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Statistical distribution and size effect of residual strength of quasibrittle materials after a period of constant load

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ABSTRACT

In preceding studies, the type of cumulative probability distribution functions (cdf) of strength and of static lifetime of quasibrittle structures, including their tails, was mathematically derived from atomistic scale arguments based on nano-scale cracks propagating by many small, activation energy-controlled, random breaks of atomic bonds in the nanostructure. It was shown that a quasibrittle structure (of positive geometry) must be modeled by a finite (rather than infinite) weakest-link model, and that the cdf of structural strength as well as lifetime varies from nearly Gaussian to Weibullian as a function of structure size and shape. Excellent agreement with the observed distributions of structural strength and static lifetime was demonstrated. Based on the same theoretical framework, the present paper formulates the statistics of the residual structural strength, which is the strength after the structure has been subjected to sustained loading. A strength degradation equation is derived based on Evans' law for static crack growth during sustained loading. It is shown that the rate of strength degradation is not constant but continuously increasing. The cdf of residual strength of one RVE is shown to be closely approximated by a graft of Weibull and Gaussian (normal) distributions. In the left tail, the cdf is a three-parameter Weibull distribution consisting of the $(n+1)$th power of the residual strength, where n is the exponent of the Evans law and the threshold is a function of the applied load and load duration. The finiteness of the threshold, which is typically very small, is a new feature of quasibrittle residual strength statistics, contrasting with the previously established absence of a threshold for strength and lifetime. Its cause is that there is a non-zero probability that some specimens fail during the static preloading, and thus are excluded from the statistics of the overload. The predictions of the theory are validated by available test data on glass–epoxy composites and on borosilicate and soda-lime silicate glasses. The size effect on the cdf of residual strength is also determined. The size effect on the mean residual strength is found to be as strong as the size effect on the mean initial strength.

1. Introduction

In most engineering applications such as bridges, dams, ships, aircraft and microelectronic components, it is essential for the design to ensure a very low failure probability such as 10^{-6} throughout the lifetime (Nkb, 1978; Melchers, 1987; Duckett,

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2005; Bažant and Pang, 2006). To calculate the failure probability of a structure, the probability density distribution of applied loads must be combined with the cumulative probability distribution function (cdf) of structural strength. This cdf must be known up to the tail region with probabilities of the order of 10^{-6}. Such small probabilities, however, are beyond direct experimental verification by repeated (histogram) tests (since at least 10^8 of identical structures would have to be tested). Therefore, the form of the cdf of strength must be established theoretically with the mean and coefficient of variation as the only free empirical parameters.

The type of cdf of strength became well understood long ago for structures that are either perfectly ductile (i.e., plastic) or perfectly brittle. In the former case, the cdf must be Gaussian (i.e. normal), based on the central limit theorem and the fact that the failure load is a weighted sum of contributions of all the representative volume elements (RVE) of material whose random strength values are mobilized along the whole failure surface simultaneously. In the latter case, the cdf of strength and lifetime must be Weibullian, which follows from the weakest-link model (with infinite number of links) and the stability postulate of extreme value statistics (Fisher and Tippett, 1928). In either case, the cdf tail is simply obtained from the mean and standard deviation. It is noteworthy that, for the same coefficient of variation (typical of concrete strength), the point of probability 10^{-6} is for Weibull cdf about twice as far from the mean than it is for the Gaussian cdf. This fact highlights the importance of accurate prediction of the tail (Bažant and Pang, 2007).

An early attempt to introduce quasibrittleness into strength statistics was made by Bažant and Xi (1991) in their nonlocal Weibull theory, which could fit the size effect but was purely phenomenological and did not directly address the problem of distribution and its tail. The problem of tail is more intricate for quasibrittle structures which behave as ductile when small and as brittle when large. Preceding studies argued that the form of cdf must be deduced theoretically from nanomechanics of fracture and experimentally verified by other theoretical predictions, among which the size effect due to material randomness is the most important and easily testable (Bažant, 2005; Bažant and Pang, 2007; Le et al., 2011). This is what has been done in the preceding studies for strength and lifetime and what will be done here for the residual strength of structures.

The residual strength is important for two reasons: (1) in real structures, the accidental overload against which the safety factor is supposed to provide protection can occur at any moment within the lifetime rather than at the beginning, and (2) in the laboratory, it is advantageous to replace lifetime tests by residual strength tests, as will be shown later. Practical examples of overload include a sudden excessive traffic load on a bridge, excessive rise of water level behind a dam, or sudden overheating of an electronic component. Knowing the statistics of residual strength should allow improvements in the safety factors taking into account the strength degradation of the structure depending on the load history and duration. It should also allow meaningful estimates of the remaining service life of structures for which maintenance design is a primary concern. This is of paramount importance from the perspective of cost reduction and safety, especially for modern large aircraft made of load-bearing quasibrittle composites (Lee et al., 2008).

The residual strength of different materials has been widely studied phenomenologically, but mainly for the case of cyclic loading. For example, Yang and Liu proposed a model for residual strength degradation and periodic proof tests for graphite-epoxy laminates under cyclic loading (Yang and Liu, 1977; Yang, 1978). A mechanistic attempt was provided by Halpin et al. (1972) based on the kinetics of fracture growth. There have been some experimental studies, e.g., for concrete (Award and Hilsdorf, 1972), but strictly deterministic. Unfortunately, no information exists in the literature on the residual strength in the statistical setting. In those few attempts that provided a statistical perspective (Kirchner and Walker, 1971; Evans, 1974; Hahn and Kim, 1975; Thomas et al., 2002; Duffy et al., 2003), the quasi-brittleness was not considered.

To ensure a tolerably low failure probability throughout the lifetime of the structure, a physically justified theory is needed to determine the form of the cdf of strength, lifetime and residual strength up to the remote tail (Bažant and Pang, 2006, 2007; Le and Bažant, 2009; Le et al., 2011). This is especially true for quasibrittle materials, which represent heterogeneous materials characterized by brittle constituents and inhomogeneities that are not negligible compared to structural dimensions. Depending on the scale of observation or application, many materials exhibit quasibrittle behavior including concrete, fiber composites, tough ceramics, rocks, sea ice, bone and other bio-materials, wood, rigid foams, stiff soils, snow slabs, and many more at micrometer scale (Bažant and Planas, 1998).

In this work, using the same framework as in the preceding ones (Bažant and Pang, 2006, 2007; Le and Bažant, 2009; Le et al., 2011), the theory is extended to deal with the probabilistic distributions of residual strength after a period under sustained constant load. A relation between strength and residual strength for one RVE is proposed and then combined with the Gauss–Weibull distribution of strength to get the cdf of residual strength. Then, the cdf for structures of any size is determined within the framework of the finite weakest link theory and validated by means of available test data on glass/epoxy composites as well as borosilicate and soda-lime silicate glasses. Finally, the size effect on the mean residual strength is studied and it is estimated to be as strong as the size effect on the initial strength.

2. Physical concepts, atomistic basis and scaling

Here we focus on the broad class of quasibrittle structures of positive geometry. These are the structures that fail, in case of load control, as soon as a macro-crack initiates from one RVE. They are characterized by a positive derivative of the stress intensity factor with respect to the crack length. Such structures are statistically equivalent to the weakest-link model, which consists of a chain of RVEs coupled in series. The crucial point, made in Bažant and Pang (2006, 2007), is that the

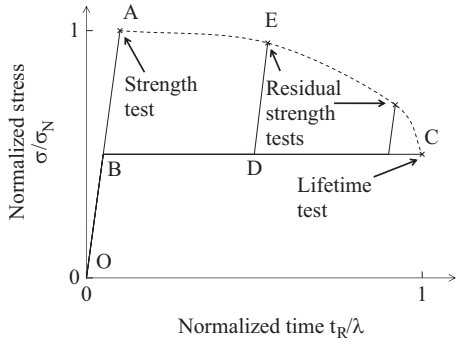

Fig. 1. Schematic of various load histories.

number of RVEs in the chain must be finite if the structure is not far larger than the RVE, which is a defining statistical characteristic of quasibrittleness. If the chain is infinite, the behavior is brittle.

