mathbf{x}_i) \rangle^n)] \tag{25}$$

or

$$\ln(1 - P_f) = \sum_{i=1}^{n_s} \ln [1 - P_{1\mathcal{N}}(\mathcal{N}_f \langle s(\mathbf{x}_i) \rangle^n)] \tag{26}$$

where n_s is the number of RVEs in the structure, $\mathcal{N}_f = \Delta\sigma_{n_0}^n N_f$, N_f is the fatigue lifetime of the structure, $\Delta\sigma_{n_0}$ is the amplitude of maximum principal stress in the structure, $s(\mathbf{x}_i)$ is the dimensionless principal stress field such that $\Delta\sigma_{n_0} s(\mathbf{x}_i)$ is the amplitude of maximum principal stress at the center of the i th RVE.

The finite weakest-link model (Equation (25)) requires subdividing the structures into a number of RVEs. Such a subdivision is often ambiguous, which might lead to an inconsistent calculation of the lifetime cdf. Recent studies [9, 10, 52] proposed a nonlocal boundary layer model, in which a boundary layer with thickness l_0 is separated from the structure. What matters for the boundary layer is the average strain over its thickness, which is approximately equal to the stress of the smoothing continuum at points of the middle surface Ω_M of the layer. For the interior domain V_I , the stress can be calculated by the conventional nonlocal continuum model [53, 54], using a nonlocal weight function that does not extend beyond the thickness of the boundary layer and thus can never protrude outside the structure. With the boundary layer model, the logarithmic form of the finite chain model (Equation (25)) can then be re-written as

$$\ln[1 - P_f(\mathcal{N}_f)] = \int_{\Omega_M} \ln\{1 - P_{1\mathcal{N}}[\mathcal{N}_f \langle s(\mathbf{x}_M) \rangle^n]\} \frac{d\Omega}{l_0^3} + \int_{V_I} \ln\{1 - P_{1\mathcal{N}}[\mathcal{N}_f \langle \bar{s}(\mathbf{x}) \rangle^n]\} \frac{dV(\mathbf{x})}{V_0} \tag{27}$$

where $V_0 = l_0^3$ is the volume of one RVE and $\bar{s}(\mathbf{x})\Delta\sigma_0$ is the amplitude of nonlocal stress for one RVE. For very large structures, the boundary layer becomes negligible compared with the structure size and the nonlocal stress in the interior becomes the local stress. Furthermore, the failure of the entire structure is governed by the

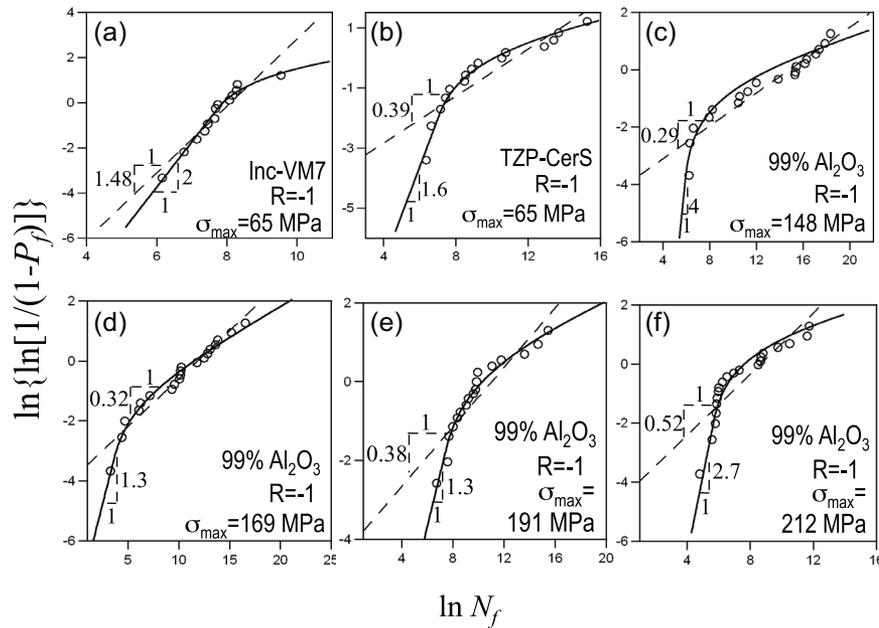


Figure 5. Optimum fits of lifetime histograms.

