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Quasibrittle Size Effect: Problems and Progress

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Abstract: Qualitatively proposed by Mariotte and mathematically formulated by Weibull, the statistical theory of size effect reigned supreme until the 1980s. However, beginning in 1976, a different, nonstatistical theory of size effect gradually emerged and was shown to be important for brittle heterogeneous materials, concisely termed "quasibrittle", which include concrete, fiber composites, sea ice, rocks, stiff cohesive soils, snow slabs, foams, wood, paper, bone and many materials on micrometer scale. In these materials, the maximum load is attained only after the stable formation of either (1) a large fracture process zone (FPZ), with distributed cracking, or (2) a large crack. The lecture first reviews the sources of these type I and II size effects, consisting of the energy release associated with stress redistribution prior to maximum load, and combined in Type I size effect with the statistical weakest link model for a finite chain. Numerical simulation, requiring nonlocal and stochastic approaches, is discussed. The asymptotic matching approach (anchored through dimensional analysis) to the development of analytical laws for Type I and Type II quasibrittle size effects, is then outlined. It is explained that while in Type 2 material randomness affects only the variance and the tail of the cumulative distribution of RVE strength is unimportant, in Type I it also strongly (though not totally) affects the mean, and the far-left tail plays a major role. This role can be theoretically predicted by nano-fracture mechanics of cracks in the atomic lattice based on crack length dependence of the activation energy barriers for the metastable states of the free energy potential of the atomic lattice. Extensions to reentrant corners, time dependence, and cyclic loading are pointed out. Various structural applications, experimental evidence, and reinterpretation of some structural disasters are discussed. Finally, some important open problems are highlighted.

1. Introduction

Size effect is a problem of scaling, which has occupied a central position in many problems of physics and engineering. In solid mechanics, the main interest of scaling is the dependence of the structural strength on the structure size. The speculations about size effect can be traced back to Leonardo Da Vinci [1]. The idea of statistical size effect, which is caused by randomness of material strength, was qualitatively proposed by Mariotte [2] and then mathematically formulated by Weibull [3, 4]. Until 1980s, all the experimentally observed size effects were explained by the statistical theory.

With the advances in research on concrete and other quasibrittle materials, it has been established that, aside from the statistical size effect, there exists a deterministic size effect which is governed by the energy release due to stress redistribution. The quasibrittle materials include concrete, fiber composites, sea ice, rocks, stiff cohesive soils, snow slabs, foams, wood, paper, bone and many materials on micrometer scale. Numerous tests have demonstrated a marked deterministic size effect, which is much stronger than exhibited by Weibull's statistical theory. Based on an approximate

asymptotic energy release analysis, a closed-form size effect law bridging the small- and large-size asymptote was proposed in [5] for the failures which occur after stable crack growth. Later, the size effect laws [6-10] for deterministic and deterministic-statistical size effect for failures at crack initiation have been developed. Numerous other models contributed to the development of size effect theory [11-26].

Recently, there are rising efforts to use modern quasibrittle materials (i.e., fiber-reinforced concrete, advanced composites) to build large infrastructures, fuel-efficient aircrafts, light crashworthy automobiles and large low-weight ships, for which there inevitably exists a large gap between the full scale structure and the laboratory-size specimen. Taking into account the size effect becomes crucial for safe, efficient and durable design. The paper gives a short review of the nature of scaling and presents a brief summary of the recent advances in quasibrittle size effect.

2. Review of Type I and II Size Effects

Quasibrittle materials are materials that 1) are incapable of purely plastic deformations, and 2) in normal use, have a FPZ which is not negligible compared to the structure size. A salient property of quasibrittle materials is that they obey on a small scale the theory of plasticity characterized by material strength, and on a large scale the LEFM characterized by fracture energy. This dichotomy leads to transitional size effect, which is deterministic in nature. In the deterministic size effect, one discerns principally the Type I size effect, occurring in structures that fail at crack initiation from a smooth surface, and Type II size effect, that occurs in structures with a deep notch or stress-free (e.g., fatigued) crack formed stably before failure.

