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# **Analysis of Crack Propagation in Concrete Structures by Markov Chain Model and R-curve Method**

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## Abstract

Due to the random nature of material properties, crack growth is a random process. A probabilistic model for randomness of the progressive crack growth in a quasibrittle material such as concrete is presented. The model consists of a Markov chain adapted to R-curve fracture behavior. The method is useful particularly for concrete structures with existing major cracks. The R-curve serves as the mean curve for the fracture process. The R-curve under monotonic loading is obtained from the given test data on the effect of structural sizes. The standard deviation of the mean curve is predicted from the standard deviation of the load carrying capacity of specimens. The parameter estimation method is formulated and applications to existing structures are illustrated.

## Introduction

The randomness of fracture properties of concrete is quite pronounced. The previous investigations concentrated mainly on the deterministic and statistical behaviors of concrete fracture at the peak load. The purpose of this paper is to develop for quasibrittle materials such as concrete a probabilistic model that can predict the probabilities of the crack growth under a given increment of loading as well as the probabilities of failure for the given increment of loading. The method will consist of a Markov chain model combined with the R-curve behaviors. The parameter estimation method for the model parameters will be formulated. Finally, some applications will be presented.

## Markov chain model for crack growth in concrete

The Markov chain model is a stochastic model commonly used to analyze many kinds of accumulative damage processes (Bogdanoff and Kozin, 1985). In this paper, the damage of concrete is considered to be a crack. An observable measure of the crack, called the damage state, can be represented by the crack length, or the crack mouth opening displacement (CMOD). The well-known basic evolution equation for the Markov chain model is

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$$p_x = p_0 P^x \quad (1)$$

$x$  is an integer, denoting the value of  $X$ , and  $X$  is the load or a load parameter proportional to the load, such as the nominal stress;  $p_0$  is the vector of initial state probability,  $p_0 = \{\pi_1, \pi_2, \dots, \pi_B, 0\}$  with  $\sum \pi_j = 1$ , in which  $\pi_j$  = probability that damage state  $j$  is initially occupied. In the present study, we always assume that initially  $\pi_1 = 1$ , with all other  $\pi_j = 0$ , which means the crack or damage always starts from state 1.  $p_x$  is the vector of the damage state probability,  $p_x = \{p_x(1), p_x(2), \dots, p_x(B)\}$ , in which  $p_x(j)$  = probability that damage state  $j$  is occupied at loading level  $x$ ;  $B$  = the failure state;  $P$  is the probability transition matrix. For a unit-jump model (crack can only jump from point  $i$  to  $i+1$ ),  $P$  is

$$P = \begin{bmatrix} p_1 & q_1 & 0 & \dots & 0 \\ 0 & p_2 & q_2 & \dots & 0 \\ 0 & 0 & p_3 & \dots & 0 \\ 0 & 0 & 0 & p_{B-1} & q_{B-1} \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad (2)$$

where  $p_i$  = probability of remaining in the state  $i$  during one loading step, and  $q_i$  = probability that in one loading step the damage moves from state  $i$  to state  $i+1$ .  $p_i + q_i = 1$  and  $r_i = p_i/q_i$ .  $r_i$  and  $B$  are the parameters to be determined.

### R-curve and Standard deviation of nominal stress before failure loads

To determine the parameters  $r_i$  and  $B$ , the deterministic equation for the nominal stress will be used as a mean curve and may generally be written in the form:

$$\bar{X} = \frac{\sqrt{R(c)E_c}}{\sqrt{\pi a} F\left(\frac{a}{d}\right)} \quad (3)$$

where  $\bar{X}$  represents the mean nominal stress (which is proportional to the applied load),  $E_c$  is initial elastic modulus,  $R(c)$  is the R-curve, which represents the energy required for crack growth as a function of the crack extension  $c$ ,  $F(a/d)$  is a geometry-dependent function available for many different geometries of specimens from fracture mechanics handbooks (e.g. Tada, 1983),  $a$  is the current crack length. The R-curve can be obtained using the size effect law proposed by Bazant and Kazemi (1990).

The other information needed for determining of the parameters is the standard deviation of load or nominal stress from the mean curve before the peak load.

Literature search has shown that for concrete there has not been much research on the statistical analysis of crack propagation. To obtain the standard deviation, three-point-bend tests have been conducted for many concrete beams. The detailed description of the tests can be found elsewhere (Xi et al., 1995; Amparano and Xi, 1996). The deviation of the load along the obtained CMOD - load curves can be approximated as a linear function of CMOD. We assume that the standard deviation of the load in terms of crack extension would be similar to standard deviation of the CMOD - load curves. As a result, the standard deviation of the load prior to the peak load may be assumed to be a linear function of the crack extension

$$s_j^2 = \frac{(c_j)}{(c_{\max})} s_{\max}^2 \quad (4)$$

where  $c_{\max}$  is the crack extension at the peak load, which can be obtained from (3);  $s_{\max}^2$  represents the variance of the peak load. From (4), the standard deviation of the entire process of crack propagation may be predicted solely from the standard deviation of the peak loads.