tail part of the lifetime cdf of one RVE. With the approximation $\ln(1 - x) = -x$, it can be easily shown that Equation (27) reduces to

$$P_f(N_f) = 1 - \exp \left[- \left(\int_V \langle s(\mathbf{x}_i) \rangle^{n\bar{m}} \frac{dV(\mathbf{x})}{V_0} \right) \left(\frac{N_f}{s_N} \right)^{\bar{m}} \right] \quad (28)$$

or

$$P_f(N_f) = 1 - \exp \left[- \left(\int_V \langle s(\mathbf{x}_i) \rangle^{n\bar{m}} \frac{dV(\mathbf{x})}{V_0} \right) \left(\frac{N_f}{s_N} \right)^{\bar{m}} \right] \quad (29)$$

This indicates that the cdf of the fatigue lifetime of large structures follows the Weibull distribution, which corresponds to a perfectly brittle failure behavior and is also consistent with the extreme value statistics (or the infinite weakest-link model) [55].

5. Optimum fits of lifetime histograms

Extensive efforts have been directed to investigate the statistics of fatigue lifetime of ceramic materials through histogram testing. Figure 5 presents the optimum fits of the measured lifetime histograms of engineering and dental ceramics by the two-parameter Weibull theory and the present theory. The details of these tests are as follows. (a)–(b) Dental ceramics which are used for dental bridges [23, 24]; glass infiltrated Al_2O_3 – ZrO_2 with feldspathic glass (Inc-VM7) (Figure 5(a)) and yttria-stabilized ZrO_2 with feldspathic glass (TZP-CerS) (Figure 5(b)). For each material, 30 specimens of size 4 mm \times 5 mm \times 50 mm were subjected to fully reversed cyclic bending. (c)–(f) Alumina ceramics (99% Al_2O_3) [34]; round bar specimens were tested under fully reversed cycles with different stress levels by a rotating bending machine. For each stress level, 20 specimens were tested.

Despite the small number of test specimens, Figure 5 clearly indicates that the measured lifetime histograms do not appear to represent a straight line on the Weibull scale. This implies that the corresponding cdfs of fatigue lifetime cannot be fitted by the two-parameter Weibull distribution. Instead, it can be seen that the lifetime histograms consist of two parts: the lower portion follows a straight line and the upper portion deviates from this straight line to the right. The present theory can fit both parts simultaneously. According to the present

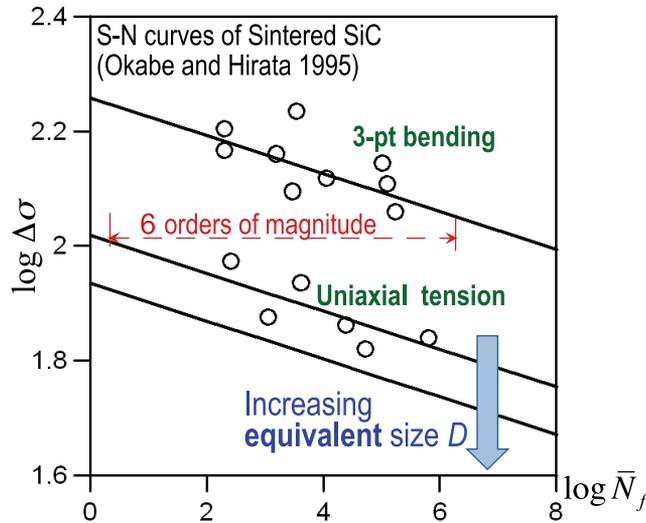


Figure 6. Size effect on the $S-N$ curve.

theory, the lower portion of the histogram can be modeled by a chain of Weibullian elements, which leads to the two-parameter Weibull distribution, and the upper portion of the histogram can be modeled by a chain of elements with a Gaussian distribution transformed by a power-law function. The grafting point separating these two parts depends on the structure size and geometry, which is a measure of the quasibrittleness of the structure.