2.1. Type I Size Effect

The Type I size effect occurs for crack initiation from a smooth surface. At crack initiation, a continuous macroscopic crack has not yet started to propagate. However, this does not mean that the size of FPZ is negligible and the size effect is purely governed by statistical theory, unless the structure size is extremely large compared to the FPZ. Because of the heterogeneity of the quasibrittle materials, the FPZ is still not negligible and the maximum load is obtained typically right after this large cracking zone coalesces into a continuous crack. Because of the existence of a large cracking zone in the boundary layer prior to the peak load, one cannot expect the Weibull theory to be applicable for finite-size structures.

Within the framework of fracture mechanics, the Type I size effect can be deduced by considering the limiting case where the energy release approaches zero with the vanishing crack length [6, 8, 10]. Because the dimensionless energy release rate function $g(\alpha)$ vanishes for $\alpha \to 0$ while its first and second derivatives do not, the third term of the large-size asymptotic series expansion of function $g(\alpha)$ must be retained to capture the Type I size effect. This gives

$$\sigma_N = \left(\frac{E' G_f}{g'(0) c_f + g''(0) c_f^2 / 2D} \right)^{1/2} = f_{r\infty} \left(1 - \frac{2D_b}{D} \right)^{-1/2} \approx f_{r\infty} \left(1 + \frac{rD_b}{D} \right)^{1/r} \quad (1)$$

where c_f is the effective length of FPZ for crack initiation, and D_b = constant = about four times of the maximum aggregate size, which roughly represents twice the thickness of the boundary layer of cracking [8].

The complete Type I size effect, which also includes the statistical effect for large-size structures, can be derived through the nonlocal Weibull theory [27]. According to Weibull's statistical theory, the structure fails as soon as one representative volume attains the strength limit. This leads to the Weibull distribution

$$P_f = 1 - \exp\left\{ -\int_V \left\langle \frac{\sigma(x) - \sigma_u}{\sigma_0} \right\rangle^m \frac{dV(x)}{V_r} \right\} \quad (2)$$

where V = volume of structure, σ_0 = scale parameter, σ_u = strength threshold, and m = Weibull modulus. To take into account the micro-crack interaction and stress redistribution due to the FPZ in quasibrittle materials, a nonlocal generalization, which imposes the spatial correlation over the representative volume element (RVE), was developed in [27,28]. In the nonlocal generalization, the probability of material failure at a material point depends not only on the stress at this point, but also on the stress in the neighboring area as well.

$$P_f = 1 - \exp\left\{ -\int_V \left\langle \frac{E\bar{\varepsilon}(x) - \sigma_u}{\sigma_0} \right\rangle^m \frac{dV(x)}{V_r} \right\} \quad (3)$$

Where $\bar{\varepsilon}$ is the nonlocal strain of the corresponding elastic stress field. The numerical calculation based on nonlocal Weibull theory is able to capture the small- and large-size asymptotes, and at the same time overcome the spurious mesh sensitivity, a tremendous advantage compared to the stochastic finite-element method (SFEM).

For large-size structures, the thickness of boundary layer is negligible compared to structure size D. Since the material strength is random, a macro-crack can initiate at many different points in the structure. Therefore, for $D / D_b \to \infty$, the Weibull's statistical size effect should be applicable. Based on the fracture mechanics and nonlocal Weibull theory, Eq. (1) is extended and a general Type I size effect law [27-29] amalgamating both the energetic and statistical size effects is generated

$$\sigma_N = f_{r\infty} \left[\left(\frac{D_b}{D} \right)^{rn/m} + \frac{rD_b}{D} \right]^{1/r} \quad (4)$$

where m = Weibull modulus and n = 1, 2, or 3 for uni-, two- or three-dimensional similarity of fracture. For small sizes and $m \to \infty$, Eq. (4) converges to Eq. (1), and for large sizes it converges to Weibull's size effect $\sigma_N \propto D^{-n/m}$.