The statistics of the structural strength then follow from the statistics of one RVE by using the joint probability theorem to express the condition that the structure survives if all the RVE′s survive. This further leads to an important finding about the size effect, viz. the location of the grafting point of the Gauss–Weibull distribution depends on the size and shape of the structure. The dependence was shown to be such that, for small structures, the distribution is mainly Gaussian except for a far-out Weibullian tail (which is a power law). As the size increases, the Weibull part gradually penetrates into the Gaussian core. In the limit of infinite size, the distribution becomes purely Weibullian because then the FPZ size is negligible in comparison to the structure dimension and the failure is brittle.

The nano-mechanical derivation of the cdf of RVE strength as well as lifetime under static and cyclic loads is based on the fact that failure probability can be exactly predicted only on the atomic scale because the bond breakage process is quasi-stationary, which means that the probability must be exactly equal to the frequency. Applying Kramer′s rule of the transition rate theory, one finds that the tail of the cdf of strength of a nano-scale element must follow a power-law with the exponent of 2 (Bažant et al., 2008; Le et al., 2011). Analysis of the multiscale transition of tail probabilities shows that the RVE must be statistically represented as a hierarchy of elements coupled alternately in series (Weibull, 1939) and in parallel (Daniels, 1945; Phoenix, 1978a, 1978b). In the series couplings, the tail exponent remains. In the parallel couplings, the tail exponents are additive, which is a universal property independent of whether, after the peak, the elements coupled in parallel fail suddenly or soften gradually, or respond plastically.

The multiscale transition of the tail probabilities of failure is based on the finding that a power-law tail is indestructible and its exponent must increase while moving up through the scales, and that scale transitions with both series and parallel couplings are required to preserve a power law tail with a deep enough reach (Bažant and Pang, 2006, 2007; Le and Bažant, 2009; Le et al., 2011). This analysis shows that the cdf must be a graft of Weibull and Gaussian distributions, in which the transition between these two cdfs is so abrupt that it can approximately be considered as a point-wise graft. A marked size effect was also noted in the form of the cdf with the grafting point moving into the core as the structure size increases. The fact that the cdf of strength of one RVE should be Gaussian except for the far-out tails can be understood from the following simple argument. The RVE is failing when a small number of remaining major bonds within the RVE are failing. Can these bonds fail in sequence, with the failure of one of them corresponding to the maximum stress in the RVE? They cannot, since then the assumed RVE would not the correct RVE. The assumed RVE would function as the weakest-link chain model, and so only a part of the assumed RVE could be the correct RVE. Hence, the last major bonds in the RVE must be failing simultaneously, which means that the strength of the RVE is a weighted sum of the strength of these bonds, which implies a Gaussian cdf except in the tails. And how many are they? Noting that, even for rectangularly distributed independent random variables, the sum of only six of them has a distribution very close to Gaussian (excluding, of course, the tail, as asserted by central limit theorem of probability), one must conclude that only about six last major bonds must be failing simultaneously (and fewer if each bond is close to Gaussian). This is physically reasonable.

The predictions from this theory agreed very well with experimental data on a wide range of quasibrittle materials. To derive the statistics of static lifetime of an RVE, it is first noted that the crack growth rate on the atomic scale must follow a power law of applied stress with the exponent of 2. Equating the time rates of energy dissipations on the RVE and on the atomic level explains why Evans′ law for subcritical macrocrack growth has a much higher exponent, typically about 10 for concrete and 30 for tough ceramics (Evans, 1972; Thouless et al., 1983; Evans and Fu, 1984). This provides the theoretical basis for Evans′ law, which had widely been used and amply justified by experiments (Bažant and Prat, 1988; Fett and Munz, 1991; Bažant and Planas, 1998; Munz and Fett, 1999; Lohbauer et al., 2002). Using Evans′ law to integrate the failure probability contributions over time yields a simple relation between the strength and static lifetime statistics (Le et al., 2011). An underlying assumption is that the mechanisms of crack growth in a strength test and a static lifetime test are the same. The argument is revisited here to extend it to the statistics of residual strength.

2.1. Relation between structural strength and static lifetime of one RVE

Evans' law for subcritical crack growth under sustained load (Evans, 1972), recently justified by atomistic arguments and scale bridging (Le et al., 2011), reads as

$$\dot{a} = A\, e^{-Q_0/kT} K_1^n \tag{1}$$

where a is the crack length, $\dot{a} = \mathrm{d}a/\mathrm{d}t$ (t is the time); A is the material constant, Q_0 is the activation energy, k is the Boltzmann constant and T is the absolute temperature. The stress intensity factor is denoted as K_1 where the subscript 1 indicates the RVE level. So, we have $K_1 = \sigma\sqrt{l_0}k_1(\alpha)$ where σ = nominal stress, which is defined as $\sigma = F/l_0^2$, $l_0 = RVE$ size, $\alpha = a_1/l_0$ = relative crack length and k_1 = dimensionless stress intensity factor. Accordingly, the above equation becomes

$$\dot{a} = Ae^{-Q_0/kT}\sigma^n l_0^{n/2} k_1^n(\alpha) \tag{2}$$

Consider now the different load histories illustrated in Fig. 1.

The load history O–A corresponds to the strength test and O–B–C to a static lifetime test. The history O–B–D–E corresponds to a residual strength test which will be considered in the next section.

First we consider load history O–A in which the load is increased linearly and rapidly until failure. Upon separating the variables from Eq. (2) we get

$$\frac{e^{Q_0/kT}}{Al_0^{(n-2)/2}}\,\frac{\mathrm{d}\alpha}{k_1^n(\alpha)} = \sigma^n\,\mathrm{d}t \tag{3}$$

Now let α_0 be the initial relative crack length and α_c be the critical relative crack length. The loading rate is expressed as $r = \sigma/t$. Integrating the above equation with appropriate limits for history O–A on both sides gives

$$\frac{1}{r}\int_0^{\sigma_N} \sigma^n\,\mathrm{d}\sigma = e^{Q_0/kT}\int_{\alpha_0}^{\alpha_c} \frac{1}{Al_0^{(n-2)/2}k_1^n(\alpha)}\,\mathrm{d}\alpha \tag{4}$$

where the substitution $\mathrm{d}t = \mathrm{d}\sigma/r$ is made. This yields

$$\sigma_N^{n+1} = r(n+1)e^{Q_0/kT}\int_{\alpha_0}^{\alpha_c} \frac{1}{Al_0^{(n-2)/2}k_1^n(\alpha)}\mathrm{d}\alpha \tag{5}$$

Next, we consider the lifetime test (history O–B–C), where the load is increased fast to σ_0 at a constant rate r and then held constant until failure. The time required for failure is the static lifetime denoted by λ. Integrating the same crack growth equation for this load history, we get

$$\frac{1}{r}\int_0^{\sigma_0} \sigma^n\,\mathrm{d}\sigma + \int_{t_0}^{\lambda} \sigma_0^n\,\mathrm{d}t = e^{Q_0/kT}\int_{\alpha_0}^{\alpha_c} \frac{1}{Al_0^{(n-2)/2}k_1^n(\alpha)}\,\mathrm{d}\alpha \tag{6}$$

where $t_0 = \sigma_0/r$. This yields

$$\sigma_0^{n+1} + r(n+1)\sigma_0^n(\lambda - t_0) = r(n+1)e^{Q_0/kT}\int_{\alpha_0}^{\alpha_c} \frac{1}{Al_0^{(n-2)/2}k_1^n(\alpha)}\,\mathrm{d}\alpha \tag{7}$$

where the case $\lambda \gg t_0$ is of main interest. Since the mechanisms of crack growth can be considered to be the same for both load histories, one may eliminate the integrals from Eqs. (5) and (7) to get a very simple relation between σ_N, λ and σ_0:

$$\sigma_0^{n+1} + r(n+1)\sigma_0^n(\lambda - t_0) = \sigma_N^{n+1} \tag{8}$$

Eq. (8) can equivalently be written as follows:

$$\sigma_N = \sigma_0^{n/(n+1)}[r(n+1)\lambda - n\sigma_0]^{1/(n+1)} \tag{9}$$

where it should be noted that for $\sigma_0 \to \sigma_N$, $\lambda \to \sigma_N/r$ while, for $\sigma_N \gg \sigma_0$,

$$\sigma_N \approx \sigma_0^{n/(n+1)}[r(n+1)]^{1/(n+1)}\lambda^{1/(n+1)} \quad \text{or} \quad \lambda = \frac{\sigma_N^{n+1}}{r(n+1)\sigma_0^n} \tag{10}$$

This is the equation for lifetime that has already been derived in Le et al. (2011). Eq. (8) can also be solved for lifetime which gives

$$\lambda = \frac{\sigma_N^{n+1}}{r(n+1)\sigma_0^n} + \frac{n\sigma_0}{r(n+1)} \tag{11}$$

Based on the foregoing result, a comment on the statistics of lifetime is in order. In Le et al. (2011), the initial loading portion of the loading history O–B–C was ignored. This resulted in a relation between the strength and lifetime which is the same as obtained here in Eq. (10). Based on this equation, the statistics of static lifetime were derived from the statistics of strength. Further it was shown that the non-existence of a threshold in the statistics of strength implied the non-existence of a threshold in the statistics of lifetime. However, these conclusions are valid only if the sustained stress σ_0 is so low

compared to the strength that the contribution to P_f from the initial short-time rising segment of load history is negligible (which is generally true for sustained loads in the service stress range).