6. Mean size effect on fatigue lifetime and its relation with the lifetime cdf

The present theory directly implies that, for quasibrittle materials, the cdf of the fatigue lifetime must depend on the structure size and geometry. Consequently, the mean fatigue lifetime \bar{N}_f must also be size and geometry dependent. Based on the weakest-link model, the mean lifetime \bar{N}_f can be calculated as

$$\bar{N}_f = \Delta\sigma_{n0}^{-n} \int_0^\infty \prod_{i=1}^{n_s} \{1 - P_{1N}(\langle s(\mathbf{x}_i) \rangle^{n_f} N_f)\} dN_f \tag{30}$$

Although an analytical expression of \bar{N}_f is impossible, an approximate expression has been proposed, based on asymptotic matching [3, 4, 9, 56]:

$$\bar{N}_f = \Delta\sigma_{n0}^{-n} \left[\frac{C_a}{D} + \left(\frac{C_b}{D} \right)^{\psi n_d / \bar{m}} \right]^{1/\psi} \tag{31}$$

where n_d is the number of spatial dimensions in which the structure is scaled, \bar{m} is the Weibull modulus of the fatigue lifetime, and C_a, C_b, ψ are constants, which can be determined from the small- and large-size asymptotes of the mean size effect curve. It is clear that Equation (31) indicates an inverse power law of the stress–life curve for fatigue loading, which agrees well with the well-known Basquin law [12] (commonly known as the ‘ $S-N$ ’ curve). Furthermore, Equation (31) implies that, on a bilogarithmic plot, the $S-N$ curve must shift horizontally to the left as the structure size increases. Figure 6 presents the experimentally measured $S-N$ curves of sintered SiC under three-point bending and uniaxial tension, which are fitted by Equation (31) [57]. When there are different stress fields for the same physical specimen size, the equivalent size of the specimen under three-point bending is much smaller than it is under uniaxial tension. As seen in Figure 6, there is a marked size effect on the $S-N$ curve.

Now we will explicitly relate the constants of the mean size effect curve to the statistical parameters of fatigue lifetime so that the fatigue lifetime cdf can directly be determined from the mean size effect analysis. Based on Equations (23) and (24), for a chosen N_0 , we need five independent constants to define the lifetime cdf of one RVE, i.e. $\bar{m}, n, \mu_G, \delta_G,$ and s_N . Constant n , which is the exponent of the Paris law, can always be

obtained by the standard tests of fatigue crack growth or $S-N$ curve. The other four constants can be determined from the mean size effect curve of the fatigue lifetime.

Assume that we have experimentally obtained the mean size effect curve on the fatigue lifetime with the four calibrated constants of Equation (31), and that the Paris law exponent is known. Clearly, the calibration directly yields the Weibull modulus of fatigue lifetime \bar{m} . At the large-size limit ($D \rightarrow \infty$), Equation (31) reduces to

$$\bar{N}_f = \Delta\sigma_{n0}^{-n} (C_b/D)^{n_d/\bar{m}} \quad (32)$$

On the other hand, the weakest-link model at the large-size limit implies that the probability distribution of N_f must approach the Weibull distribution and its mean can be calculated as

$$\bar{N}_f = \Delta\sigma_{n0}^{-n} s_N \Gamma(1 + 1/\bar{m}) \Psi^{-1/\bar{m}} / D^{-n_d/\bar{m}} \quad (33)$$

where $\Psi = \int_V s(\mathbf{x}_i)^{n\bar{m}} dV/dV_0$. By comparing Equations (32) and (33), we could calculate s_N from C_b .