2.2 Type II Size Effect

When a quasibrittle structure has a deep notch or a large traction-free (i.e., fatigued) crack, a continuous macroscopic crack will start to propagate from the notch or crack tip

during fracture growth. There is no chance for the dominant crack to initiate elsewhere in the structure volume, which means that material randomness cannot cause any size effect. For positive structure geometry, the failure happens right after the FPZ gets fully developed. Thus, the size effect on the mean nominal strength of structure is essentially energetic, while statistics can affect only the standard deviation of structure strength, not its mean [27].

The Type II size effect law can be derived by using asymptotic approximation of the energy release function for the propagating crack based on the equivalent linear elastic fracture mechanics (LEFM) [6, 8, 10, 30], or the J-integral [8, 20]. Truncating the expansion of energy release function $g(\alpha)$ after the second term, one can express the Type II size effect law:

$$\sigma_N = \sqrt{\frac{E' G_f}{g(\alpha_0 + c_f / D)D}} \approx \sqrt{\frac{E' G_f}{g'(\alpha_0)c_f + g(\alpha_0)D}} = \frac{\sigma_0}{\sqrt{1 + D / D_0}} \tag{5}$$

where α_0 = relative initial crack length, c_f = effective length of PFZ (considered as a material parameter), and E' = effective Young's modulus.

4. Overview of Recent Results on Dimensional Analysis and Asymptotic Matching

For the size range of practical structures, a direct analytical solution for size effect law is generally difficult, if not impossible, to obtain. However, by scaling the structure down to vanishing size, or up to infinite size, one can get a ductile or brittle response, either of which is much easier to solve. With the knowledge of these asymptotic limits, an approximate failure prediction for the middle range of practical interest can be obtained by a sort of interpolation between the opposite infinites, called the asymptotic matching.

Dimensional analysis is a powerful tool to obtain the asymptotic limits. The application of dimensional analysis to size effect rests on two hypotheses: 1) the major cracks at failure are geometrically similar for different sizes, and 2) the fracture process zones at the tips of a notch or a crack give approximately the same energy dissipation rates, regardless of structure size. Both hypotheses are not valid for Type I size effect. However, for Type II size effect, both hypotheses are good approximations, as supported by the cohesive crack model [8].

According to the Π-theorem of dimensional analysis [31, 32], two asymptotes of opposite infinities can be obtained easily. From the governing parameters of the failure problem, σ_N, D, f_t', G_f and E, one may form two independent dimensionless parameters expressed as

$$F(\Pi_1, \Pi_2) = 0, \quad \Pi_1 = \sigma_N^2 D / E' G, \quad \Pi_2 = \sigma_N^2 / f_t'^2 \tag{6}$$

where $F(\Pi_1, \Pi_2)$ is the governing function of failure problem. While other polynomials could serve as dimensionless parameters, these are the only ones with the property that one vanishes at $D \to 0$ and the other at $D \to \infty$. For the size effect, the other parameters characterizing the geometry in the third dimension may be ignored because they remain constant when the structure is scaled up or down. If the structure is approaching vanishing size, the size of FPZ becomes much larger than the entire cross section. Hence

there is no crack propagation and Π_1 should be removed from function $F(\Pi_1, \Pi_2)$. Since all the other parameters are constant for the scaled geometry, σ_N must also be constant. Thus one obtains the asymptotic limit for the small sizes. On the other hand, if the structure size tends to infinity, the FPZ becomes a point compared to the structure size. So there exists a stress singularity, which means that the local material strength f_t' does not matter. Therefore, Π_2 must be removed from governing function. In this way, the large-size asymptote, i.e. $\sigma_N \propto D^{-1/2}$, is obtained [33].