If σ_0 is not low enough, then Eq. (11) needs to be used. This equation then implies the existence of a certain threshold in the statistics of the lifetime. The physical justification of the threshold is given by the fact that there is a non-zero probability of failure during the initial short-time increase of the load, i.e., the fact that some specimens may have a strength even lower than σ_0. This probability is related to the additional term in Eq. (11). However, for common n values ($n=20$–30, as indicated in Kawakubo (1995), for example), Eq. (10) gives a very good approximation when $\sigma_0 \leq 0.85\sigma_N$ and the threshold is negligible.

2.2. Relation between structural strength and static residual strength

We now extend the preceding arguments to the residual strength after a period of sustained loading. We consider the loading history O–B–D–E in which the stress is first raised rapidly from 0 to $\sigma_0 = t_0 r$, where r is the loading rate, then sustained for the period $t_R - t_0$, and at time t_R raised up to failure at a stress representing the residual strength σ_R. Integration over this history provides

$$\frac{1}{r}\int_0^{\sigma_0} \sigma^n \, \mathrm{d}\sigma + \int_{t_0}^{t_R} \sigma_0^n \, \mathrm{d}t + \frac{1}{r}\int_{\sigma_0}^{\sigma_R} \sigma^n \, \mathrm{d}\sigma = e^{Q_0/kT} \int_{\alpha_0}^{\alpha_c} \frac{1}{A l_0^{(n-2)/2} k_1^n(\alpha)} \, \mathrm{d}\alpha \tag{12}$$

This yields

$$\sigma_R^{n+1} + r(n+1)\sigma_0^n (t_R - t_0) = r(n+1) e^{Q_0/kT} \int_{\alpha_0}^{\alpha_c} \frac{1}{A l_0^{(n-2)/2} k_1^n(\alpha)} \, \mathrm{d}\alpha \tag{13}$$

Now, substituting from Eq. (5), we get

$$\sigma_R^{n+1} + r(n+1)\sigma_0^n \left(t_R - \frac{\sigma_0}{r}\right) = \sigma_N^{n+1} \tag{14}$$

or

$$\sigma_R = [\sigma_N^{n+1} - \sigma_0^n (n+1)(r t_R - \sigma_0)]^{1/(n+1)} \tag{15}$$

This is the equation for the degradation of the residual strength as a function of two independent (deterministic) variables, applied load σ_0 and time t_R of sustained load application. This equation also represents a link between the short-time strength and the residual strength. The short-time strength test and the lifetime test can be identified as two limiting cases of the above equation. For the strength test, $\sigma_0 = 0$ and $t_R = t_0$. Substituting these values, Eq. (15) reduces to $\sigma_R = \sigma_N$. Similarly, for the lifetime test, $t_R = \lambda$, and thus substitution of Eq. (10) into Eq. (14) yields $\sigma_R = \sigma_0$.

2.3. Analysis of residual strength degradation for one RVE

We now proceed to analyze the effect of various parameters on Eq. (15) for the residual strength degradation. Fig. 2a shows the degradation in strength of one RVE under static load for various values of n for $\sigma_0 = 0.5\sigma_N$. The time of load application normalized with respect to the lifetime is shown on the horizontal axis. For the sake of convenience, σ_N is assumed to be unity and the loading rate is taken as 0.5 MPa/s. As expected, the normalized residual strength decreases with the applied load until it reaches the lower limit σ_0/σ_N. In other words, the end condition that $\sigma_R = \sigma_0$ at $t_R = \lambda$ is satisfied.

As it can be seen, the rate of strength degradation is negligible initially but progressively increases. Rapid strength degradation is seen in the end. This effect gets more and more pronounced for higher values of n. For, e.g., $n=6$, the residual strength σ_R drops to 90% of the original strength σ_N after a hold time of $t_R = 0.543\lambda$, whereas for $n=20$ and $n=26$, this occurs after a hold time of $t_R = 0.873\lambda$ and $t_R = 0.929\lambda$, respectively. Based on this observation, the degradation curve could be roughly divided into two regions, one of relatively slow degradation and one of rapid degradation. The distinction between these two regions becomes more pronounced with high values of n.

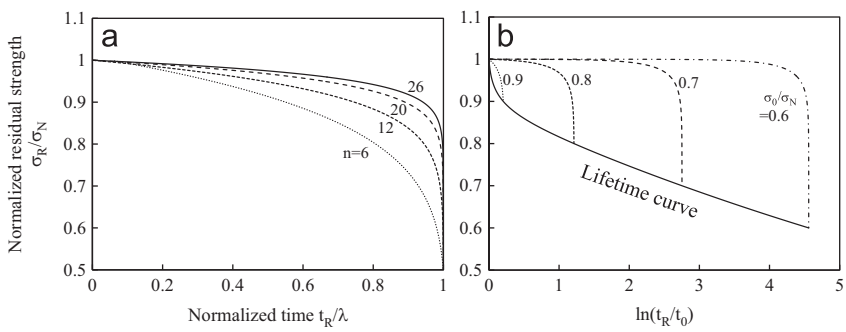

Fig. 2. Predicted degradation curves for (a) various values of static crack growth exponent n and (b) various values of applied load.

We now consider the effect of applied load on the predicted degradation in strength. Fig. 2b shows the normalized residual strength as a function of $\ln(t_R/t_0)$ for different values of applied load σ_0. Again, σ_N is assumed to be unity and the loading rate is taken as 0.5 MPa/s. A sub-critical crack growth exponent $n=26$ is assumed here, which is typical for ceramics such as alumina (Fett and Munz, 1991; Kawakubo, 1995). For a given applied load, increasing the time of load application leads to a decrease of the residual strength until the lower limit σ_0 is reached, which occurs at the intersection with the lifetime curve. Again, it can be seen that the rate of degradation continuously increases. When $\sigma_0=0.6\sigma_N$, the residual strength is almost equal to the structural strength for about 90% of its lifetime. Then it drops rapidly towards the σ_0 value. As the applied load is increased, the lifetime is reduced and the curves become steeper and steeper. For $\sigma_0=0.9\sigma_N$, the residual strength drops to σ_0 reaching the lifetime in a small fraction of time.

To document the large effect of the applied load level on the lifetime, it may be noted that the ratio of the lifetimes of the last and the first curves (i.e., those for $\sigma_0=0.6\sigma_N$ and $\sigma_0=0.9\sigma_N$, respectively) is about 1.5×10^4. A similar behavior was reported in Yang and Liu (1977), Yang (1978) and Diao et al. (1995) for strength degradation of composites under cyclic loading, in Nielsen (1996) for wood under static and cyclic loading, in Park and Lee (1997) for ceramics, and in Yavuz and Tessler (1993) for silicon carbide ceramics subjected to sustained loading at high temperatures.

This study reveals the usefulness of Eq. (15). For a given applied load, this equation can help determine the portion of lifetime for which the strength degradation is negligible. The only other parameter needed for this determination is the exponent n of static crack growth law.

3. Formulation of statistics of residual strength

For quasibrittle structures, the probabilistic aspects of strength and lifetime are more complex than they are for brittle or ductile structures. In Bažant and Pang (2006, 2007) and Le et al. (2011), the cdfs of strength and lifetime of a structure were derived from the statistics of one RVE using the finite weakest-link theory. Similarly, we now proceed to derive the cdf of residual strength for one RVE.