The remaining two constants μ_G and δ_G can be calibrated from the two asymptotic conditions at the small-size limit: $[\bar{N}_f]_{D=l_m}$ and $[d\bar{N}_f/dD]_{D=l_m}$. Note that l_m denotes the size of the smallest specimens for which the finite weakest-link model makes sense. For example, a recent study [58] showed that beams under three-point or four-point bending should have a minimum depth of four RVEs. Based on Equation (31), we have

$$\bar{N}_f|_{D=l_m} = \Delta\sigma_{n0}^{-n} \frac{C_a^{1/\psi}}{l_m^{1/\psi}} \quad (34)$$

$$d\bar{N}_f/dD|_{D=l_m} = -\frac{\Delta\sigma_{n0}^{-n}}{\psi} \left[\frac{C_a}{l_0^2} + \frac{n_d\psi}{\bar{m}l_m} \left(\frac{C_b}{l_m} \right)^{n_d\psi/\bar{m}-1} \right] \left[\frac{C_a}{l_m} + \left(\frac{C_b}{l_m} \right)^{n_d\psi/\bar{m}} \right]^{1/\psi-1} \quad (35)$$

On the other hand, $[\bar{N}_f]_{D=l_m}$ and $[d\bar{N}_f/dD]_{D=l_m}$ can be calculated from the finite weakest-link model. At the small-size limit, the Weibull part of the lifetime cdf is very short, which can be safely ignored. Therefore, we have

$$\bar{N}_f|_{D=l_m} = \Delta\sigma_{n0}^{-n} \int_0^\infty \prod_{i=1}^{N_m} \left[1 - \Psi \left(\frac{\gamma_N N^{1/n} \langle s(\mathbf{x}_i) \rangle - \mu_G}{\delta_G} \right) \right] dN = f_1(\mu_G, \delta_G) \quad (36)$$

$$d\bar{N}_f/dD|_{D=l_m} = -\Delta\sigma_{n0}^{-n} \int_0^\infty (dP_f/dD) dN \quad (37)$$

$$= -\frac{n_d \Delta\sigma_{n0}^{-n}}{l_m} \int_0^\infty \prod_{i=1}^{N_m} \left[1 - \Psi \left(\frac{\gamma_N N^{1/n} \langle s(\mathbf{x}_i) \rangle - \mu_G}{\delta_G} \right) \right] \quad (38)$$

$$\times \sum_{i=1}^{N_m} \ln \left[1 - \Psi \left(\frac{\gamma_N N^{1/n} \langle s(\mathbf{x}_i) \rangle - \mu_G}{\delta_G} \right) \right] dN \quad (39)$$

$$= f_2(\mu_G, \delta_G) \quad (40)$$

where N_m is the number of RVEs of the smallest structure, $\Psi(x)$ is the standard Gaussian distribution. By comparing Equations (34) and (36) and Equations (35) and (40), we obtain two nonlinear equations for μ_G and δ_G , which can be solved numerically. Knowing the entire set of statistical parameters for Equations (21) and (22), we can then calculate the cdf of the fatigue lifetime N_f for structures of any size and geometry.

The present analysis shows that the cdf of the fatigue lifetime can be determined based on the size effect curve of the mean fatigue lifetime and the Paris law exponent. This is much more efficient than the conventional histogram testing, which usually involves testing several hundred identical specimens (and even that may not be good enough for determining the tail of lifetime cdf). By contrast, the size effect test requires specimens of four sizes and only 3–6 specimens are needed for each size.

7. Conclusions

We draw the following conclusions.

1. The probability distribution of the fatigue lifetime of quasibrittle structures can be calculated from a finite weakest-link model. The model shows that the probability distribution of the fatigue lifetime as well as the mean fatigue lifetime depend on the structure size and geometry.
2. The size and geometry dependence of the probability distribution of the fatigue lifetime implies that the safety factors guarding against the fatigue failure of large quasibrittle structures, such as bridges, aircraft, ships, must vary with the structure size and geometry.
3. The model further indicates that the statistics of fatigue lifetime can be efficiently determined from the mean size effect analysis.

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