If the failure function F is assumed to be smooth, one can approximate it by a Taylor series centered about some state (D, σ_N) in the middle of the size range. It follows

$$\Delta F = F^* + F_1 \Delta\Pi_1 + F_2 \Delta\Pi_2 \tag{7}$$

where $F_i = \partial F / \partial \Pi_i$ (i = 1,2), $\Delta\Pi_1 = (\sigma_N^2 D - \sigma_N^{*2} D^*) / EG_f$ and $\Delta\Pi_2 = (\sigma_N^2 - \sigma_N^{*2}) / f_t'^2$. Substituting these values into Eq. (7) and solving for σ_N, one obtains an equation of the form of Bažant's classic size effect law [5]. Of course, different forms, which satisfy both the first- and second-order asymptotic properties of the cohesive crack model, can be chosen for dimensionless variables Π_1 and Π_2. However, they lead to more complex size effect formulas differing from Eq. (5) only by third- and higher-order terms of the asymptotic expansions [34]. Thus, if dimensional analysis is combined with asymptotic matching, the resulting size effect law is unique up to the second-order term.

5. New Results on Size Effect on Strength and Lifetime Distribution of Quasibrittle Structures

For quasibrittle structures following the weakest-link model, the Weibull distribution cannot be applied due the fact that the size of the representative volume element (RVE) is not negligible compared to the structure size. The type of probability distribution is derived from the transition rate theory on the atomic scale [35]. It turns out, that on the RVE scale, the strength distribution is Gaussian, with a Weibull tail grafted on the left at failure probability cca 10^{-3}. As the structure size increases, the grafted Weibull tail expands and the Gaussian core shrinks, until in the large size limit, the entire distribution becomes Weibullian. The model also indicates a strong size effect on the mean strength, which agrees well with the predictions of several models such as nonlocal Weibull theory, nonlocal damage model, crack-band model, and cohesive crack model [36, 37].

The theory has recently been extended to model the lifetime distribution of quasibrittle structures. The randomness in the strength and lifetime of one RVE are correlated by applying Evans' power law for the rate of growth of a dominant subcritical crack on the RVE level. Similar to strength distribution, the type of lifetime distribution is found to be size- and geometry- dependent. The size effect on the mean structure lifetime for the case of creep-rupture is shown to be more pronounced than the size effect on mean strength. The theory permits calculation of the safety factor on the basis of tolerable failure probability (generally $<10^{-6}$) and specified lifetime, which is shown to be size-dependent.

6. New Results on Influence of Reentrant Corner, Loading rate and Cyclic Loading

Reentrant corner – A general size effect law has been developed for the quasibrittle structures with reentrant corners symmetrically loaded in tension. The stress singularity $(1-\lambda)$ caused by the sharp but finite notch angle is weaker than the crack-tip singularity, which implies a vanishing energy release rate at the corner tip. The asymptotic properties of the size effect law can be analytically obtained by the linear elastic fracture mechanics. The singular elastic stress field can be obtained by the Williams' solution. The simplest fracture criterion is to consider that the structure reaches its peak load once the normal stress at the point $r = c_f$ (i.e., at the middle of FPZ) becomes equal to the material strength f_t'. It yields: $\sigma_N \sim kf_t'\left(c_f / D\right)^{1-\lambda(\gamma)}$. Note that σ_N depends not only on D but also on the notch angle γ. By asymptotic matching, one obtains the following size effect law [38]:

$$\sigma_N = \sigma_0\left(1 + D / D_{0\gamma}\right)^{1-\lambda(\gamma)} \tag{8}$$

where $D_{0\gamma}$ is transitional size for notch of angle γ. The same type of equation is shown to be applicable to describe the size effect on the strength of metal-composite hybrid joint, in which λ denotes the real part of the exponent of the displacement singularity at the bimaterial corner [39].