3.1. Formulation of statistics of residual strength for one RVE

The analysis of interatomic bond breaks and multiscale transitions to the RVE has shown that the strength of one RVE must have a Gaussian distribution transitioning to a power law in the tail of probability within the range of 10^{-4} to 10^{-3}. This may be described by the following grafted Gauss–Weibull distribution:

$$P_1(\sigma_N)=1-\exp(-\langle\sigma_N\rangle^m/s_0^m)=\Phi_W(\sigma_N)\quad \sigma_N<\sigma_{N,gr} \tag{16}$$

$$P_1(\sigma_N)=P_{gr}+\frac{r_f}{\sqrt{2\pi}\delta_G}\int_{\sigma_{N,gr}}^{\sigma_N} e^{-(\sigma'-\mu_G)^2/2\delta_G^2}\,\mathrm{d}\sigma'=\Phi_G(\sigma_N)\quad \sigma_N\geq\sigma_{N,gr} \tag{17}$$

where P_1 is the probability of the strength of one RVE; $\langle x\rangle=\max(x,0)$, μ_G and δ_G are, respectively, the mean and standard deviation of the Gaussian core if considered extended to $-\infty$, s_0 and m are, respectively, the scale and shape parameters of the Weibull tail, r_f is a scaling parameter required to normalize the grafted cdf such that $P_1(\infty)=1$, and P_{gr} is the grafting probability, $P_{gr}=1-\exp[-\sigma_{gr}^m/s_0^m]$. The continuity of the probability density function (pdf) at the grafting stress requires that $(\mathrm{d}P_1/\mathrm{d}\sigma)|_{\sigma_{gr}^+}=(\mathrm{d}P_1/\mathrm{d}\sigma)|_{\sigma_{gr}^-}$.

Based on the statistics of strength, the probability distribution of lifetime was derived in Le et al. (2011) using Eq. (10). Similarly, starting from the cdf of strength, it is now possible to determine the cdf of residual strength for one RVE by means of Eq. (15). This yields

$$P_{1,R}(\sigma_R)=1-\exp\left[-(\langle\sigma_R^{n+1}+\sigma_A\rangle/s_R)^{\overline{m}}\right]\quad \sigma_0\leq\sigma_R<\sigma_{R,gr} \tag{18}$$

$$P_{1,R}(\sigma_R)=P_{gr}+\frac{r_f}{\sqrt{2\pi}\delta_G}\int_{\sigma_{R,gr}}^{(\sigma_R^{n+1}+\sigma_A)^{1/(n+1)}} e^{-(\sigma'-\mu_G)^2/2\delta_G^2}\,\mathrm{d}\sigma'\quad \sigma_R\geq\sigma_{R,gr}>\sigma_0 \tag{19}$$

where $\sigma_A=\sigma_0^n(n+1)(rt_R-\sigma_0)$, $\sigma_{R,gr}=(\sigma_{N,gr}^{n+1}-\sigma_A)^{1/(n+1)}$, and $s_R=s_0^{n+1}$, $\overline{m}=m/(n+1)$; $P_{1,R}$ represents the probability of failure of one RVE under an overload, and $P_{1,R}(\sigma_0)$ represents the probability of failure of one RVE before the overload is applied. Note that only the part of the cdf where the residual strength is defined, i.e. where $\sigma_R\geq\sigma_0$, is considered.

Unlike the strength distribution, the residual strength cdf of one RVE does not have a pure Weibull tail. It is noteworthy that Eq. (18) describes a three parameter Weibull distribution in the variable, σ_R^{n+1}, which has a finite threshold. Although it was proved that there can be no finite threshold in the distribution of strength (Pang et al., 2008; Le et al., 2011), the same does not hold true for the residual strength. The existence of a threshold value, σ_A, in the cdf stems from the fact that some specimens could fail already during the period of sustained preload, which excludes them from the statistics of the overload. These are the specimens for which $\lambda<t_R$ or $\sigma_N<\sigma_0$. This is a crucial difference between the cdfs of strength and of residual strength. The threshold is described by σ_A which is a function of the deterministic parameters σ_0 and t_R. Note also that when t_R or σ_0 becomes sufficiently small, the threshold σ_A becomes negligible because the number of specimens with $\sigma_N<\sigma_0$ or $\lambda<t_R$ tends to zero. Then, the statistics of residual strength resembles the statistics of structural strength.

Similar to the cdfs of strength and lifetime, the cdf of residual strength is also a graft of two distributions. It was shown in Le et al. (2011) that although the cdf of strength is a Gauss–Weibull graft and the tail of the lifetime cdf is Weibullian, the rest of the lifetime cdf did not exactly follow the Gaussian distribution. Similarly, Eq. (19) here indicates that the rest of the residual strength distribution of one RVE does not exactly follow the Gaussian distribution. Also, note that the Weibull moduli of the cdfs of lifetime and of the $(n+1)$th power of the residual strength are the same.

3.2. Formulation of residual strength cdf for structures of any size

It has been demonstrated in various ways that the behavior of quasibrittle materials transitions from quasi-plastic to brittle as the structure size increases (Bažant, 1984, 1997, 2004, 2005; Bažant and Kazemi, 1990; Bažant and Planas, 1998). This transitional behavior was shown to be general, holding, in particular, even for the statistics of strength and lifetime of quasibrittle structures (Bažant and Pang, 2006, 2007; Le et al., 2011). In this theory, quasi-brittleness means that a structure fails if a single finite-size RVE fails. Accordingly, the structure can be considered to behave like a finite chain of RVEs. Using this hypothesis, it was shown that as the structure size D (or the number N of RVEs in the chain) increases, the strength distribution transitions from Gaussian to Weibullian.

Just like strength and lifetime, the residual strength of a chain is equal to the smallest residual strength among its links. So, the weakest-link model can be used to compute the residual strength cdf of a structure of any size, i.e., a structure consisting of any number of RVEs. Therefore, we proceed to derive the cdf of residual strength of a structure in a similar fashion.

Some explanations on statistics of quasibrittle failure may be appropriate. Once the cdf of residual strength related to one RVE is found, the cdf of failure of a structure of any size and geometry can be determined by means of the weakest link theory. The general applicability of this theory for brittle, ductile or quasi-brittle structures is guaranteed by the definition of RVE itself and the fact that failure is considered to occur at macro-crack initiation. One RVE is defined as the smallest part of the structure whose failure causes the failure of the entire structure. Thus, the RVE statistically represents a link (the failing RVE is the weakest link) and the structure can be statistically treated as a chain.

The key point for quasi-brittle structures is that the number of RVEs constituting the chain is finite, which stems from the fact that the fracture process zone size is not negligible compared to the structure size. Different from Weibull theory, which implicitly assumes an infinite number of RVEs, we apply a finite weakest link theorem, in which the number of elements constituting the chain, and thus the structure statistics, does depend on the RVE size. The two limit cases of this behavior are (i) a purely brittle behavior, in which case the structure represents an infinite chain, i.e., has an infinite number of point-wise RVEs, and (ii) a ductile (or quasi-plastic) behavior, where the entire structure consists of just one link undergoing distributed damage and its size is equal to or smaller than one RVE defined as a fully developed FPZ.

It should also be noted that, in three-dimensional structures, the weakest-link theory must often be applied in two dimensions, because of the restrictions of mechanics of failure. For example, in the bending of a beam of length l and depth D, mechanics dictates that all the elements across the beam width b must undergo damage and fail together, nearly simultaneously. They thus represent only one link in two-dimensional analysis. This greatly reduces the relevant number of RVEs in the structure (Le et al., 2011).

Similar to the definition of nominal strength, we define the nominal applied stress, $\sigma_0 = c_n P/bD$ or $c_n P/D^2$ for two- or three-dimensional scaling, respectively, where P is the applied load. Then, by applying the joint probability theorem to the survival probabilities, the residual strength distribution of the structure can be expressed as

$$P_{f,R}(\sigma_R) = 1 - \prod_{i=1}^{N} \{1 - P_{1,R}[\langle \sigma_0 s(\mathbf{x}_i)\rangle, t_R, \sigma_R]\} \tag{20}$$

where $s(\mathbf{x})$ is the dimensionless stress field and $\mathbf{x}$ is the position vector. Similar to the chain model for the cdf of structural strength, the residual strength of the i-th RVE is here assumed to be governed by the maximum average principal stress $\sigma_0 s(\mathbf{x}_i)$ within the RVE, which is valid provided that the other principal stresses are fully statistically correlated. Furthermore, similar to the calculation of strength cdf and of lifetime cdf, the residual strength of a structure of any geometry can be calculated by using the nonlocal boundary layer model (Le et al., 2011, 2012).