#### Markov chain model combined with R-curve

Based on (3) and (4), we first divide the damage states  $j = 1, \dots, B-1$  into  $j$  groups as follows:  $1, \dots, B_1 - 1; B_1, \dots, B_2 - 1; \dots$ ; and  $B_j, \dots, B-1$ . Then we assume  $r_1$  for  $1, \dots, B_1 - 1$ ;  $r_2$  for  $B_1, \dots, B_2 - 1; \dots$ ; and  $r_j$  for  $B_j, \dots, B-1$ . The expressions for  $r_i$  and  $B_j$  are

$$B_j = \frac{(\bar{X}_j - \bar{X}_{j-1})^2}{(\bar{X}_j - \bar{X}_{j-1}) + (s_j^2 - s_{j-1}^2)} + B_{j-1} \quad r_j = \frac{\bar{X}_j - \bar{X}_{j-1}}{B_j - B_{j-1}} - 1 \quad (5)$$

For  $j = 1$ , we have  $B_1 = \bar{X}_1 / (s_1^2 + \bar{X}_1) + 1$  and  $r_1 = \bar{X}_1 / (B_1 - 1) - 1$ . After all the parameters,  $B_j$  and  $r_j$ , have been determined, we can use (1) and (2) to calculate the damage state probability.

Consider now, as an example, notched three-point-bend concrete beam specimens. The details of the test can be found in Gettu et al. (1991). The maximum nominal stress  $X_{\max} = 66$  psi, and the maximum crack extension  $c_{\max} = 0.274$  in. To be able to draw the crack extension  $c$  and the nominal stress  $X$  in a two-dimensional square mesh with the probability as the third dimension, the crack extension must be manipulated by a transformation,  $N_j = j c_{\max} / X_{\max}$  ( $j = 1, 2, \dots, 66$ ). The three-dimensional graph of  $N_j$ ,  $X$  and the probability is shown in Fig. 1.

#### Conclusions

The discrete Markov chain model can be used to predict the probabilistic structure of progressive cracking under monotonic loading in materials characterized by R-curve behavior. The R-curve obtained by size effect analysis of the measured peak loads is employed as the mean curve. Three-point-bend concrete beam tests are performed in order to determine the variation of the load along the loading path. Based on the current test results, it may be reasonably assumed that the variance is a linear function of the crack extension. Thus, the standard deviation of the entire process of crack propagation may be predicted solely from the standard deviation of the peak loads of specimens. The probabilistic information on the damage evolution process predicted by the present model covers the entire monotonic loading history.

### References

Bazant, Z.P., and Kazemi, M.T. (1990), "Size Effect in Fracture of Ceramics and Its Use To Determine Fracture Energy and Effective Process Zone Length", *J. Am. Ceram. Soc.*, 73(7), 1841-1953.

Bogdanoff, J.L. and Kozin, F. (1985), "Probabilistic Models of Cumulative Damage", John Wiley & Sons, New York.

Gettu, R., Bazant, Z.P., and Karr, M.E. (1990), "Fracture Properties and Brittleness of High-Strength Concrete", *ACI Material Journal*, Nov.- Dec., 87, 608-618.

Tada, H. (1983), "The Stress Analysis of Cracks Handbook", Del Research Corp., St. Louis, MI 63105.

Xi, Y., Amparano, F.E., and Li, Z. (1995) "Effect of Aggregate Content on Fracture Behavior of Concrete", *Proc. of Materials Research Society Fall Meeting, Symposium Q*.

Ampanano, F.E., and Xi, Y. (1996) "Experimental Study on the Effect of Aggregate Content on Fracture Behaviors of Concrete", to be submitted.

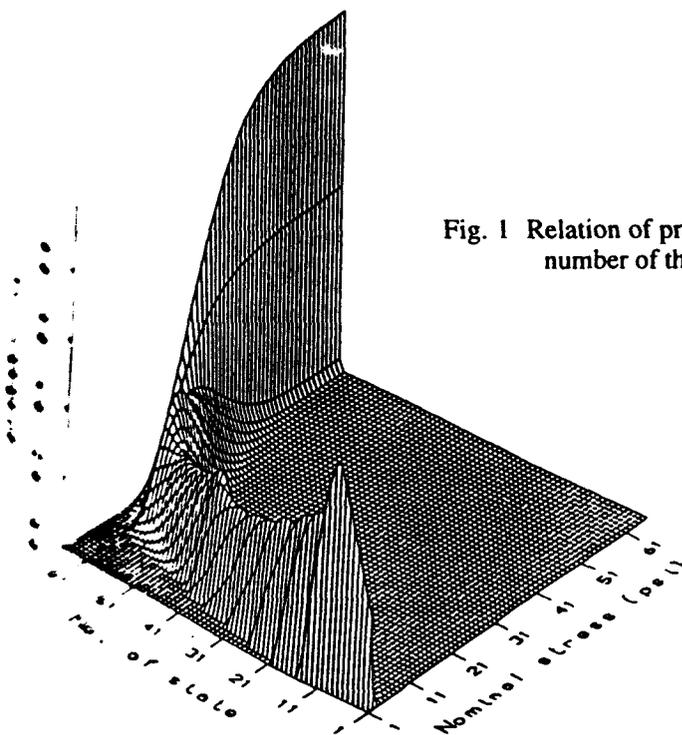


Fig. 1 Relation of probability, nominal stress, and the number of the state in the crack extension process