Loading rate – the dependence of fracture on the loading rate basically results from: 1) viscoelasticity of material; and 2) bond rupture. Predominantly, the first one is the source for polymers and the second one for rocks and ceramics. In other quasibrittle materials such as concrete, both sources of time dependence are important [40-42]. The effect of viscoelasticity can be introduced to the cohesive crack model on the basis of elastic-viscoelastic analogy. For the bond rupture, it can be considered on the basis of the rupture of an interatomic bond, which is a thermally activated process. The net difference between frequency of bond rupture and bond reformation can be written as:

$$\Delta f \propto \sinh(c\sigma / RT)e^{-Q/RT} \tag{9}$$

where T = absolute temperature, R = gas constant and ε = energy of the vibrating atom. The rate of the opening w of the cohesive crack may be assumed to be approximately proportional to Δf. In this way, the following rate-dependent formula for the cohesive crack model can be deduced [41, 42]

$$w = g\left[\sigma - \kappa e^{Q/RT}\operatorname{asinh}\left(\dot{w} / c_0\right)\right] \tag{10}$$

Cyclic loading – Cracks slowly grow under cyclic (fatigue) loading. After a certain number of cycles, a crack can reach a critical length that leads to the failure. For materials like metal, with a negligible fracture zone, the crack growth can be correlated to the amplitude of the cyclic stress intensity factor and described well by Paris law. However, for quasibrittle materials which have a large fracture process zone, it is found that the Paris law needs to be combined with the size effect law for monotonic loading. This yields the following generalization of Paris law in which the size effect of structure size D is taken into account [43]:

$$\frac{\Delta a}{\Delta N} = \kappa\left(\frac{\Delta K_I}{K_{IC}}\sqrt{1 + \frac{D_0}{D}}\right)^n \tag{11}$$

where a = crack length, N = number of cycles, K_I = stress intensity factor, and κ and n = dimensionless empirical constants.

7. Experimental Evidence, Structural Applications and Consequences

During the last 30 years, a marked size effect, which is much stronger than Weibull's statistical theory, was observed in quasibrittle materials. Fig. 1 shows the results of size effect tests on various quasibrittle materials such as concrete [44], SiC ceramics [45], sea ice [46], carbon composites [47], vinyl foam [48], Al alloy, limestone [49], and vinyl foam-laminate sandwich beams [50]. The corresponding deterministic size effect laws of Type I and Type II agrees well with the experimental results.

The deterministic size effect is governed by fracture behavior of quasibrittle materials. Therefore, the size effect can be used to identify the fracture characteristics. By calibrating Eqs. (1) and (4) with the size effect tests, one can obtain the values of G_f and c_f. The size-effect method to measure the fracture characteristics has been adopted for an international standard recommendation for concrete [10], and has also been verified and used for many other quasibrittle materials. The advantage of this method is that the tests, measuring only the maximum loads, are easy to carry out.

The large values of the safety factor (about 3 to 8 for small sizes structures) implied by concrete design codes make it difficult for structures to collapse solely by size effect. However, when multiple errors happen in practice, the size effect will be an important contributing factor in structural failure. Recent investigation shows size effect has played an important role in collapses of Malpasset Dam (1959) [51], St. Francis Dam in L.A. (1928), warehouse in Wilkins Airforce base (1955) and Sleipner oil platform in Norway (1991). A preliminary study of the collapse of the Koror-Babeldaob Bridge in the Republic of Palau also shows a major strength reduction due to size effect may have played a role.

8. Closing Comment

Although the size effect on the mean strength of structures is now widely considered in the design of large and sensitive concrete structures, and is automatically captured by commercial finite element codes when the crack band approach is used, the size effect on the safety factors is still ignored in practice. This is a problem where further progress is imperative, in order to guarantee the expected survival probability ($1-10^{-6}$) of large concrete structure, and large aircraft and ships made of fiber composites.