For small values of σ_A and sufficiently large structures (strictly speaking, for $N \to \infty$), the weakest-link model shows the residual strength cdf to be determined by the far-left tail of the residual strength cdf of one RVE: $P_{1,R}(\sigma_R) = (\langle \sigma_R^{n+1} + \sigma_A\rangle / s_R)^{\overline{m}}$. Therefore, noting that $r = s(\mathbf{x})\mathrm{d}\sigma/\mathrm{d}t$, Eq. (20) can be rewritten as

$$P_{f,R}(\sigma_R) = 1 - \exp\left[-\int_V \langle s(\mathbf{x})\rangle^m \frac{\mathrm{d}V(\mathbf{x})}{V_0} (\langle \sigma_R^{n+1} + \sigma_A\rangle / s_R)^{\overline{m}}\right] \tag{21}$$

where $V_0 = l_0^3$ is the volume of one RVE, and l_0 is the size of one RVE, which is a material property (material length).

As described in Bažant and Pang (2007) and Le et al. (2011), the strength of a specimen or structure under a non-uniform stress field may conveniently be defined by the equivalent number of RVEs, N_{eq}, which is the number of RVEs for which a uniform stress field gives the same strength distribution. In that case, each RVE is weighted according to the stress in that RVE. For instance, when $m=24$ and the stress of some RVE is $0.75 f_c$, then its weight is only 0.1% of the weight of an element whose stress is f_c (in detail, see Bažant and Pang, 2007 and Le et al., 2011). Similarly, for residual strength, we define $N_{eq,\sigma_R} = \int_V \langle s(\mathbf{x})\rangle^m \, \mathrm{d}V(\mathbf{x})/V_0$, so that N_{eq,σ_R} would represent the number of RVEs under uniform stress for which σ_R gives the

same cdf of residual structure strength as does Eq. (20) for the given σ_0 and t_R. In principle, the definition of N_{eq,σ_R} is different from the definition of $N_{eq,\sigma}$ for strength, but it turns out that the equivalent numbers of RVEs for the strength and residual strength have exactly the same values.

Starting from the previous analysis, the mean residual strength, considering also the specimens that do not survive to the constant applied load, can be computed by the following expression:

$$\overline{\sigma}_R = \int_{P_{f,R}(\sigma_0)}^{1} \sigma_R \, \mathrm{d}P_{f,R} + \int_0^{P_f(\sigma_0)} \sigma_N \, \mathrm{d}P_f + \sigma_0 \left[P_{f,R}(\sigma_0) - P_f(\sigma_0) \right] \tag{22}$$

where $P_{f,R}(\sigma_R)$ and $P_f(\sigma_N)$ represent the structure cdf of the residual strength and strength, respectively. On the right-hand side of Eq. (22), the first term refers to failures that occur after the application of the overload, the second term to failures occurring before the sustained constant load is reached, and the third term to failures occurring during the sustained constant loading.

The proposed relation between the structure strength and the residual strength indicates an efficient way to obtain either the lifetime or the residual strength distribution without any testing of the lifetime or the residual strength histograms, which is time consuming and costly, and for realistically long lifetimes virtually impossible. Aside from exponent n, one merely needs the Weibull modulus m of strength distribution, which can most easily be determined by tests of the mean size effect (Pang et al., 2008). Exponent n of power-law for crack growth can be obtained by the standard test of not too long duration, which measures the mean crack growth velocity (see, e.g., Ritter, 1974). In this way, either the distribution of lifetime or the distribution of residual strength for any kind of load histories can be fully characterized.

Alternatively, the present theory can be used as an efficient way to predict the lifetime distribution by means of tests of the strength and residual strength. In fact, all the required parameters can be calibrated by size effect tests of strength and residual strength by much fewer data than by histogram testing. Moreover, for sufficiently high applied loads, the tests of mean residual strength are far less time consuming than the lifetime tests. Of course this is true provided that the large size asymptote of the size effect plot can be discerned from laboratory scale tests.

4. Results and discussion

4.1. Comparison between strength, residual strength and lifetime distributions

The statistical formulation developed in the previous sections is used to predict the distribution of residual strength and compare it to distributions of strength and lifetime for a practical case. For the purpose, strength and lifetime histograms of four-point bend beams made of 99.6% Al_2O_3 reported by Fett and Munz (1991) are used to determine the parameters of the grafted Gauss–Weibull distribution.

Thirty specimens were used for each histogram testing. Fig. 3a and b shows, respectively, the experimentally observed strength and lifetime histograms plotted in Weibull scale with the optimum fits by the Gauss–Weibull distribution. Despite the relatively low number of the specimens tested, it is clear that both the strength and lifetime cdfs are not straight lines as required by the two-parameter Weibull distribution. Neither can the three-parameter Weibull distribution fit the whole distribution. As shown by the solid curves in Fig. 3a and b instead, the grafted Gauss–Weibull distribution gives a very good fit considering the scatter of the data. By optimum fitting, the Weibull moduli, m and $\overline{m}$, for strength and lifetime are estimated to be about 30 and 1.1, respectively (Le et al., 2011). According to Eq. (10), the exponent n of the power-law for the creep crack growth is found to be about 26. It should be emphasized that, except for n, all the parameters of the lifetime distribution are determined by the strength histograms. Thus, with the same set of parameters, the theory gives an excellent fit of the strength histograms and an excellent prediction of the lifetime distribution.

In view of the remarkable prediction of lifetime provided by the theory, the same parameters are used to predict the distribution of residual strength. This choice is supported by noting that, in the sense of load history, the residual strength

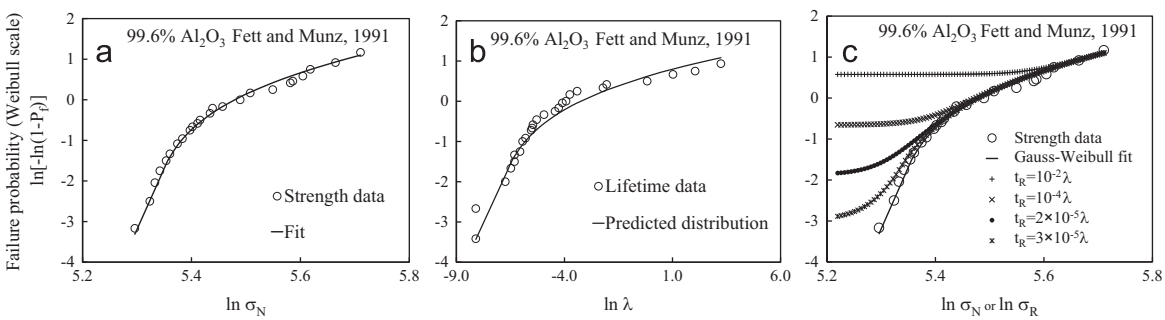

Fig. 3. (a) Optimum Gauss Weibull fit of strength histogram, (b) predicted lifetime distribution, and (c) predicted distribution of residual strength for various values of hold time (for 99.6% Al_2O_3).

test is an intermediate case of the strength and lifetime tests. Fig. 3c shows the Weibull plot of residual strength for different times of load application. The time is normalized to the mean lifetime, $\overline{\lambda}$, while the applied sustained load $\sigma_0 = 0.78\overline{\sigma}_N$ is chosen to be consistent with the lifetime histogram tests. For reference, the strength data are shown in the same figure. Only the part of the cdf related to the specimens that survive the sustained constant preload (i.e., those for which $\sigma_R \geq \sigma_0$) is considered.

