Engineer's Digest of $E = mc^2$

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Engineering students occasionally wonder: why not $\frac{1}{2}mc^2$? While most have no need for Einstein's special theory of relativity, this theory nevertheless offers them a shining example of the power of mathematical deduction and beautiful simplicity. At its 100th anniversary in 2005, I gave my students a one-page pencil note explaining this glorious gem of a theory. The note became popular, and here, upon invitation, I give a simplified derivation, trying to waste no word and achieve utmost brevity, a boost to clarity.

Everything follows from three hypotheses: (**I**) The speed of light, *c,* is constant regardless of the speed of moving observer (as revealed in 1887 by Michelson–Morley experiment at Case Western, Cleveland), (**II**) momentum $p = mv$ is conserved (though not $F = ma$, as erroneously assumed before Einstein), and (**III**) energy $E = \int F \cdot v \, \mathrm{d}t$ is conserved (t, time; m, mass; v, velocity; a, acceleration; F, force).

Imagine a light-based clock in which time t is measured by the distance traveled by a light ray bouncing between two parallel mirrors at distance h (Fig. 1(*a*)). If the clock is stationary, $t = nh/c$, where n is the number of light reflections.

Then, imagine the clock to move at constant velocity v in direction x aligned with the mirrors (Fig. 1(*c*)). According to hypothesis I, the ray travels a distance which, according to the vector triangle in Fig. 1(*b*), satisfies the equation $(ct)^2 = (ct')^2 - (vt')^2$, from which

$$t/t' = \sqrt{1 - v^2/c^2} \tag{1}$$

So, *relative* to time t measured on the fixed clock, time t' in the moving clock runs slower.

Now, rotating Fig. 1(*a*) by 90 deg, we imagine the mirrors to be normal to the v-direction x; Fig. 1(*d*). Eq. (1) must hold again. The number n of light ray reflections counted on the stationary and moving clock must be the same, and so (from hypothesis I), the distance between the mirrors must change from h to h', such that $n = t/(h/c) = t'/(h'/c)$. Substituting for t from Eq. (1), we see that $h/h' = \sqrt{1 - v^2/c^2}$ at a given time t in the fixed clock, i.e., $h(t)$ contracts *relative* to $h'(t)$. Since hypothesis I implies nonexistence of any preferred reference frame, the same applies at any given time t' (instead of t) in the moving clock *relative* to the fixed clock, i.e., $h'(t')$ contracts equally *relative* to $h(t')$:

$$h'/h = \sqrt{1 - v^2/c^2} \tag{2}$$

Consider now the transformation between systems S and S' of coordinates (x, y, z, t) and (x', y', z', t') moving at relative velocity v in the direction x. For the movement of S *relative* to S', Eq. (2) (or hypothesis I) implies that coordinate x' gets contracted, $x = vt + x'\sqrt{1 - v^2/c^2}$. From this,

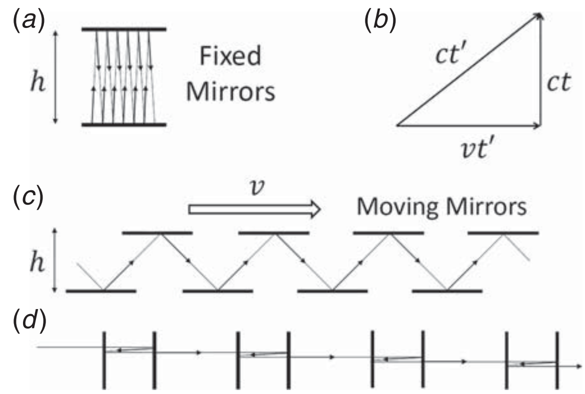

Fig. 1 Light ray path seen by fixed observer in his time *t*

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{3}$$

In the movement of S' *relative* to S, x gets contracted, and so (according to hypothesis I) $x\sqrt{1 - v^2/c^2} = vt' + x'$. Expressing x' from this equation and substituting it into Eq. (3), one gets

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \tag{4}$$

Equations (3) and (4) (with $y' = y$ and $z' = z$) represent the Lorentz (1904) transformation.[1]

From hypothesis II, $\mathrm{d}(mv)/\mathrm{d}t' = \mathrm{d}(m_0 v)/\mathrm{d}t = [\mathrm{d}(m_0 v)/\mathrm{d}t'](\mathrm{d}t'/\mathrm{d}t)$. So, at constant v, $\mathrm{d}(mv)/\mathrm{d}t' = [\mathrm{d}(m_0 v)/\mathrm{d}t'] / \sqrt{1 - v/c^2}$. Hence,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad \text{(Einstein 1905)} \tag{5}$$

Equation (5) yields $m^2c^2 = p^2 + m_0^2 c^2$. By differentiation:[2] $\mathrm{d}(m^2c^2)/\mathrm{d}t = \mathrm{d}(p^2)/\mathrm{d}t$ or $2m\,\mathrm{d}(mc^2)/\mathrm{d}t = 2p\,(\mathrm{d}p/\mathrm{d}t)$. Substituting $p = mv$ and $\mathrm{d}p/\mathrm{d}t = F$, we get $\mathrm{d}(mc^2)/\mathrm{d}t = F \cdot v = \mathrm{d}E/\mathrm{d}t$ (hypothesis III). Integration with initial condition $m = m_0$ at $v = 0$ yields:[3,4,5]

$$E = mc^2 \quad \text{(implied in Einstein 1905)} \tag{6}$$

References

[1] Landau, L.D., and Lifshitz, E.M., 1974. *The Classical Theory of Fields*, Pergamon, New York.

[2] Feynman, R. P., Leighton, R. B., and Sands, M., 1963, *The Feynman Lectures on Physics*, Vol. I, Addison-Wesley, Redwood City, CA.

[3] Einstein, A., 1905, "Zur Elektrodynamik bewegter Körper," Annalen der Physik, **17**, p. 891.

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[1]It is interesting that Lorentz originally introduced his transformation to make Maxwell equations of electromagnetism invariant. Its salient property is $x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$, as may be checked by substituting Eqs. (3) and (4). Thus, the vector (x, y, z, ict) preserves its length upon rotation, behaving like a vector in a 4D linear Euclidean space ($i^2 = -1$).

[2]Here, we must implicitly allow v, along with p, to be variable. As pointed out to me by JS Ben-Benjamin, v may better be kept everywhere constant and hypotheses II and III dispensed with if one assumes, more fundamentally, that Hamilton's equation $v = \partial E/\partial p$ applies relativistically [1].

[3]For $m > m_0$, Eq. (6) calls for deeper discussion; briefly, m_0 is properly regarded as the magnitude of a conserved four-vector $(\vec{p}, iE)$ ($\vec{p}$ = momentum vector), and Eq. (6) follows from $\vec{p} = 0$.

[4]A much more thorough, yet still simple, derivation is given in Ref [2]; rigorously, cf. [1]; original, [3].

[5]Thanks for helpful comments are due to Marlan O. Scully and Jonathan S. Ben-Benjamin of Texas A&M known for new relativistic quantum radiation sources at black holes and to Petr Vogel of Caltech.