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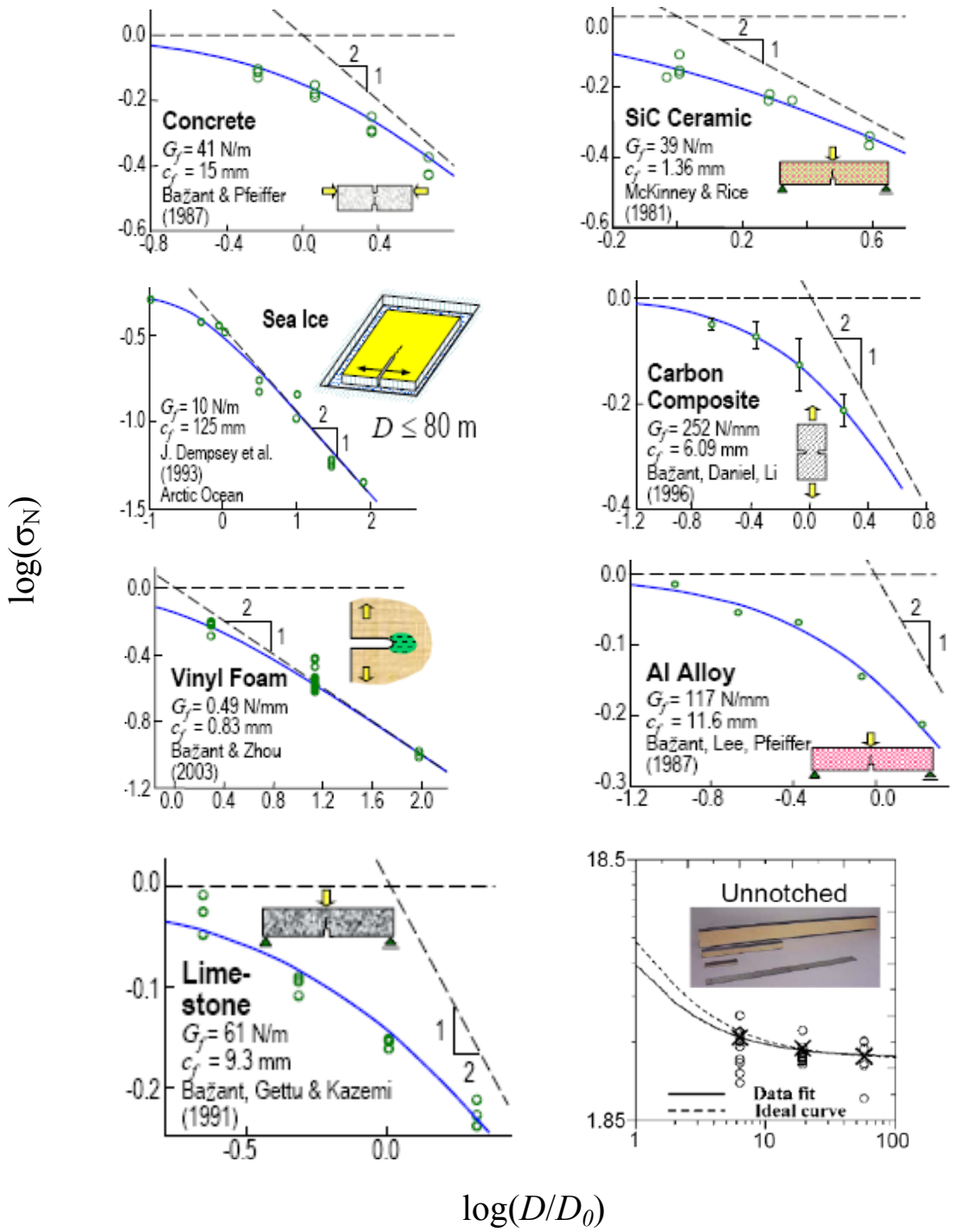

Figure. 1 Size effect tests on quasibrittle structures