The plot shows that, for high values of σ_R, the cdf of residual strength coincides with that of strength. This part of the distribution results from a chain of elements with sufficiently high strength compared to the applied load to show almost no degradation. For lower values of σ_R, the distribution of residual strength diverges from that of strength and rapidly reaches the lowest probability $P_{f,R}(\sigma_0)$. This probability stems from a chain of elements that fail before the overload is applied. For a given value of σ_0 the point at which the distribution begins to diverge shifts to the left for lower values of t_R. For sufficiently low values of t_R, the divergence occurs below the grafting probability. Above the point of divergence, the residual strength distribution almost resembles the Gaussian and a part of the Weibullian distribution. In such a case, the grafting probability values of strength and residual strength are exactly the same. This is due to the fact that the equivalent numbers of RVEs for strength and residual strength have the same values. Because the grafting probabilities of a structure of any size can be calculated as $P_{gr} = 1-[1-P_{gr,1}]^{N_{eq}}$, and because $P_{gr,1}$ values are the same (see Eqs. (16) and (18)), the grafting probability of the structure also must be the same. In the limit case where $t_R \to t_0$, the two distributions are, of course, identical.

4.2. Optimum fits of strength and residual strength histograms of borosilicate glass

In the previous section, the parameters of the grafted Gauss–Weibull distribution were determined by fitting the strength histograms and then they were used to predict the cdf of lifetime and residual strength. We now repeat the procedure to predict the residual strength distributions and compare them to the histograms for borosilicate glasses obtained from experiments by Sglavo and Renzi (1999). Fig. 4a–d show the experimentally observed strength and residual strength histograms plotted in the Weibull scale. All the data considered were determined by conducting, in deionized water, four-point bend tests of borosilicate glass rods with a nominal diameter of 3 mm and length of 100 mm. The loading rate was set to about 60 MPa/s and different sustained load durations were used.

Since glass is a brittle material and its RVE size is very small compared to the tested specimen size, the distribution of strength is virtually indistinguishable from the Weibull distribution, as can be seen in Fig. 4a–d. By the optimum fitting of strength and residual strength, a Weibull modulus m of about 6 and a value of n of about 30 have been estimated. The fit predicted by the statistical formulation, shown by the solid line curves, is seen to be in good agreement with the experimental results. Except for the 1 h case, all the other plots show the deviation of the residual strength distribution from the strength distribution to reach the probability value $P_{f,R}(\sigma_0)$. It should be emphasized that, despite the scatter and a low number of data, all the residual strength distributions are predicted using the same set of parameters. These predictions can be considered as a validation of the present theory.

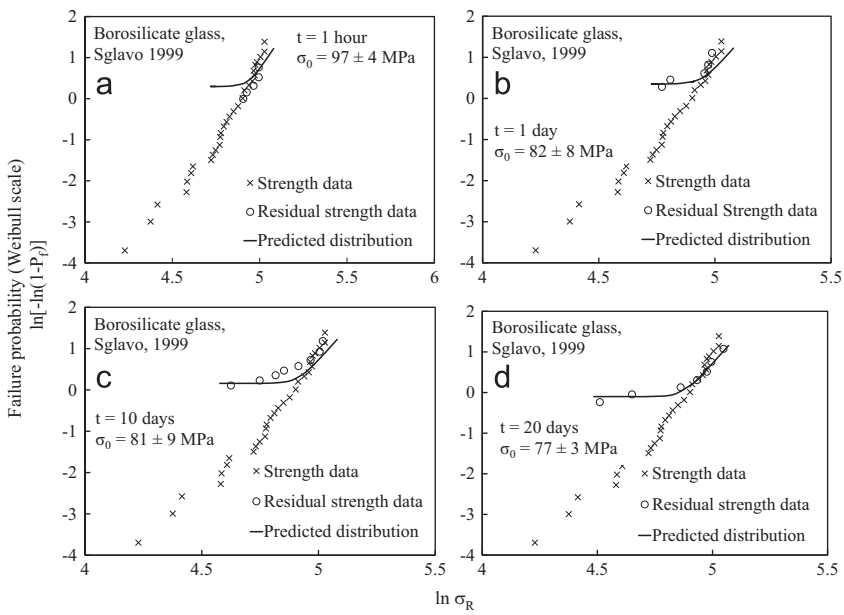

Fig. 4. Optimum fits of residual strength histograms for borosilicate glass hold times : (a) 1 h, (b) 1 day, (c) 10 days and (d) 20 days.

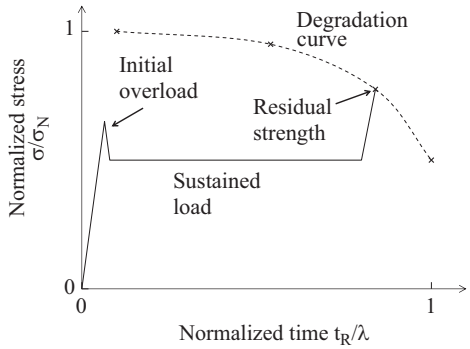

Fig. 5. Load history in the tests by Hahn and Kim (1975).

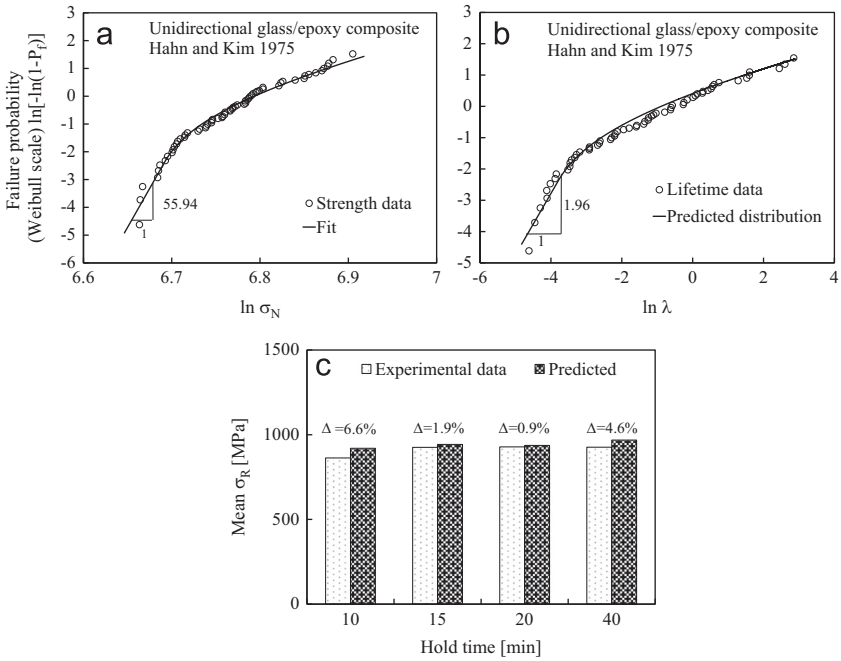

Fig. 6. (a) Optimum Gauss Weibull fit of strength histogram, (b) predicted lifetime distribution, and (c) comparison of predicted mean residual strength for unidirectional glass/epoxy composite (Hahn and Kim, 1975).

4.3. Optimum fit of strength histograms and prediction of lifetime and mean residual strength for unidirectional glass/epoxy composites

The methodology of the previous sections is now pursued for the strength, lifetime and residual strength data on unidirectional glass–epoxy composites reported by Hahn and Kim (1975). Each specimen analyzed consisted of 8 unidirectional plies. Seventy-one specimens were tested to obtain the strength and lifetime distributions. A constant sustained load $\sigma_0 = 758$ MPa was applied for all the lifetime tests. Unlike the previously considered cases, an initial overload σ_p was applied in the residual strength tests. After that, the load was decreased to σ_0, then held for time t_R and then monotonically increased up to failure. None of the specimens broke before the application of the final load (see Fig. 5).

Fig. 6a shows the fit of strength histograms by means of the grafted Gauss–Weibull distribution in the Weibull scale. Similar to the fits for alumina, this fit shows a kink in the curve corresponding to the transition from Weibull to Gaussian distribution. A value of m equal to 56 and a value of n equal to 27 are estimated by least-square optimum fitting. Then, with the set of parameters derived from strength, the lifetime distribution is predicted. As can be seen from Fig. 6b, the prediction represented by the solid line agrees very well with the experiments despite the scatter of the data.

It needs to be mentioned that Hahn and Kim (1975) proposed a methodology for predicting the cdf of lifetime from the cdf of strength. But its success was limited due to adopting a purely Weibullian distribution of strength, which corresponds to an infinite, rather than finite, weakest-link chain and ignores the quasibrittle behavior of the composites. The present theory overcomes this limitation by considering a finite weakest-link chain.

Now that the required parameters of the distribution have been identified, the theory is applied to predict the mean residual strength and compare it to the experimental data. The comparison is made only for the mean since the number of available data is not sufficient to study the entire cdf. Moreover, since the load history considered differed slightly from the one considered in this paper, as shown in Fig. 5, the following equation is used instead of Eq. (15) to compute the strength degradation:

$$\sigma_R = \left[\sigma_N^{n+1} - \sigma_0^n (n+1) r t_R' + 2(\sigma_0^{n+1} - \sigma_p^{n+1})\right]^{1/(n+1)} \tag{23}$$

Here σ_p is the initial overload and t_R' is the time of application of the sustained constant load measured after the initial overload, as reported in Hahn and Kim (1975). Eq. (23) is derived from Eq. (1) similarly as Eq. (15) was.

The resulting cdf of residual strength is then used to compute the mean values. No failures before the application of final overload were reported, which is explained by the small number of specimens tested. Therefore, the means computed from these tests considered only the surviving specimens. The probability of failure of the specimens surviving at the start of overload, denoted as P_f', can be determined as follows:

$$P_f' = \frac{P_{f,R} - P_0}{1 - P_0} \tag{24}$$

where $P_0 = P_{f,R}(\sigma_0)$ is the probability of failure before applying the final load. The results are shown in Fig. 6c for the different initial overloads and durations considered. Note that the predictions agree with the experiments, the difference being always less than 7%. The agreement provides another support for the present theory.

4.4. Prediction of strength degradation curve for soda-lime silicate glasses

The experimental data for verifying the distributions of residual strength after a static preload are relatively scarce. It is nevertheless possible to use other predictions for further verification of the theory. We now consider the existing data for the mean residual strength of soda-lime glass, as reported in Sglavo and Green (1995).

The strength histograms were determined by four-point bend tests in deionized water of soda-lime silicate glass rods with a nominal diameter of 3.2 mm and length 90 mm. The loading rate was set to about 47 MPa/s. The mean residual strength was studied for different durations and different applied loads. In this case, the set of parameters of the cdf is determined by fitting the histograms of strength. A value of m equal to 5.7 is determined while the parameter n is assumed to be equal to 24, since this is a typical value for such materials.

The mean residual strength is then computed according to Eq. (22) and compared to the experimental data in Fig. 7a–d where it is shown as a function of σ_0 for different durations t_R of initial load rise. As can be noted, the mean degradation curve of strength agrees very well with the available experimental data. It is shown that, for a given duration, the strength is initially almost unaffected by the applied load. Later, the effect of the applied load on the strength rapidly increases,

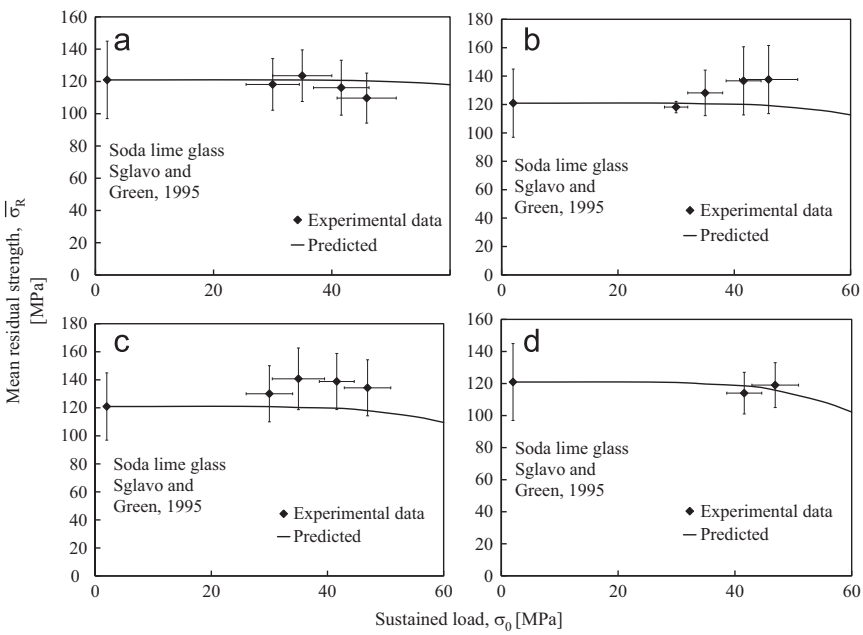

Fig. 7. Comparison of predicted strength degradation curves with experimental values for soda lime silicate glass at (a) 1 h, (b) 1 day, (c) 3 days and (d) 20 days.

lowering the mean residual strength of the structure significantly. Also, a more pronounced effect of the applied load can be seen on the strength degradation at higher values of t_R.

This and the previous fits are considered as a successful validation of the theory. It is seen that it can predict the statistical distributions, the means and the degradation curves with satisfactory agreement with experiments.

4.5. Comparison between the size effect on mean strength, residual strength and lifetime

A more severe check on the theory would be to test the size effect on the mean lifetime and residual strength. However, no such test data seem to be available in the literature. It is nevertheless interesting to predict the size effect on the mean residual strength using the theory. Once the distributions of strength and lifetime are determined, it is possible to compute their mean values as follows:

$$\overline{\sigma}_N = \int_0^1 \sigma_N \, \mathrm{d}P_f \tag{25}$$

$$\overline{\lambda} = \int_0^1 \lambda \, \mathrm{d}P_{f,\lambda} \tag{26}$$

where P_f and $P_{f,\lambda}$ are the cdfs of the strength and of the lifetime, respectively. When the specimens that do not survive the sustained preload are considered, the residual strength can similarly be computed from Eq. (22).

Fig. 8a and b show the size effect on both the mean structural strength and lifetime of the 99.6% Al_2O_3, predicted on the basis of the strength and lifetime cdfs that were calibrated by Fett and Munz′s histograms used in Section 4.1. Note that, for the large size limit, the curves of size effect on both the mean strength and lifetime tend to straight lines in the logarithmic plot. This agrees well with the power-law size effects of Weibull statistics because the strength and lifetime cdfs approach the Weibull distribution as the structure size increases (Bažant and Pang, 2006, 2007; Le and Bažant, 2009; Le et al., 2011). Fig. 8c shows the calculated size effect on the mean residual strength based on the set of parameters determined in Section 4.1 from the strength and lifetime histograms. An applied load $\sigma_0 = 0.78\overline{\sigma}_N$ is considered. Different times of load application are used, as reported in the figure, depending on the mean strength, i.e., $rt_R = \beta\overline{\sigma}_N$. Note that, for a given t_R, the mean residual strength shows a similar trend as the strength and lifetime for the large size limit. In fact, the means tend to a straight line with the same slope as the mean strength.

With reference to Eq. (22), this can be explained as follows. Since, in the large size limit, the Weibull statistics applies, the mean strength scales according to $\overline{\sigma}_N = \overline{\sigma}^W_{N,0}(D_0/D)^{n_d/m}$ where $\overline{\sigma}^W_{N,0}$ is the intercept of the Weibull asymptote on the Y-axis ($D = D_0$) and n_d is the number of dimensions. Now, if the applied load is expressed as a fraction of the mean strength, i.e., $\sigma_0 = \alpha\overline{\sigma}_N$, then the

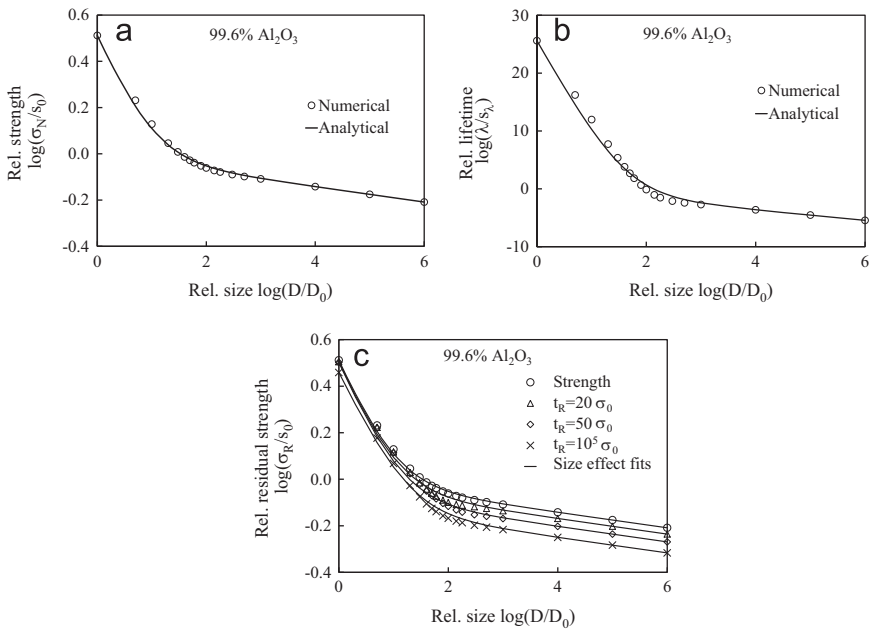

Fig. 8. Calculated size effect curves on the mean (a) strength, (b) residual strength at different hold times, and (c) lifetime for 99.6% Al_2O_3.

probability of failure during the initial load application, $P_f(\sigma_0)$, does not depend on the size. In fact, for size D,

$$P_f(\sigma_0)|_D = 1-\exp\left[-\left(\frac{D}{D_0}\right)^{n_d}\left(\frac{\alpha\overline{\sigma}_N}{s_0}\right)^m\right] = 1-\exp\left[-\left(\frac{\alpha\overline{\sigma}^W_{N,0}}{s_0}\right)^m\right] = P_f(\sigma_0)|_{D_0} \tag{27}$$

The same holds true for the probability of failure during the application of the sustained load, $P_{f,R}(\sigma_0)$. Since $rt_R = \beta\overline{\sigma}_N$, one has $\sigma_A = \overline{\sigma}_N^{n+1}(n+1)(\beta-\alpha) = \chi\overline{\sigma}_N^{n+1}$. Accordingly,

$$P_{f,R}(\sigma_0)|_D = 1-\exp\left[-\left(\frac{D}{D_0}\right)^{n_d}(\alpha+\chi)^{m/n+1}\left(\frac{\overline{\sigma}_N}{s_0}\right)^m\right] = 1-\exp\left[-(\alpha+\chi)^{m/n+1}\left(\frac{\overline{\sigma}^W_{N,0}}{s_0}\right)^m\right] = P_{f,R}(\sigma_0)|_{D_0} \tag{28}$$

Thus, the third term in Eq. (22) is proportional to $\alpha\overline{\sigma}_N$ and it scales according to $\alpha\overline{\sigma}^W_{N,0}(D_0/D)^{n_d/m}$. Now, for a given probability of failure, the strength and the residual strength scale as $\sigma_N|_D = (D_0/D)^{n_d/m}\sigma_N|_{D_0}$ and $\sigma_R|_D = (\sigma_N^{n+1}-\beta\overline{\sigma}_N^{n+1})|_D^{1/n+1} = (D_0/D)^{n_d/m}(\sigma_N^{n+1}-\beta\overline{\sigma}_N^{n+1})|_{D_0}^{1/n+1} = (D_0/D)^{n_d/m}\sigma_R|_{D_0}$, respectively. Thus, in the large size limit, all the terms in Eq. (22) scale with the same power law and the mean residual strength can now be expressed as

$$\overline{\sigma}_R = (D_0/D)^{n_d/m}\overline{\sigma}^W_{R,0} \tag{29}$$

where $\overline{\sigma}^W_{R,0}$ is the intercept of the Weibull asymptote on the Y-axis ($D = D_0$). Thus it is shown that, in the large size limit, the trend of mean residual strength is similar to the one of strength except for the fact that the mean residual strength is lower depending on the applied load and time of load application. Upon increasing the applied load and the time of load application, the mean residual strength decreases until it reaches the value of the sustained applied load. This happens when, for all the sizes, most specimens fail before the application of the overload.

It is impossible to obtain analytical expressions for $\overline{\sigma}_N$ and $\overline{\lambda}$. However, sufficiently accurate analytical formulas have been obtained by asymptotic matching (Bažant, 2004, 2005; Le et al., 2011):

$$\overline{\sigma}_N = \left[\frac{N_a}{D}+\left(\frac{N_b}{D}\right)^{\psi/m}\right]^{1/\psi} \tag{30}$$

$$\overline{\lambda} = \left[\frac{C_a}{D}+\left(\frac{C_b}{D}\right)^{\phi/m}\right]^{(n+1)/\phi} \tag{31}$$

where the size effect exponent of strength, m, must be equal to the Weibull modulus of strength distribution, n is the exponent of the power-law for creep crack growth and N_a, N_b, ψ, C_a, C_b and ϕ can be determined by matching the following six asymptotic conditions (Le et al., 2011):

(1) $[\overline{\sigma}_N]_{D\to l_0}$,
(2) $[d\overline{\sigma}_N/dD]_{D\to l_0}$,
(3) $[\overline{\sigma}_N D^{1/m}]_{D\to\infty}$,
(4) $[\overline{\lambda}]_{D\to l_0}$,
(5) $[d\overline{\lambda}_N/dD]_{D\to l_0}$ and
(6) $[\overline{\lambda}_N D^{(n+1)/m}]_{D\to\infty}$.

Since the large size asymptotic behavior of mean residual strength resembles that of mean strength and the shape of the size effect curve is similar, the size effect can reasonably be approximated by an equation similar to Eq. (31):

$$\overline{\sigma}_R = \left[\frac{M_a}{D}+\left(\frac{M_b}{D}\right)^{\eta/m}\right]^{1/\eta} \tag{32}$$

where m is the Weibull modulus of the cdf of strength and, similar to the size effect on the mean strength, M_a, M_b and η can be derived by matching three asymptotic conditions:

(1) $[\overline{\sigma}_R]_{D\to l_0}$,
(2) $[d\overline{\sigma}_R/dD]_{D\to l_0}$, and
(3) $[\overline{\sigma}_R D^{1/m}]_{D\to\infty}$.

As can be noted from Fig. 8c, the approximation given by Eq. (32) is rather good for all the different times of load application.

In deriving the foregoing result, the two ratios, i.e., of the sustained applied load to strength and of the hold time to lifetime, were kept constant across the sizes. Although trivial, it should be noted that if the absolute value of the applied load or the hold time, or both, is kept constant, the size effect will of course be much stronger. However, in this case, the mean residual strength does not resemble the strength curve and it cannot be described by Eq. (32).

An important practical merit of the previous theory (Bažant and Pang, 2006, 2007; Le et al., 2011) and its present extension is that they provide a way to determine the strength, residual strength and lifetime distributions without any

histogram testing. In fact, if size effect tests on the mean strength and the crack growth rate are available, they can be used to calibrate the grafted distributions. Once the set of parameters of the distribution are known, they can be used to obtain not only the cdf of strength but also the cdf of residual strength and the cdf of lifetime. Moreover, the curve of mean size effect can be calibrated with much fewer tests since the mean has a much smaller coefficient of variation, $1/\sqrt{N_d}$ smaller where N_d is the number of individual data points.

Alternatively, the present theory can be used as an efficient way to predict the lifetime distribution by means of the tests of strength and of residual strength. For sufficiently high applied loads, the tests of mean residual strength are far less time consuming than the lifetime tests.

5. Conclusions

1. A theory for predicting the probabilistic distributions of residual strength after a period of static load has been developed and validated against test data. An important practical merit of the present theory combined with predecessors (Bažant and Pang, 2006, 2007; Le et al., 2011) is that it provides a way to determine the strength, residual strength and lifetime distributions without any histogram testing.
2. The rate of degradation of strength under a constant static load is not constant. Initially it is very slow and in the end very rapid. This effect is more pronounced for higher static crack growth exponents.
3. The cdf of residual strength of quasibrittle materials may be closely approximated by a graft of Gaussian and Weibull distributions. In the left tail, the distribution is a three parameter Weibull distribution in the variable σ_R^{n+1}. Unlike the cdfs of strength and lifetime, the cdf of residual strength has a finite threshold, albeit often very small.
4. The finiteness of the threshold is explained by the fact that some specimens may fail during the sustained static preload and are thus excluded from the statistics of overload.
5. An expression for the size effect on the mean residual strength is derived using asymptotic matching. It is shown that the size effect on the residual strength is as strong as the size effect on strength.
6. Good agreement with the existing test data on glass–epoxy composites and on borosilicate and soda-lime silicate glasses is demonstrated.

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